Homework 8, Astro 1

Solutions

Some formulas come with references on the right hand side if you'd like to track each step or discuss a particular step in the solutions with your TA.

1. (U11-21.27)

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1}$$

Setting $\Delta t' = 10$ years and $\Delta t = 8$ years we have

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{8}{10} \tag{2}$$

Squaring both sides gives

$$1 - \frac{v^2}{c^2} = .64 \tag{3}$$

$$\frac{v^2}{c^2} = .36 \tag{4}$$

Taking the square root of both sides shows that $v = .6c$

2. (U11-21.30) Since the question is posed from the perspective of the astronaut, then we know that they are the observer. Using the time dilation formula:

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We can then plug in $\Delta t = 15$ & $v = 0.8c$.

So, $\Delta t' = 25$. An observer on Earth would measure the trip as taking 25 years.

Now that we have the travel time from the astronaut's perspective, T_a , and the travel time from Earth's perspective, T_E, we can calculate the distance from each perspective.

Using the formula $d = vt$, we can obtain the distance from the known speed and travel time.

$L_a = v \times T_a = 0.8c \times 15 years = 12ly$

$L_E = v \times T_E = 0.8c \times 25 years = 20ly$

3. (U11-21.60)

$$r_s = \frac{2GM}{c^2} \tag{7}$$

Using this let us first calculate the three requested figures. The given mass of the earth, the sun, and the black hole of NGC 4261 are 5.97 $\times 10^{24} kg$, 1.99 $\times 10^{30} kg$ (1 $M_{\odot}$), and 1.2 $\times 10^9 M_{\odot}$, respectively. Direct substitution into the equation above gives the values (a),(b),(c) for the earth, the sun, and the black hole to be

(a)The Schwarzschild radius is given by

$$r_s = 8.87 \times 10^{-3} m \tag{8}$$

Now, to calculate the corresponding density we imagine a spherical black hole with volume V = $\frac{4}{3}\pi r_s^3$. Then the density is just $\rho = \frac{M}{V}$. Direct substitution into this equation yields the value

$$\rho = 2.04 \times 10^{30} kg/m^3 \tag{9}$$

(b)repeating for the sun gives a Schwarzschild radius

$$r_s = 2970m \tag{10}$$

with a corresponding density of

$$\rho = 1.82 \times 10^{19} kg/m^3 \tag{11}$$

(c)Lastly, for the super-galaxy we find a Schwarzschild radius

$$r_s = 3.56 \times 10^9 km \tag{12}$$

with a corresponding density of

$$\rho = 12.6 kg/m^3 \tag{13}$$

4. (U11-21.63) Let us begin with the proof. In the previous solution, it was stated that the volume of a black

hole is given by $V = \frac{4}{3}\pi r_s^3$. To find out the mass dependency of this equation we need to recall that the Schwarzschild radius is defined as $r_s = \frac{2GM}{c^2}$. Substituting into our equation for V we find

$$V = \frac{4}{3}\pi(\frac{2GM}{c^2})^3 = \frac{32}{3}\frac{\pi G^3}{c^6}M^3 \tag{14}$$

Then, recalling that density is given by $\rho = M/V$ we have

$$\rho = \frac{M}{V} = \frac{M}{\frac{32}{3}\frac{\pi G^3}{c^6}M^3} = \frac{1}{M^2}\frac{3c^6}{32\pi G^3} \tag{15}$$

Which is what we set out to show. Now to answer the second part of the problem we must set $\rho = 1000kg/m^3$ and solve for the given mass.

$$1000\frac{kg}{m^3} = \frac{1}{M^2}\frac{3c^6}{32\pi G^3} \tag{16}$$

Thus, we have

$$M = \sqrt{\frac{1}{1000}\frac{3c^6}{32\pi G^3}}kg = 2.7 \times 10^{38}kg \tag{17}$$

5. (U11-22.26)

The orbital velocity is $v = 400$ km s^{-1} $= 400 \times 10^3$ m s^{-1}.
The radius of the orbit is $r = 20000$ pc $= 20000(3.09 \times 10^{13})(10^3)$ m $= 6.18 \times 10^{20}$ m.

(a) To find the orbital period P,

$$P = \frac{2\pi r}{v} \quad \Rightarrow \quad P = \frac{2\pi(6.18 \times 10^{20})}{400 \times 10^3} = 9.71 \times 10^{15} \text{ s} = \boxed{3.08 \times 10^8 \text{ years}}$$

(b) To find the mass of the galaxy, we use the equation in Box 22-2 in *Universe*,

$$M = \frac{rv^2}{G} \quad \Rightarrow \quad M = \frac{(6.18 \times 10^{20})(400 \times 10^3)^2}{6.67 \times 10^{-11}} = 1.48 \times 10^{42} \text{ kg} = \boxed{7.45 \times 10^{11} M_\odot}$$

6. (U11-22.41)

(a) Sagittarius A* is a supermassive black hole with $M_\bullet = 4.1 \times 10^6 M_\odot$, and therefore comparatively, the mass of the stars S0-2 and S0-19 is negligible.

$$M_1 + M_2 = \frac{a^3}{P^2} \quad \Rightarrow \quad M_\bullet = \frac{a^3}{P^2} \quad \Rightarrow \quad a = (M_\bullet P^2)^{1/3}$$

For S0-2, with $P = 14.5$ years, $a = ((4.1 \times 10^6)(14.5^2))^{1/3} = \boxed{952 \text{ AU}}$.
For S0-19, with $P = 37.3$ years, $a = ((4.1 \times 10^6)(37.3^2))^{1/3} = \boxed{1790 \text{ AU}}$.

(b) To calculate the angular size of the semi-major axis as seen from Earth, use the small-angle formula,

$$D = \frac{\alpha d}{206265} \quad \Rightarrow \quad \alpha = \frac{206265 D}{d}$$

which we will set $D = a =$ semi-major axis. Since Sagittarius A* is at the Galactic center, and the distance between the Milky Way and the Galactic center, i.e. Sagittarius A*, is around 8 kpc, we set $d = 8$ kpc $= 8(10^3)(3.26)(63240) = 1.65 \times 10^9$ AU.

For S0-2, with $a = 952$ AU,

$$\alpha = \frac{206265(952)}{1.65 \times 10^9} = \boxed{0.119 \text{ arcsec}}$$

For S0-19, with $a = 1790$ AU,

$$\alpha = \frac{206265(1790)}{1.65 \times 10^9} = \boxed{0.224 \text{ arcsec}}$$

High-resolution of infrared images are required due to the presence of dust, and the angular sizes of the two stars are very small.

7. (U11-22.46)

(a) Volume of a cylinder $= \pi r^2 h$, where r is the radius, h is the height of cylinder. Given the Galaxy disk has diameter of 50 kpc, and thickness of 600 pc, we have $r = 50/2 = 25$ kpc, and $h = 600$ pc.

$$V = \pi r^2 h \quad \Rightarrow \quad V_{\text{disk}} = \pi(25 \times 10^3)(600) = \boxed{1.18 \times 10^{12} \text{ pc}^3}$$

(b) Volume of a sphere $= 4\pi r^3/3$, where r is the radius of the sphere. With $r = 300$ pc,

$$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad V_{\text{sphere}} = \frac{4}{3}\pi(300)^3 = \boxed{1.13 \times 10^8 \text{ pc}^3}$$

(c) The region 300 pc around the Sun is the spherical region we considered in part (b).

$$\text{Probability of the supernovae to occur 300 pc around the Sun} = \frac{V_{\text{sphere}}}{V_{\text{disk}}} = \frac{1.13 \times 10^8}{1.18 \times 10^{12}} = \boxed{9.58 \times 10^{-5}}$$

If there are about 3 supernovae per century in our Galaxy, there is 1 supernova per 33.3 years on average ($100/3 = 33.3$ years). For 1 supernova to occur in the sphere of 300 pc in radius, there will be $1/9.58 \times 10^{-5} = 1.04 \times 10^4$ supernovae in the whole Galaxy.

$$\text{Time interval for 1 supernova to occur in the sphere} = (1.04 \times 10^4)(33.3) = \boxed{3.46 \times 10^5 \text{ years}}$$

8. (U11-22.47)

(a) For the RR Lyrae star, we are given $L = 100 L_\odot$ and $b = 1.47 \times 10^{-18} b_\odot$. Let $d_\odot$ be the Sun-Earth distance, i.e. $d_\odot = 1$ AU and use $b = L/(4\pi d^2)$,

$$\frac{b}{b_\odot} = \frac{L}{L_\odot}\left(\frac{d_\odot}{d}\right)^2 \quad \Rightarrow \quad d = d_\odot\sqrt{\left(\frac{L}{L_\odot}\right)\left(\frac{b_\odot}{b}\right)} = (1 \text{ AU})\sqrt{(100)\left(\frac{1}{1.47 \times 10^{-18}}\right)} = \boxed{8.25 \times 10^9 \text{ AU}}$$

(b) To also express the distance in parsec, note that 1 pc = 3.26 ly, and 1 ly = 63240 AU,

$$d = 8.25 \times 10^9 \text{ AU} = (8.25 \times 10^9 \text{ AU})\left(\frac{1 \text{ ly}}{63240 \text{ AU}}\right)\left(\frac{1 \text{ pc}}{3.26 \text{ ly}}\right) = \boxed{40000 \text{ pc}}$$