Gary's larger dimensions on Numerical R
turbulent influence

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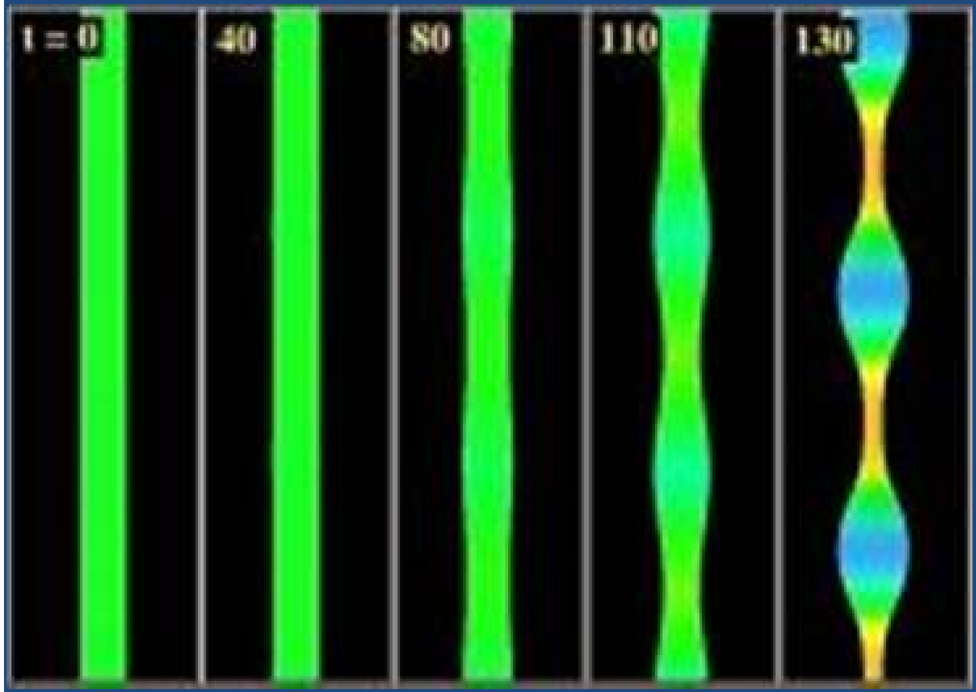
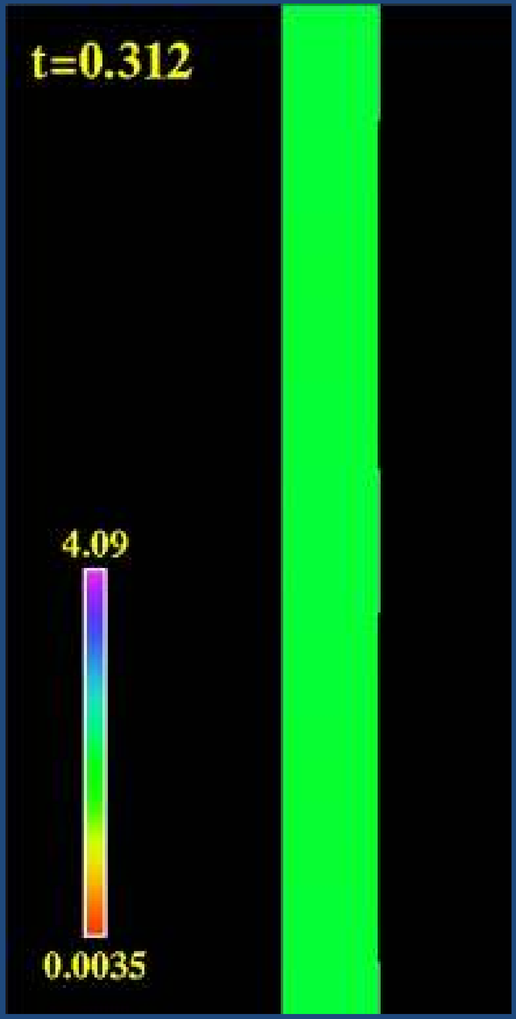
Last ‘Gary’ story & Num Rel…

- Black strings. “Can you consider a black hole in 10d?” (ITP 1999).
  - Final fate of black strings subject to the GL instability ? [Horowitz-Maeda ‘01]



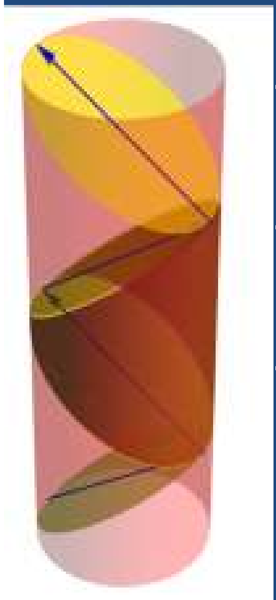
[Choptuik etal ’03]



[LL-Pretorius ’10]

Motivation…

- The AdS/CFT correspondence relates a (d)−QFT with a (d+1)−dimensional theory of gravity.
  - Any gravitational phenomena should have an equivalent CFT analog, and vice-versa.
  - A natural arena to study field theory open questions: transport properties in strongly coupled field theories, quantum turbulence, etc.
  - Plenty of applications. Most of which in equilibrium situations and in the probe limit (phase space analysis) (e.g. CMT applications)
  - Long list (and growing!) of efforts in dynamical settings [Chesler,Yaffe;Das,Nishioka,Takayangi,Basu;Bhattacharya,Minwalla;Romatschke, Bantilan,Gubser,Pretorius;Abajo,Aparicio,Lopez;Albash,Johnson,Ebrahim,Headrick,Balasubramanian,Bernamonti,deBoer,Copland,Craps,Keski,Mueller,Shaffer,Shigemor,Staessens,Galli,Schwelinger,Caceres,Kundu,Wi,Gauntlett,Simons,Wiseman,Sonner,Myers,Buchel,LL,vanNikerke,Abajo,daSilva,Lopez,Mas,Serantes,Dias,Santos,Marolf, *Horowitz*]



[Image: J. Santos]

- *Black holes have become the 'harmonic oscillator of the 21st century'* [A. Strominger]

Holographic *path to the promise land*

- Goal: understand properties of out of equilibrium phenomena and its eventual thermalization. Is there a universal behavior?

- AdS/CFT offers a way into: strongly coupled field theories, *provides a real-time analysis*, allows for considering finite temperature setups and is amenable to general spacetime dimensions. From a gravity perspective, interesting excuse to push intuition

- Dynamical qns involve time-dependence (solve PDEs)
  - Quark-gluon plasma: thermalization? Hydrodynamization?

    [Chesler,Yaffe,Heller,Romatschke,Mateos,vanderSchee,Fernandez,Bantilan,Gubser,Pretorius,….]
  - ``Quantum quenches'': universal properties as a response to fast and slow quenches in the Hamiltonian of a system (N=4 SYM <-> mass-deformed gauge theory   [Balasubramanian etal; Buchel,Myers,vanNiekerk,LL;Das,Das….]

- Out of equlibrium behavior from a given initial state will involve transfer of energy
  – How does it take place?
  – What’s its time scale?
  – How and where to does energy flow?
  – [Note: many problems are essentially the same in gravitational terms]

- Starting with a perturbation off a thermal state, thermalization time scale given by : perturbation time scale (adiabatic case), or a ~ scale consistent with the slowest –triggered- black hole QNM (abrupt case)
  – Perhaps perturbative arguments do provide the answer. In particular we can perturb BHs and analyze QNMs. Can also analyse perturbatively pure AdS and obtain relevant time scales. Is this all?

- *Perhaps not….*
  - Perturbative analysis & conclusions are notoriously delicate [e.g. abuse takes place all too often, phenomena might be obscured through an unfortunate choice of perturbative scheme. Also what a good scheme is might be a ``moving target’’]
  - QNMs aren’t a basis even in AdS [Warnick 2013]
  - QNMs for relevant cases might not yet be known (e.g. d=4,5 Kerr-AdS [Cardoso,Dias,Santos,Harnet,LL dec 2013])

- *And worse yet…*
  - Kerr-AdS is not known to be stable [in fact math arguments for the opposite]. Further, if not ‘linearly-stable’, we can’t use QNMs in a straightforward way
  - Math arguments for `pure’ AdS being unstable and specific illustrations. This is good ‘academically speaking’, but AdS/CFT demands more in regards to thermalization. Will all ``non-pure states in the CFT’’ yield configurations always leading to BH formation? If not, what’s the path to a thermal state?

Turbulence (in hydrodynamics)

some would say: "that phenomena you know is there when you see it''

For Navier-Stokes (incompressible case):

- Breaks symmetry (recovered only in a 'statistical sense')
- Exponential growth of (some) modes [not linearly-stable]
- Global norm (*non-driven case*): Exponential decay possibly followed by power law, then exponential
- Energy cascade (direct d>=3, inverse/direct d=2)
- $E(k) \sim k^{-p}$ (*5/3 and 3 for 2+1*)

‘Turbulence’ in gravity?

- Perhaps there isn’t… (arguments against it, mainly in 4d)
  - Perturbation theory (e.g. QNMs)
  - Numerical simulations (e.g. ‘scale’ bounded)
  - (hydro has shocks/turbulence, GR no shocks)
- Perhaps there is…
  - AdS/CFT <-> AdS/Hydro (turbulence?! [Van Raamsdonk 08])
  - Applicable if $LT >> 1$ $L\,(\rho/\nu) >> 1$ $L\,(\rho/\nu)\ v = Re >> 1$
  - (membrane paradigm? / Blackfolds)
- List of questions…
  - Tension in the correspondence or gravity?
  - Reconcile with QNMs expectation? (and perturb theory?)
  - If there is, does it have similar properties to hydro case?
  - What’s the analogue `gravitational’ Reynolds number?

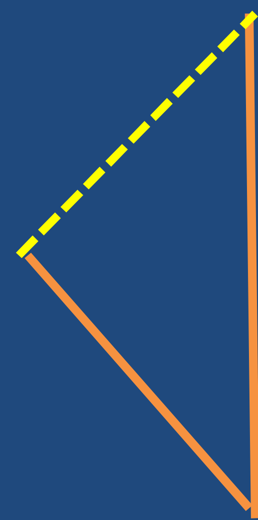
If there is turbulence….

- Multiple scales would ``pop up'' dynamically
- Linearized analysis is insufficient
- Self similarity of spacetime fractal structure
- Spectra of energy might leave particular relics in, e.g. grav waves, matter/energy structure, etc.
- Can play a role as a 'virtual' censor depending on decay properties
- Can help understand turbulent behavior in hydro
- Out of equilibrium behavior might show clearly spacetime dimensionality, etc…

- AdS/CFT gravity/fluid correspondence *(definition?)*

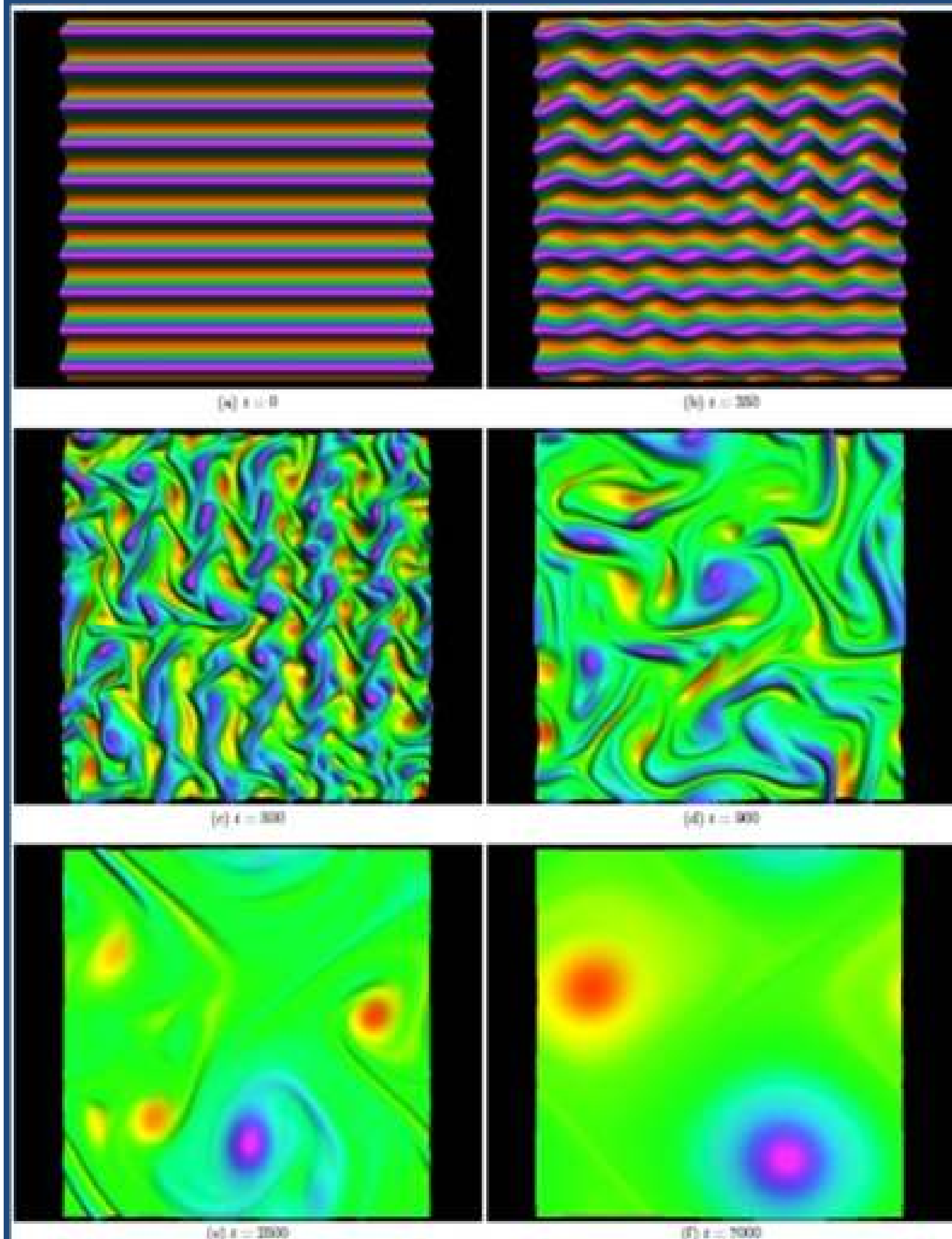
[Bhattacharya,Hubeny,Minwalla,Rangamani; VanRaamsdonk;
Baier,Romatschke,Son,Starinets,Stephanov]

$$ds^2_{[0]} = -2u_\mu dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{1}{(br)^d} u_\mu u_\nu \right) dx^\mu dx^\nu .$$



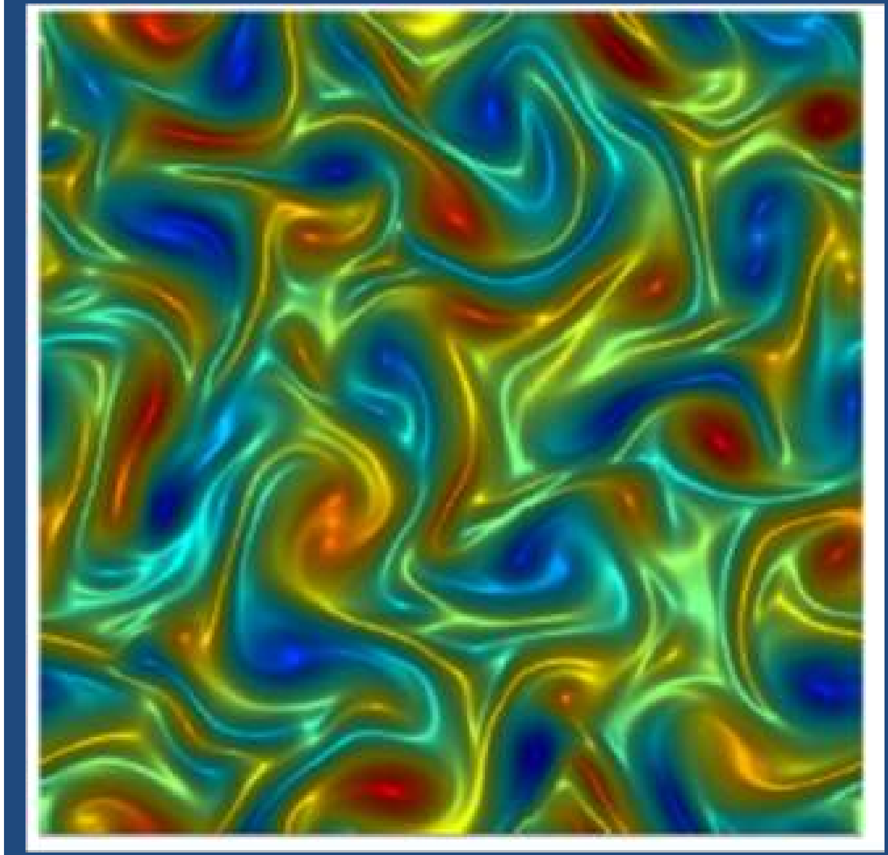
- $T_{ab} = T_{ab} = \frac{\rho}{d-1}(d u_a u_b + \eta_{ab}) + \Pi_{ab}$
- Subject to :
  - $u_a u^a = -1$; $T^a{}_a = 0$; $\Pi_{ab} = -2\eta\sigma_{ab} + \cdots$
  - $\nabla_a T^{ab} = 0$.
- Do these eqns/eos give rise to turbulence?
  - Non-relativistic limit → Navier-Stokes eqn. why wouldn't they?
  - If so, NS eqns have indirect cascade for 2+1 dimensions. Why? There exists a conserved quantity: *enstrophy. Does it exist for these eqns/eos?*

[Carrasco,LL,Myers,Reula,Singh 2012]

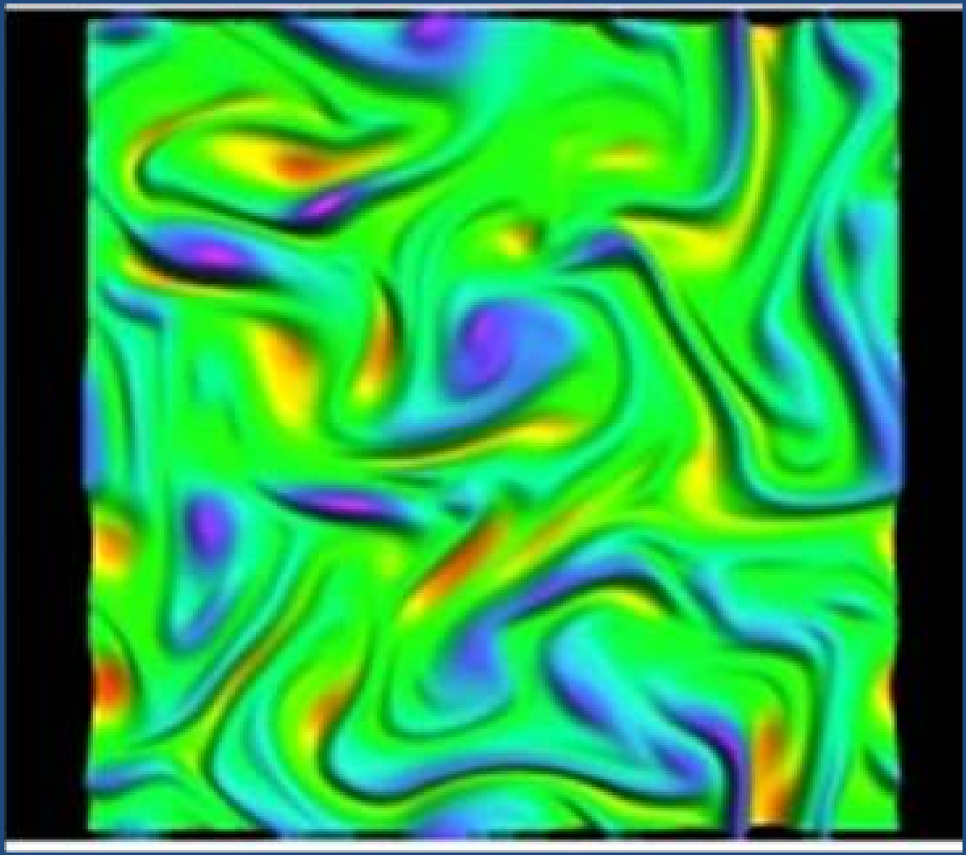


- Using correspondence on can reconstruct the spacetime
- Spacetime describes gravitational ‘tornadoes’ connecting boundary with horizon.
- Also, 3+1 hydro with conformal eos leads to direct cascade.

Bulk & holographic calculation



[Adams,Chesler,Liu PRL 2014]



[Green,Carrasco,LL, PRX 2013]

observations

- Inverse cascade carries over to relativistic hydro and so, gravitational turbulence in 3+1 and 4+1 move energy in opposite directions

(…warning for particular studies imposing symmetries that can eliminate relevant phenomena).

- Consequently 4+1 gravity (relative to QNM differences) equilibrates more rapidly (direct cascade dissipation at viscous scales which does not take place in 3+1 gravity)

- *Note 1: GR-Hydro correspondence established in the regime where slow QNMs dominate. How is the transition to such regime?*

- *Note 2… there are always limits to numerical solns!*

- From a hydro standpoint: geometrization of hydro in general and turbulence in particular:
  - Provides a new angle to the problem, might give rise to scalings/Reynolds numbers in relativistic case, etc. Answer long standing questions from a different direction. *However, to actually do this we need to understand things from a purely gravitational standpoint. Obvious first targets:*
  - *What mediates vortices merging/splitting in 2 vs 3 spatial dims?*
  - *Can we interpret how turbulence arises within GR?*
  - *Can we predict global solns on hydro from geometry considerations? (e.g. Oz-Rabinovich '11)*
  - *Can we indentify what triggers this phenomena 'outside' the long-wavelength regime assumption?*
  - *Is AdS really needed?*

Hydro analysis?

- Must go beyond linear level. Obtain linear modes (sound, shear), then write $\mathbf{u}(\mathbf{k},t) = A(\mathbf{k},t)\hat{u}(\mathbf{k},t)$, Navier Stokes eqn:

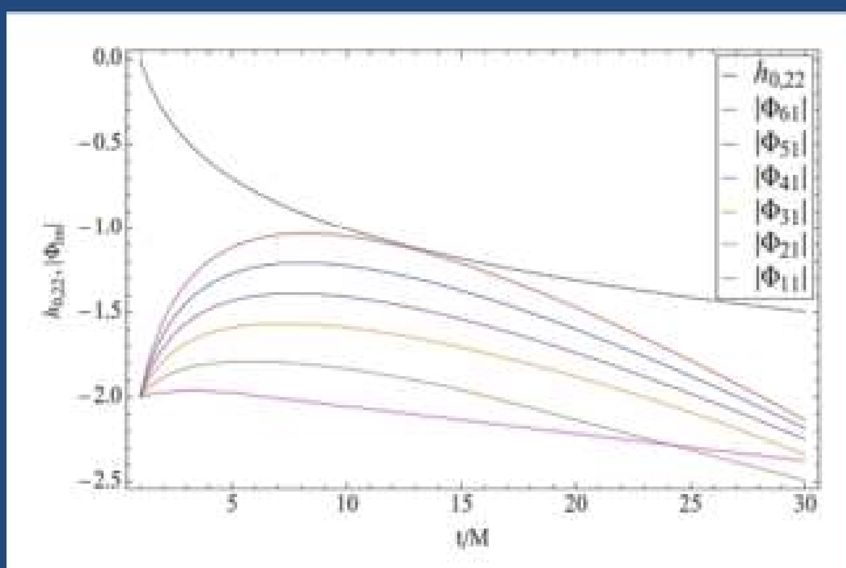
$$\left(\frac{\partial}{\partial t} + \nu k^2\right) A(\mathbf{k},t) = i \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \kappa(\mathbf{k},\mathbf{p},\mathbf{q}) A(\mathbf{p},t) A(\mathbf{q},t)$$
$$= i \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \{[\hat{u}(\mathbf{k},t)\cdot\hat{u}(\mathbf{p},t)][\mathbf{k}\cdot\hat{u}(\mathbf{q},t)] + [\hat{u}(\mathbf{k},t)\cdot\hat{u}(\mathbf{q},t)][\mathbf{k}\cdot\hat{u}(\mathbf{p},t)]\} A(\mathbf{p},t) A(\mathbf{q},t).$$

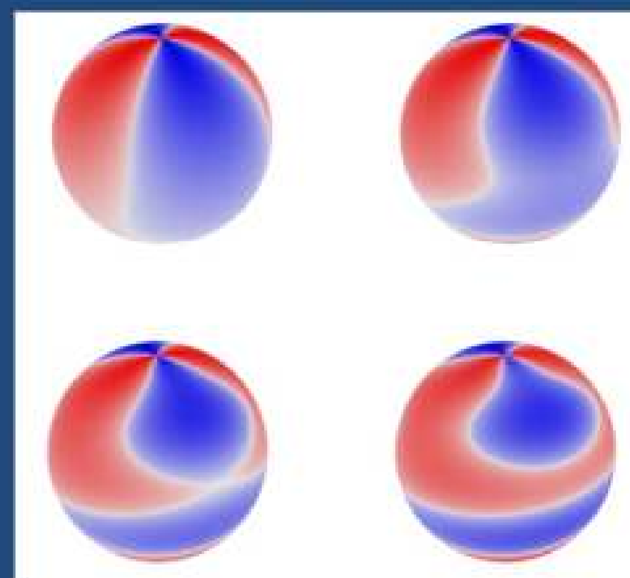
- κ(k,p,q) determine the couplings. These satisfy:
- $\kappa(\mathbf{k},\mathbf{p},\mathbf{q}) + \kappa(\mathbf{q},\mathbf{k},\mathbf{p}) + \kappa(\mathbf{p},\mathbf{q},\mathbf{k}) = 0$ conservation of energy
- and if: $k^2\kappa(\mathbf{k},\mathbf{p},\mathbf{q}) + q^2\kappa(\mathbf{q},\mathbf{k},\mathbf{p}) + p^2\kappa(\mathbf{p},\mathbf{q},\mathbf{k}) = 0$. cons. of enstrophy

Inverse cascade in 2+1 dimensions

- Ultimately what triggers gravitational turbulence?
  - AdS ‘trapping energy’ → slowly decaying QNMs & turbulence
  - Or slowly decaying QNMs → time for non-linearities to ``do something’’?
- Take rapidly spinning BH. To 2nd order $[Box + g(t)]f = 0$
- → ``parametric instability’’ with behavior analog to turbulence. The instability ``turns on’’ if the decay of perturbations is sufficiently slow *even in AF spacetimes*.
- “gravitational’ Reynolds number definition R ~ $h/(m\ w_I)$

[Yang-Zimmerman-LL]



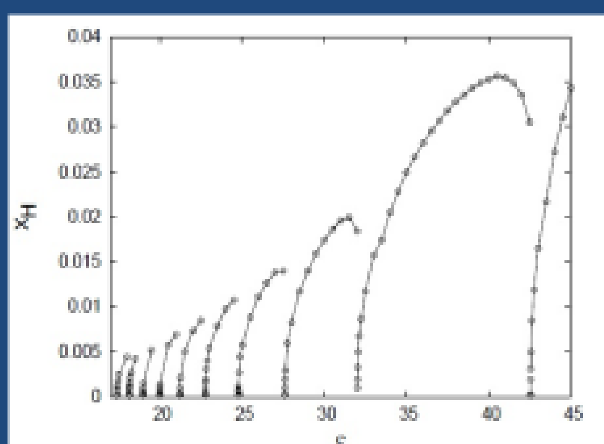


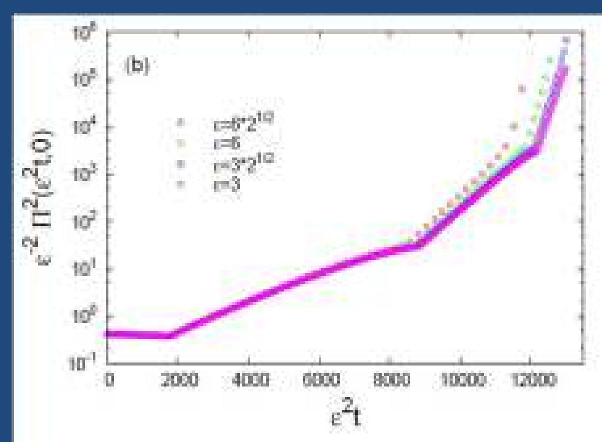
pure AdS and a path to thermalization

- What if we don't start with a BH?. Consider an out-of-equilibrium scenario in a CFT. One path to thermalization, from a holographic perspective, is through the formation of a black hole.

- Thus, studying the dynamics of 'pure-AdS' perturbed by suitable fields provides a way to probe (in a suitable limit) how this can be achieved.

- Bizon-Rostworowski revisited Choptuik's problem in (spherically symmetric) AdS . Dias-Horowitz-Santos for gravitational wave case.

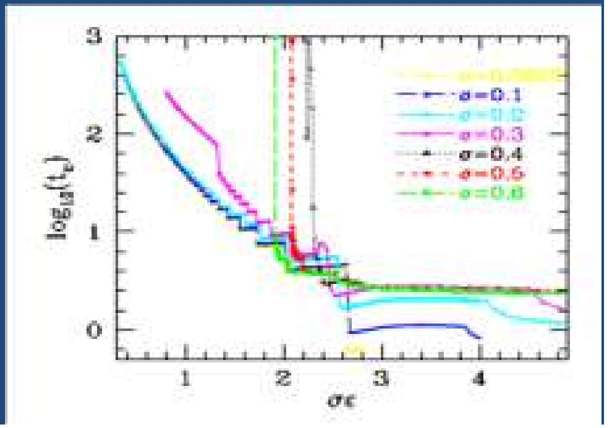




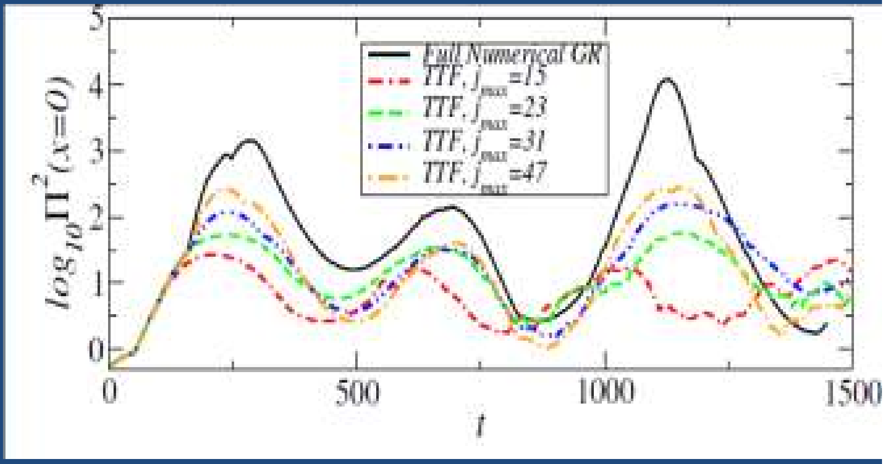
- BR result is rather convenient   Any amount of energy on the CFT side forms a BH in timescale ~ 1/energy, then it'd evaporate yielding a thermal state

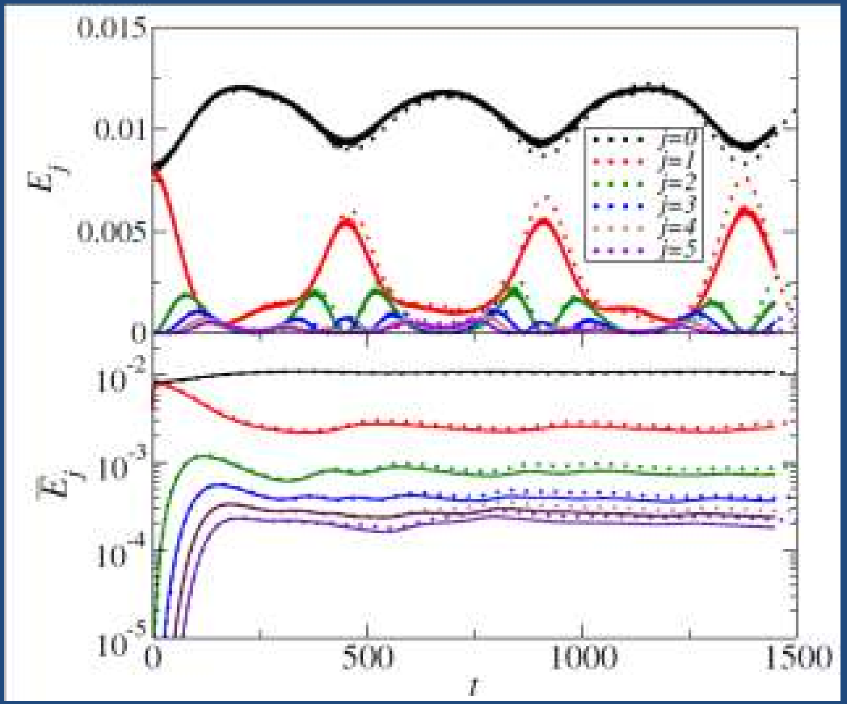
- Why does the collapse take place? (or, why is bounce #2 different from bounce #23?). What sets the timescale?
  – First: identify in the probe limit eigenfunctions and note the spectrum is fully resonant. Then: perform perturbative analysis including leading order backreaction , not all resonances can be absorbed by frequency shifts breakdown of perturbation at timescales ~ 1/energy.

- The above are compelling arguments but numerical solns showed
  – Many families of stable (stationary and quasi-stationary solns) exist: 'boson stars', 'oscillons', and even the same as used by BR with slightly different initial profiles. [& geons in the grav case –UCSB-]
  – How do they avoid collapse? What goes on?



- ‘Stable’ solutions for small enough amplitude → perturbative analysis should capture the behavior. An improved perturbative analysis (including a second time) gives:
- $\dot{A}_j \sim \sum \kappa^{j}_{klm} A_k A_l A_m$
  - Effect of resonances is accounted for capturing energy exchange among modes . Existence of 2 extra conserved quantities can be shown and also a Hamiltonian for the system.
- $E = \sum w_j^2 A_j^2$; $N = \sum w_j A_j^2$
- → cascade in both directions
  - Interestingly: eoms are the same as the ‘Fermi-Pasta-Ulam (Tsinglou)’ problem



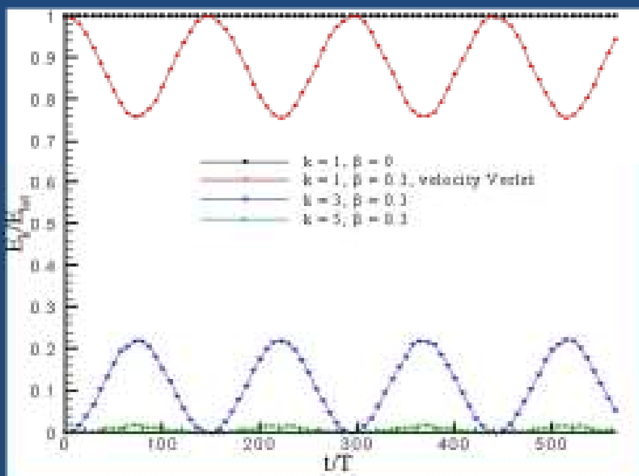


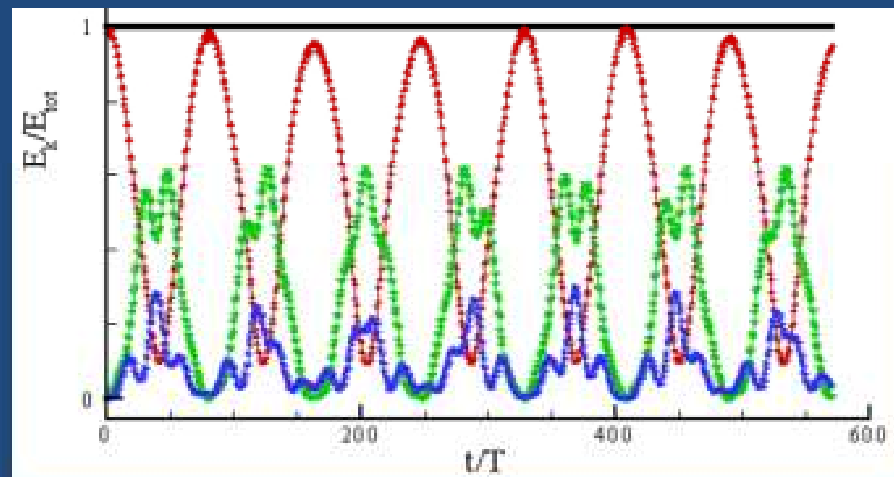
[Balasubramanian,Buchel,Green,LL,Liebling] also
[Craps,Evnin,Vanhoof]
[Dimikatropulos,Freivogel,Lippert,Yang]

Some FPU examples

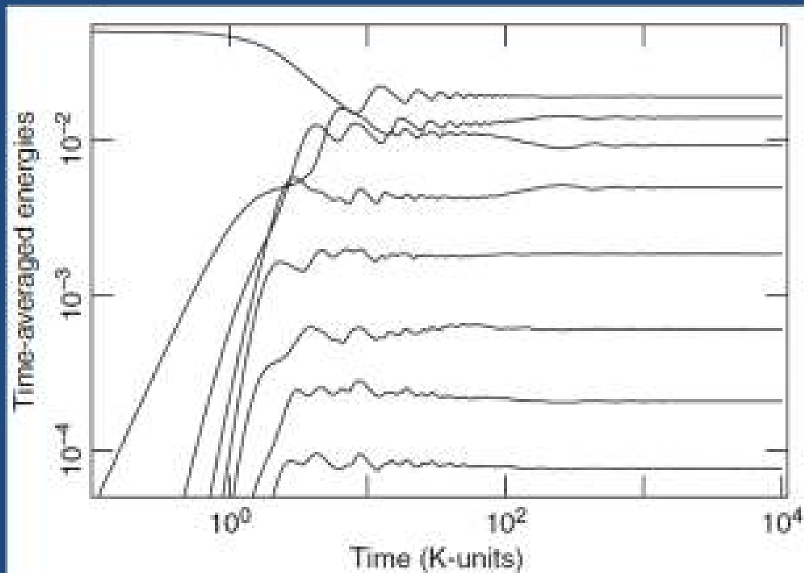
- Consider

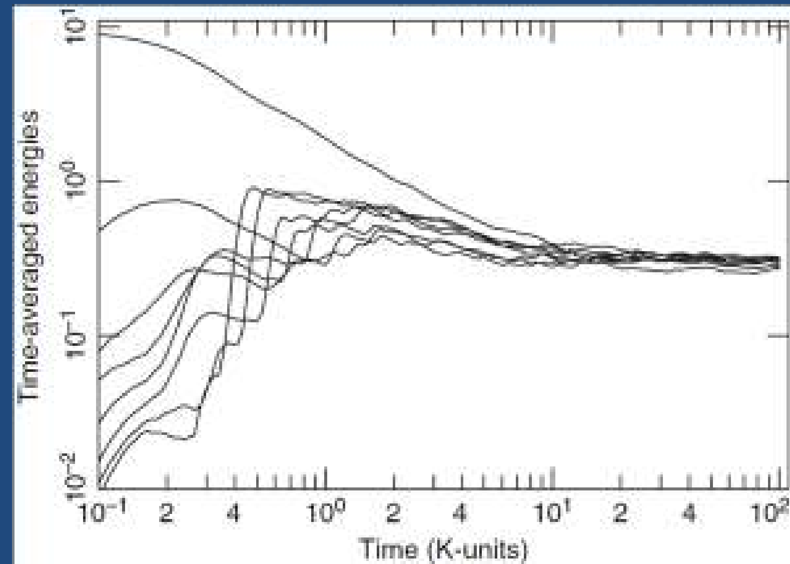
$$\ddot{x}_n = (x_{n+1} - 2x_{n+1} + x_{n-1}) + \beta([x_{n+1} - x_n]^2 - [x_{n-1} - x_n]^2)$$





- Resolution? Integrability or ergodicity dependent on the initial energy of the system
- Chirikov (55): ‘stochasticity’ [dynamical chaos]: Threshold of energy above which thermalization takes place





- Since GR in AdS (and spherical symmetry) FPUT problem and numerical results *imply many states display lack of thermalization through BH formation*.

- Further, a Floquet analysis can be performed to identify stability of ‘quasi-periodic’ solutions and *recurrence* period [Green,Maillard,LL]
- *‘Enhanced’ perturbative analysis provides: conserved quantities, cascade intuition, stable QP solutions as potential islands of stability (minima of Hamiltonian), direct calculation of recurrent times…*

Taking a step back

[Fang,Green,Yang,LL]

- In general scenarios, *at the linearized level*, one can identify special modes in the system. E.g. QNMs, normal modes, etc.
- Expand EEs as $G1(g_B, h) + G2(g_B, h) \sim 0 \; (g = g_B + h)$
- Express $h \sim \sum A(t_s) \; e^{-iwt} Z^{-}(\mathrm{x}) + B(t_s) e^{iw*t} Z^{+}(\mathrm{x})$
- $\rightarrow d_t \, A_j \sim \sum \gamma_{jkl} A_k A_l + \gamma A.B + \gamma B.B$
- In particular $d_t \, A_j \sim (\sum \gamma_{jkj} A_k) \, A_j + extra\, stuff$ -> parametric res
- Also, for long wavelengths → γ_{jkl} same coefficient as derived from fluid eqns.
- For scalar coupling → same eqns as TTF for AdS

Can capture non-linear behavior through a non-linear coupled oscillator model. Key to understand coupling constants

Final comments

- Holographic studies certainly interesting/rich and motivating. Dynamical studies of AdS fascinating with intriguing/compelling consequences *[definitively on the GR side at least]*

- Numerical simulations employed to uncover new phenomena and provide guidance for analytical (perturbative) followup which reveal further structure. In particular that the system can be probed through a coupled, non-linear, oscillator model. This, in turn, translates the dynamical problem onto understanding the coupling coefficients.

  Harmonic oscillators refuse to be left in the 20^{th} century!