Homework 7 Solutions

1. Since we know the expansion rate v, we can assume that the radius has been increasing at a constant rate and, if we know its current radius R, find the expansion time t via $R \approx vt$.

   The diameter D of the Ring nebula is related to its angular size α (in arcsec) and distance d via the small angle formula:

   $$D = \frac{\alpha d}{206265}$$

   We multiply both sides by 1/2 to get the radius ($R = D/2$), set the expression equal to vt, and solve for t:

   $$R = \frac{1}{2} \cdot \frac{\alpha d}{206265} = vt$$
   $$t = \frac{\alpha d}{2 \cdot v \cdot 206265}$$

   If we take $\alpha = 1.0$ arcmin, convert α to arcsec, d from ly to km, we get an expansion age of

   $$\begin{aligned} t &= \frac{1}{2(20 \text{ km/s})(206265)} \cdot \left(1.0' \times \frac{60''}{1'}\right) \left(2700 \text{ ly} \times \frac{9.46 \times 10^{12} \text{ km}}{1 \text{ ly}}\right) \\ &= 1.857 \times 10^{11} \text{ s} \\ &= 5900 \text{ yr} \end{aligned}$$

   If we take $\alpha = 1.4$ arcmin, then the expansion age is $1.4 \times (5900 \text{ yr}) = 8300$ yr.

   Hence the central star shed its outer layers at a time $\boxed{\text{between 5900 and 8300 years}}$ ago.

2. Both a white dwarf and a brown dwarf are supported by degenerate-electron pressure, whereas a red dwarf is supported by ideal gas pressure. Objects supported by degenerate-electron pressure obey a mass-radius relation where a more massive object has a smaller radius. As a white dwarf is more massive than a brown dwarf, a white dwarf has a smaller radius than a brown dwarf. A red dwarf is more massive than a brown dwarf and has a larger radius than a brown dwarf as it is being supported by ideal gas pressure. Hence its radius is also smaller than that of a white dwarf.

3. (a) For two objects orbiting around each other, we can relate the orbital separation a to their individual masses (M_1 and M_2) to the orbital period P via Kepler's law, which is expressed as follows in SI units:

   $$P^2 = \frac{4\pi^2}{G(M_1 + M_2)} a^3$$

One could also use Kepler's third law but in units relevant to the Earth's orbit (orbital separation a in AU, orbital period in yr, and masses in $M_\odot$):

$$(P[\mathrm{yr}])^2 = \frac{(a[\mathrm{AU}])^3}{M_1[M_\odot] + M_2[M_\odot]}$$

We solve for a, convert the given orbital period from days to years, to find

$$\frac{a}{\mathrm{AU}} = \left[\left(\frac{P}{\mathrm{yr}}\right)^2 \cdot \left(\frac{M_1}{M_\odot} + \frac{M_2}{M_\odot}\right)\right]^{1/3}$$
$$\frac{a}{\mathrm{AU}} = \left[\left(\frac{19.56 \text{ day} \times 1 \text{ yr}/365.25 \text{ day}}{\mathrm{yr}}\right)^2 \cdot \left(\frac{18\, M_\odot}{M_\odot} + \frac{34\, M_\odot}{M_\odot}\right)\right]^{1/3}$$
$$a = \boxed{0.53 \text{ AU}}$$

(b) The semimajor axes of the orbits of Mercury, Venus and Earth are 0.387, 0.723 and 1 AU respectively. The orbital separation we found in part (a) is 1.38 times the semimajor axis of the orbit of Mercury, and 0.733 and 0.53 times those of Venus and Earth.

4. (a) The brightness of a star b is related to its luminosity L and distance d from us via the inverse square law:

$$b = \frac{L}{4\pi d^2}$$

We can read off the peak luminosity of a type II supernova from the light curve shown in Figure 20-11. Luminosity is plotted in regular intervals of powers of 10, i.e. in log intervals. The peak of the type II supernova light curve lies roughly two-thirds between $10^8\, L_\odot$ and $10^9\, L_\odot$, i.e. its logarithm is roughly $8 + 2/3 \approx 8.7$, so we can roughly estimate its peak luminosity as $L_{\mathrm{peak}} \approx 10^{8.7} \approx 5 \times 10^8\, L_\odot$.

We now take the ratio of the peak brightness of Betelgeuse to the brightness of the Sun, since we know their luminosities (L_{peak} and $L_\odot$ respectively) and distances from us ($d_{\mathrm{Betelgeuse}} = 425$ ly and $d_\odot = 1$ AU respectively):

$$\frac{b_{\mathrm{peak}}}{\mathrm{b}_\odot} = \frac{L_{\mathrm{peak}}}{4\pi d_{\mathrm{Betelgeuse}}^2} \cdot \frac{4\pi d_\odot^2}{L_\odot}$$
$$= \frac{L_{\mathrm{peak}}}{L_\odot} \cdot \left(\frac{d_\odot}{d_{\mathrm{Betelgeuse}}}\right)^2$$
$$= (5 \times 10^8) \left(\frac{1 \text{ AU}}{425 \text{ ly} \times \frac{63240 \text{ AU}}{1 \text{ ly}}}\right)^2$$
$$b_{\mathrm{peak}} \approx \boxed{7 \times 10^{-7}\, \mathrm{b}_\odot}$$

As a side note, the difference in apparent magnitudes of two stars is related to the ratio of their brightness via

$$m_2 - m_1 = 2.5 \log_{10} \left(\frac{b_1}{b_2} \right)$$

If we take object 2 as the Sun ($m_\odot = -26.7$), and object 1 as the type II supernova produced by Betelgeuse, at peak brightness, then

$$\begin{aligned} m_\odot - m_{\text{peak}} &= 2.5 \log_{10} \left(\frac{b_{\text{peak}}}{\text{b}_\odot} \right) \\ m_{\text{peak}} &= m_\odot - 2.5 \log_{10} \left(\frac{b_{\text{peak}}}{\text{b}_\odot} \right) \\ &= -26.7 - 2.5 \log_{10}(7 \times 10^{-7}) \\ m_{\text{peak}} &\approx -11 \end{aligned}$$

A full moon has an apparent magnitude of -12.6, so the supernova formed by Betelgeuse may become almost as bright as a full moon!

(b) The brightness of the supernova is about $\boxed{\text{700 times}}$ that of Venus.