Experimental test for the curvature of space using the Pythagorean theorem

John M. Martinis
Department of Physics, University of California, Santa Barbara, CA 93106, USA
(Dated: December 5, 2012)

The Pythagorean theorem is expected to deviate from its mathematical formula for curved space. Here, an experiment tests the theorem by making squares with sides equal to the lengths of a right triangle, and then weighing their mass difference. The null result is consistent with the radius of curved space being greater than 1.1 meter, as expected.

INTRODUCTION

The Pythagorean theorem is a widely used relationship in mathematics and physics, as it relates the lengths a and b of two sides of a right triangle to its hypotenuse c via the formula $a^2 + b^2 = c^2$. The mathematical proof assumes the triangle lies on a flat Euclidean space [1]. Because general relativity causes matter to curve space [2], one expects the Pythagorean theorem to not precisely hold in the real physical world.

In this letter, I perform a simple test of the Pythagorean theorem in the length scale 3 - 30 cm [3]. This provides a simple, but unfortunately not very sensitive test of general relativity.

THEORY

The violation of the Pythagorean theorm can be understood and modeled for curved space corresponding to the surface of a sphere, as shown in Fig. 1(b). For right triangles that are much smaller than the radius R of the sphere, the local surface looks flat and the theory holds. For triangle distances of order the radius, the curvature of space becomes important. For the case of a right triangle with sides $a = b = R(\pi/2)$, corresponding to a length one-fourth the circumference of the sphere, the sides of the triangle run from the pole to two points on the equator. The length between the equator points is $c = R(\pi/2)$, giving the relation $a = b = c$ in violation of the Pythagorean theorem. Note here that the 3 angles of the triangle are all 90° and do not sum to the usual value 180°.

For the case $b = 0$, the Pythagorean theorem reduces to $a^2 = c^2$, which is expected to hold in any curved space because the line segments for a and c are identical. From symmetry arguments, one expects the maximum violation for sides of equal length $a = b$. Only this case is considered in this work.

For curved space given by a sphere and the case $a = b$, the length of the hypotenuse is given by [4]

$$c = \sqrt{2}\, a \cos(a/2R) \quad (1)$$

$$\simeq \sqrt{2}\, a\,(1 - a^2/8R^2) \quad \text{for } a \ll R\ . \quad (2)$$

This formula gives $c = a$ for the equator case discussed previously, and $c = 0$ for lines going from pole to pole, as expected. It is equal to the result from the Pythagorean formula for $a \ll R$, with only second order corrections due to the curvature of space R.

EXPERIMENTAL METHODS

As illustrated in Fig. 1(a), the Pythagorean theorem can be tested by comparing the sum of the areas a^2 and b^2 with the area of the hypotenuse c^2. To do so experimentally, we first cut from a sheet of paper a triangle with equal side lengths $a = b$. Two nominally equal squares were then cut with lengths a, and one square with side length c. The weight of the two smaller squares were then compared with the weight of the larger square on a differential balance, which determines precisely the difference

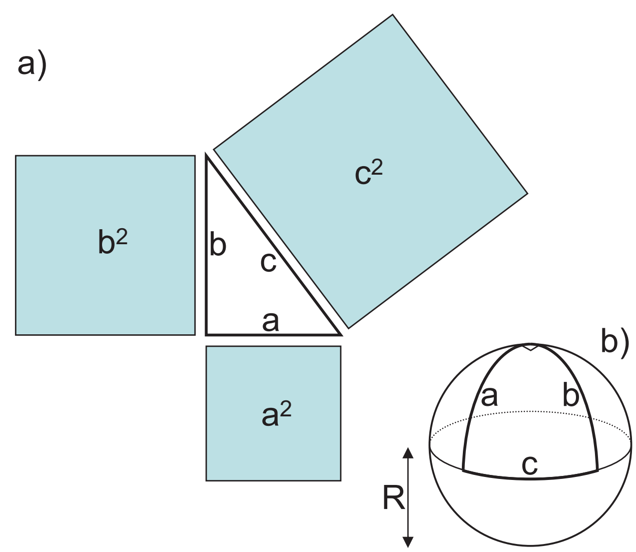

FIG. 1: (a) Drawing of a right triangle with side lengths a, b and c. Also shown at their edges are corresponding squares, with areas a^2, b^2 and c^2. The Pythagorean theorem states that the areas of the two sides sums to the area at the hypotenuse. (b) Drawing of the Pythagorean theorem in curved space, here modeled as a sphere of radius R. In the curved 2-dimensional surface of the sphere, a right triangle with equal sides $a = b = R(\pi/2)$ runs from the pole to the equator. Here, the length of the hypotenuse at the equator is $c = a = b$.

in mass ΔM_c. The mass of the larger square was also measured so as to obtain an fractional error $\Delta M_c/M_c$.

The dominant error in the experiment was imprecision to the cut dimensions, which we estimated to be about $\Delta x = \pm 1\,\mathrm{mm}$. The fractional uncertainty in the area is $c\Delta x/c^2$ for each cut of the larger square. This gives a total uncertainty to the fractional difference in mass

$$\left(\frac{\Delta M}{M_c}\right)^2 = 2\left(\frac{c\Delta x}{c^2}\right)^2 + 4\left(\frac{a\Delta x}{c^2}\right)^2 \quad (3)$$

$$= 2\left(\frac{\Delta x}{a}\right)^2 \quad (4)$$

where we have included two cuts for the large square and four cuts for the two smaller squares, with all uncertainties added together in quadrature because errors in cuts are uncorrelated.

Another source of error is deviations from 90° in the corner angles of the squares. To minimize errors, we cut the squares from the corners of the paper sheets, which we measured to have an angle error $\Delta\theta \lesssim 1°$. The difference between the area of a square and parallelogram gives a fractional area error

$$\Delta A/A = 1 - \cos\Delta\theta \quad (5)$$

$$\simeq (\Delta\theta)^2/2 \quad (6)$$

$$\lesssim 1.4 \times 10^{-4}\ . \quad (7)$$

This error is significantly smaller than the cut errors, so can be neglected.

In order to test for the variation of the density of the paper, we also compared the weight of the two identical small squares of area a^2. As shown in Fig. 2(a), the misbalance of these squares is well accounted for by the cut errors $\Delta M/M_a = 2\,\Delta x/a$, implying density errors are negligible.

RESULTS

The results of the experiment are plotted in Fig. 2(b) for three values of a. In all cases the small mass differences can be accounted for by the uncertainties in the cut dimensions, so the null results are in good agreement with the Pythagorean theorem.

The mass of the large squares is $M_c = 0.011$, 0.099 and 1.10 grams for the $a = 3$ cm, 10 cm and 100 cm trials, respectively. As these weights are only used for comparison of the fractional weight differences, their uncertainty of about 1 mg is unimportant.

ANALYSIS

We can use Eq. (2) and the uncertainty of the largest trial to compute a bound on the radius R of curved space.

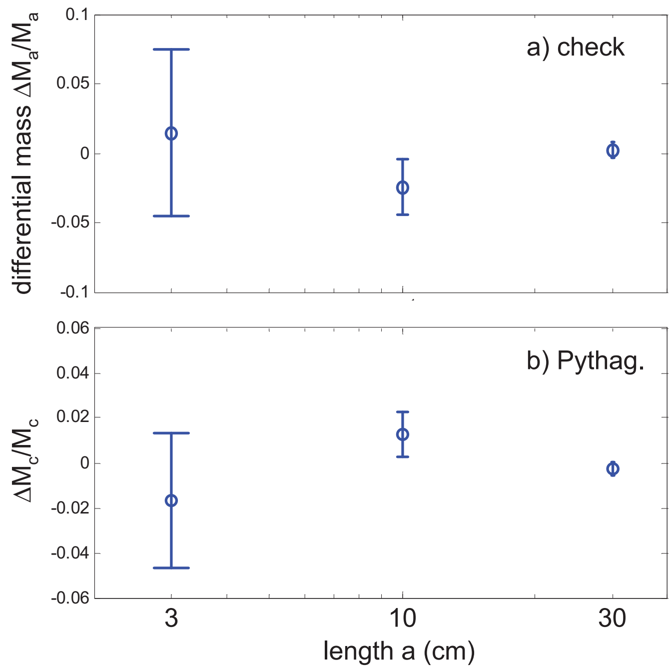

FIG. 2: Plot of differential mass measurements, normalized to mass, versus length of side a for 3 different trials. Error bars are from cut uncertainties $\Delta x = \pm 1\,\mathrm{mm}$. (a) Plot of results of check experiment, where we compare two squares both of area a^2 and mass M_a. (b) Test of Pythagorean theorem, where we compare sum of two squares each of area a^2 against square of area c^2 and mass M_c. Both plots show near-zero mass differences that are within expected 1-σ uncertainties.

From the relation

$$\frac{\Delta M}{M_c} > \frac{a^2}{4R^2} \quad (8)$$

we find from the estimated uncertainty of Eq. (4) a bound

$$R > 2.2\,\mathrm{m}\ . \quad (9)$$

Since one expects the radius of curvature to be roughly the size of the universe, this does not put an interesting or useful bound on the physics of general relativity.

CONCLUSION

The Pythagorean theorem has been tested using a weight balance method, giving results that agree with the formula. The uncertainty of the measurement has been used to put a bound on the radius of curvature of space $R > 1.1$ meter. From the scaling of this bound on parameters, it is not expected that this experimental method can usefully test general relativity.

APPENDIX

In this appendix I comment on how this manuscript was written to more clearly explain the elements of a scientific paper. Please note that I am writing for an extremely simple experiment that will purposely not take much time to read and understand: real papers will be longer and contain more detailed information. This paper took about 6 hours to write.

Title. I use the words "experimental test" to indicated this is an experimental work, "curvature of space" to indicate a general relativity test, and "Pythagorean theorem" to suggest the general experimental method.

Abstract. The first sentence summarizes the introduction and theory section. The second sentence gives the general experimental method. The third sentence summarizes the results and analysis section. I have tried to make this as concise (short) as possible. Note I have not mentioned that only the case $a = b$ is considered since this is hard to explain concisely and not an important detail for the abstract.

Introduction. I start the introduction with a statement of the Pythagorean theorem, which I do not derive since I assume the reader can look it up, and because its a mathematical theorem and does not express any physics. I then explain the physics of this paper, a test for the curvature of space from general relativity. Note in the introduction I am only making general statements, and that a fuller explanation comes later.

In the second paragraph, I want to explicitly tell the reader what the paper is about, and why it is important for them to read. Note I have introduced (without explanation) a subtle concept that we test general relativity on some length scale, which will be explained later in the theory section.

Theory. The first paragraph introduces the concept of how the Pythagorean theorm could be violated using curved space. I show a simple example so that the basic violation is clearly stated. Note that I have not gone into a detailed description of how general relativity works, mostly because this experiment is pretty lame and will not give a useful limit. So I instead try to make the theory very clear.

The second paragraph deals with an important experimental issue: I don't want to waste my time testing the Pythagorean theorem for all possible parameters. I discuss how the maximum violation is for the case $a = b$, so that is all that needs to be tested.

The third paragraph now discusses the effect of curved space on all possible length scales, not just the special case discussed in the first paragraph. Here I just state the result, but also give an expansion formula for small a since that will be needed for the analysis section.

Figure 1. Here I draw a right triangle, which defines the parameters a, b and c. I draw squares of areas a^2, b^2 and c^2 since my experimental method is to compare the weight of these areas. In subpanel (b), I also draw the case for curved space, defining R and the simple case described in the theory.

Experimental Methods. The first paragraph details how the experiment was done by cutting paper and weighing the pieces with a differential balance. This description is straightforward, but the discussion of error takes more effort and space, as expected for a "precision measurement" experiment.

I next talk about the dominant error in the experiment, which is particularly important here since it limits the performance of this method. I explicitly define the most important uncertain parameter Δx. I then write down the first formula so that one can trace back how the terms were included, given counting arguments in the text after the formula. Note that I explicitly assume that all cut errors are uncorrelated.

I then explain that other errors can be neglected.

The last paragraph could be put into the results section, since it is a differential test of the two equal masses. However, since it is a test of the experimental method and implies that density variations don't matter, I include it in the methods section. Note that this test is really useful as it provides a global test to the accuracy of the differential weight measurement, so that if I forgot some error it would probably show up in this check experiment. Note this is another reason to test the case $a = b$.

Figure 2. I could have plotted both ΔM_a and ΔM_b data on the same plot, but the overlaps would have been confusing. This way a) can be labeled as a check experiment, and b) as the Pythagorean theorem test. Note I explicitly define the x and y scales, and discuss the conclusion from the data.

Results. I refer to the data in Figure 2, and explicitly state that the small differences are accounted for by uncertainties. Note that since the figure has a plot of mass differences relative to M_c, I also include these masses so that one can figure out the actual differential weights that were measured.

Analysis. I combine some equations already in the paper to put a bound on R. Note that since I have to take the square of Eq. (2) to obtain the area uncertainty, I have written this equation in the form of a differential area so that it states the needed formula.

Conclusion. Here I just review the basic ideas of the experiment, and summarize the conclusion. I also add a comment as to the feasibility of future experiments using this method: since it provides a very weak bound on space curvature, I am honest and say this is not a very useful experimental method.

[1] Euclid (1956). Translated by Johan Ludvig Heiberg with an introduction and commentary by Sir Thomas L. Heath. ed. The Elements (3 vols.). Vol. 1 (Books I and II) (Reprint of 1908 ed.)

[2] Charles W. Misner, Kip. S. Thorne, John A. Wheeler (1973), Gravitation, W. H. Freeman, ISBN 0-7167-0344-0.

[3] Note that this is an example paper for the physics class 13A/15A, so the experiment was not actually performed.

[4] Because I am busy, I only guessed this formula. It is correct for the three important limiting cases given in the text.