Topological Quantum Error Correction

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Abstract
Interest in developing quantum computers stems from their ability to solve in polynomial
time several important problems that are only classically solvable in exponential time. One
significant hurdle facing the actual implementation of a quantum computer is the need for error
correction code to make reliable quantum memory in the presence of errors, such as measurement
errors and decoherence. Here we discuss one way of accomplishing this task within the context
of the Toric Code model. We show that error recovery for this model is related to an energy
minimization condition for a two dimensional statistical mechanical model.

1 Introduction to Quantum Error Correction
Quantum error correction is a fundamental component of any quantum computing scheme. The
use of error correction, although not unique to quantum computing, is made more difficult in the
quantum regime for several reasons. First is the fact that one cannot simply make multiple copies a
given qubit, as a classical computer might do. This follows from a fundamental theorem of quantum
mechanics called the no-cloning theorem. Secondly, and probably a more significant challenge, is
that quantum systems must deal with errors due to decoherence, an effect that is not present in a
classical system. Decoherence refers to the loss of phase information of a quantum system through
interaction with it’s environment. In order to illustrate this effect more clearly, we consider the
following example.

Example: Let’s first intitialize a qubit to the state $|\psi\rangle = |0\rangle$ and the environment to the state
$|E_0\rangle$. Applying a Hadamard gate to the qubit gives the state $H|\psi\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$. Now if we wait
some time before measuring our qubit, it may interact with the environment in a non-destructive
way, becoming entangled with the environment as follows:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|E_0\rangle + |1\rangle|E_1\rangle)$$  

(1)

where $E_1$ is another orthogonal state of the environment. We can write the density matrix of this
final state as

$$\rho = \frac{1}{2}(|0\rangle|E_0\rangle\langle 0| + |1\rangle|E_0\rangle\langle 1| + |0\rangle|E_1\rangle\langle 0| + |1\rangle|E_1\rangle\langle 1|)$$  

(2)
If we trace over the states of the environment (meaning the environment must end up in a pure state) we get a density matrix with no off diagonal elements.

\[ \rho = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]  

Measurement of the system will now yield $|0\rangle$ 50% of the time and $|1\rangle$ for 50% of the time, so it becomes just a complete mixture of qubit states. The new state here merely reproduces a classical probabilistic system, with all the quantum phase information being lost.

Other sources of error in a quantum computer include measurement errors and qubit loss errors. Several quantum error correction schemes have already been developed, dating back to Shor’s original 9 qubit error correction. Here we will introduce an important class of error correcting codes called stabalizer codes, and describe a specific implementation of them called the Toric Code.

### 2 Stabalizer Formalism and Quantum Error Correction

A binary stabalizer code is characterized by a set of mutually commuting operators with eigenvalue +1, rather than in terms of the basis states themselves. This is similar to the Heisenberg representation of quantum mechanics in terms of time evolved operators and fixed basis kets. We say an arbitrary operator K is stabalized by some state $|\psi\rangle$ if $K|\psi\rangle = |\psi\rangle$. The formal definition is given in ref. (3) as

\[ G = \{ G_i | G_i|\psi\rangle = |\psi\rangle, [G_i, G_j] = 0 \forall (i,j) \} \]  

In order to develop some intuition behind this definition, we consider the following example.

**Example:** The GHZ state can be written in the stabalizer formalism as:

\[ |GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}} \]  

We can check that the following three operators, which are tensor product of the Pauli operators, form a stabalizer group of GHZ state

\[ G_1 = XXX \]  
\[ G_2 = ZZI \]  
\[ G_3 = IZZ \]  

First we see that all three of these operators have the GHZ state as an eigenvector with eigenvalue +1. Similarly, all of these operators satisfy the mutual commuting condition: i.e.

\[ [G_1, G_2] = [XXX, ZZI] = XXX \cdot ZZI - ZZI \cdot XXX = (XZ) \otimes (XZ) \otimes (XI) - (ZX) \otimes (ZX) \otimes (IX) \]
Using the anti-commutation relation for the Pauli matrices, $X, Z = 0$, we get

$$= (-ZX) \otimes (-ZX) \otimes (XI) - (ZX) \otimes (ZX) \otimes (IX) = 0$$

(9)

The commutation relations for the other three operators follow similarly.

Stabilizer codes, such as the previous example, are very useful for quantum error correction because they allow for correction operations that can be performed directly on the encoded data, rather than on a some auxiliary qubits.

### 2.1 Error Correction in Stabilizer Codes

We start by considering an arbitrary state of our system, $|\psi\rangle$, which by definition is an eigenvector with eigenvalue +1 of all the operators in our stabilizer group. Let $E$ be the operator which produces the error in our eigenstate $|\psi\rangle$. The goal of the error correction is to first identify where the error has occurred, and second be able to correct for the error. A general error operator will satisfy the condition

$$K_i E |\psi\rangle = (-1)^m E K_i |\psi\rangle = (-1)^m E |\psi\rangle$$

(10)

for $m \in \{0, 1\}$. The new state will have an eigenvalue of either -1 if $E$ and $K_i$ anti-commute and +1 if the commute. Since the initial state has eigenvalue of +1 for all the stabilizer operators, by operating the error state, $E |\psi\rangle$, with all the group operators we can determine which ones evaluate to -1 and hence locate the position of the error in our code. To see this working in practice, we consider the following example on the GHZ state.

**Example**: In the previous example, we listed the three mutually commuting operators which have the GHZ eigenstate: $G_1 = XXX$, $G_2 = ZZI$, and $G_3 = IZZ$. Let’s consider first a case when the error operator is $E_1 = XIX$, so $E|GHZ\rangle = 1/\sqrt{2}(|100\rangle + |011\rangle)$. Now, applying the three stabilizer operators gives $G_1 E|GHZ\rangle = |GHZ\rangle$, $G_2 E|GHZ\rangle = -|GHZ\rangle$, and $G_3 E|GHZ\rangle = |GHZ\rangle$. For the other possible error cases when $E_2 = IXI$ and $E_3 = IIX$, we can use the anti-commutation relations with the stabilizer operators to determine which ones will evaluate to -1. Since this mapping of the error operators to the set of stabilizer operators which anti-commute turns out to be unique (not proved here), it presents a way to determine which error has occurred by simply operating the stabilizers on their mutual eigenkets.

We will now introduce a specific type of stabilizer code, called the Toric Code model, and briefly demonstrate how it can be used for quantum error correction.

### 3 The Toric Code

The Toric Code is a particular implementation of the general class of stabilizer codes described above. This model can be visualized as a 2-dimensional lattice with qubits located at each link. There are two stabilizer operators for this group, the site and plaquette operators, which are called
check operators. These are defined in terms of operators on the lattice links as follows:

\[ A_s = \bigotimes_{l \in \text{star}(s)} X_l \]  
\[ B_p = \bigotimes_{l \in \partial p} Z_l \]

where \( X \) and \( Z \) are the Pauli matrices:

\[
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Here the site operator \( A_s \) acts on the four links that meet at a given site \( s \) and the plaquette operator \( B_p \), acts on the four links that border a plaquette \( p \). The mapping that assigns an element \{0,1\} to each site of the lattice is called a \( Z_2 \) valued 1-chain and forms a vector space over the lattice. The term 1-chain also stands for the set of links which evaluate to +1 under the \( A_s \) and \( B_p \) operators.

**Theorem:** The set of operators \( A_s \) and \( B_p \) are mutually commuting.

**Proof:** We are interested in showing the commutator \([A_s, B_p] = 0\) for any given lattice site \( s \) and a plaquette \( p \). Two possibilities exist for the overlap of the site and the plaquette: either they share no edges or they share two edges. In the case that they share no edges, then the commutator is trivially zero since any operator commutes with the identity. For the case when they share two edges, then the two minus signs coming from the anticommutation relations, \( X, Z = 0 \) on each of the shared edges edges will cancel to give an overall commuting set of operators.

Given that the two operators commute, we expect to be able to find a mutual eigenstate of the two operators of eigenvalue +1. In order to define the eigenstates, it is convenient to define the plaquette flux operator, \( w_p(s) = \sum_{j \in \partial p} s_j \mod 2 \), where a flux of 0 is a \( B_p \) eigenstate of +1 and flux of 1 is \( B_p \) eigenstate of -1. The +1 eigenstates of the toric code are states \( |s\rangle \in \{0,1\}^N \) such that \( w_p(s) = 0 \) for all plaquettes in \( s \). We can write this ground state as

\[ |\Psi\rangle = \sum_{\{s:w_p(s) = 0\}} |s\rangle \]  

The effect of applying the operator \( A_s \) to this ground state is to flip all the spins that connect to the lattice site \( s \). Since each bordering plaquette always shares two links with the site \( s \), then flipping these two spins will not change the flux (since it is defined only modulo 2). This very simple argument shows (albeit not in the most rigorous fashion) that this ground state is an eigenstate of the stabilizer code defined by the check operators \( A_s \) and \( B_p \).

A final important comment to be made about this ground state is that it has a four fold degeneracy. If we multiply the ground state by the product of \( Z \) operators acting on links corresponding to a nontrivial cycle of the lattice, then this new state will be another ground state of the system. We define the loop operators, \( w_l(s) = \prod_{j \in l} Z_j \), for the two nontrivial loops \( l_1 \) and \( l_2 \) around the torus.
Then since there are two possible values of $\pm 1$ for each loop operator, there are four possible ground states of the toric code.

$$|\Psi\rangle = \sum_{\{s: w_p(s) = 0 \forall p\}} C_{w_{l_1}w_{l_2}} |s\rangle$$  \hspace{1cm} (15)

where the coefficients $C_{w_{l_1}w_{l_2}}$ are $\pm 1$ depending on the paths $l_1$ and $l_2$. It should be pointed out that the loop functions trivially commute with the $B_p$ stabilizer operators, and commute with the $A_s$ operators because each site operator intersects the loop at either zero or two links, thereby preserving the commutation relation. Furthermore, the loop operators are not in the stabalizer group since they have no boundary, and hence cannot be written as product of plaquette operators. The fourfold degeneracy of the ground state enables us to define two qubits for a toric code of genus $g$.

3.1 Defects and Error Correction

As demonstrated in section (2.1) for a rather simple example, the great virtue of the stabalizer codes lies in their ability to implement efficient error correction schemes during a quantum computation. To understand this process in the toric code, we must first define what types of syndromes we need to look for to identify errors. Recall that the toric code is initialized such that each check operator takes a value of $+1$ when acting on the ground state. This corresponds to the flux at each site being set to $+1$. There are then two types of defects that can occur, one in which the plaquette operators evaluate to $-1$, and one in which the site operators evaluate to $-1$.

![Figure 1: The electric and magnetic path operators from ref. (3)](image)

The site defects are defined via the electric path operator, which operates on the spins of a trivial path $l$ with endpoints at the lattice sites $s_1$ and $s_2$, as in Figure 3.1,

$$W_l^{(e)} = \prod_{j \in l} Z_j$$  \hspace{1cm} (16)

The new eigenstate, $|\psi\rangle = W_l^{(e)} |\psi_0\rangle$, contains two defects at the endpoint sites $s_1$ and $s_2$ because $W_l^{(e)} A_{s_i} = -A_{s_i} W_l^{(e)}$ for $i = 1, 2$. The commutator at all other sites is zero since the $A_s$ operator overlaps the path $l$ at either zero or two links. Similarly, we can define a plaquette defect on the dual lattice as

$$W_{l^*} = \prod_{j \in l^*} X_j$$  \hspace{1cm} (17)

for a path $l^*$ with endpoints at $p_1$ and $p_2$ (here we can simply imagine flipping the spins on links
that intersect the path $l^\ast$). Similar to the site defects, we have here that $W^{(m)}_i B_{p_i} = -B_{p_i} W^{(m)}_i$ for $i = 1, 2$.

The two types of defects outlined above classify the potential errors of the toric code qubits. The goal of the error correction scheme will be to form a path that closes the original error path $l$ ($l^\ast$) on the lattice (dual lattice) to form a trivial cycle. This will then eliminate the sites at which the check operators evaluate to -1 and restore the system to its ground state. A trivial cycle is one which borders a closed region of the lattice.

The chain where the check operators evaluate to -1 is called the syndrome chain, and is given by a path $S$ in the code. The error chain of actual defects is given by a path $E$. We write $E = S \cup C = S + C$, for some cycle $C$ with no boundary. As previously described, the defects can be of two types: charged and magnetic. The goal of the error chain is to choose the recovery path that has the highest probability of being in a nontrivial homology class. We develop this idea further below.

We can add a path $E' = S + C'$ to the original path $E$ such that together they form a closed (trivial) cycle, $E + E' = C + C'$. In order to accomplish this, we need to maximize the probability that the error correcting path $C'$ belongs to the same homology class as $C$. This involves calculating a conditional probability, which gives the probability that an event $B$ occurs given that an event $A$ has occurred, $P(B|A) = P(A \cap B)/P(A)$. The probability that, given a syndrome $S$, the path $C'$ belongs to the same homology class $h$ as $C$ is

$$\text{prob}(h|S) = \frac{\sum_{C' \in h} \text{prob}(S + C')}{\sum_{C} \text{prob}(S + C')}$$ (18)

The way to proceed from here is then to use the fact that the homology class $h$ of $C$ is the class that maximizes the probability in eq. (18). The technique for accomplishing this is outlined in the next section.

### 3.2 Statistical Mechanics Model

Here we will follow the analysis of Ref. (3) in determining the probabilities of eqn. (18), while at the same time filling in several of the missing details and explanations from their analysis. We are considering an error chain $E$ in our stabilizer code that contains electric or magnetic excitations of the kind described above. We make the simplifying assumption that the probability of an error occurring at a given link is always $p$. If we define a function, $n_E(l)$ to equal 1 if the link contains an error qubit and 0 otherwise, then the probability of a given chain defined by a set of links $l \in E$ is a product of the individual probabilities (for uncorrelated links) over the entire lattice

$$\text{prob}(E) = \prod_l (1 - p)^{1 - n_E(l)} p^{n_E(l)}$$ (19)

We are interested in finding the probability that another error chain $E'$, will have the same boundary as the original chain, $E$. From this, we will be able to find a chain that corrects the error $E$ (i.e. by forming a closed cycle from $E$) with the highest probability of being in the same homology class as $C$. The reason it is important for $C'$ to be the same homology class as $C$ is because the qubits of our
system are associated with non-trivial cycles of the lattice, so we don’t want to create any spurious loops that may be confused for qubits. Let \( E' = E + C \) for a cycle \( C \). We can identify links of the lattice that belong to \( E' \) and not to \( E \) by finding the probability that \( n_C(l) = 1 \) and \( n_E(l) = 0 \)

\[
\text{prob}(l \in E, l \notin E') = \left( \frac{p}{1-p} \right)^{n_C(l)}
\]

Alternatively, the probability of that a given error link is in \( E \) and not in \( E' \), i.e. \( n_C(l) = 1 \) and \( n_E(l) = 1 \) is given by

\[
\text{prob}(l \in C, l \in E) = \left( \frac{1-p}{p} \right)^{n_C(l)}
\]

Here we want to calculate the probability of having a chain \( E' \) with the same boundary of \( E \), given our original chain \( E \). Using the previously obtained probabilities, eqn. (24) and (25), we have

\[
\text{prob}(E'|E) \propto \prod_l e^{J_l u_l}
\]

where we follow the definitions of ref.(3) and let \( u_l = 1 - 2n_C(l) \) and define a coupling parameter \( J_e \) such that

\[
e^{-2J_e} = \begin{cases} \frac{p}{(1-p)} & 1 \notin E \\ \frac{(1-p)}{p} & 1 \in E \end{cases}
\]

Now, we can relate this model to the more familiar Ising model by defining "spins" on the dual lattice (i.e. associated with each plaquette). The quantity \( u_l \) takes values -1 inside the area enclosed by \( E + E' \). Since this is a closed cycle, plaquettes on the interior of \( C, P^* \), occur in an even number and we have

\[
\prod_{l \in P^*} u_l = 1
\]

We now introduce variables \( \sigma_i \in \{1, -1\} \) on each plaquette (site of the dual lattice) and write \( u_{ij} = \sigma_i \sigma_j \) for \( i \) and \( j \) nearest neighbor sites. This allows us to describe the fluctuations of the error chains \( E' \) that share a boundary with \( E \) through the statistical mechanical model

\[
Z[J, \eta] = \sum_{\sigma} \exp \left( J \sum_{(i,j)} \eta_{ij} \sigma_i \sigma_j \right)
\]

where \( e^{-2J} = \frac{p}{(1-p)} \). The error chains occur with probabilities

\[
\eta_{ij} = \begin{cases} 1 & 1 \notin E^* \text{ with prob. } 1-p \\ -1 & 1 \in E^* \text{ with prob. } p \end{cases}
\]

where \( E^* \) describes the region of the dual lattice enclosed by the boundary of \( E + E' \), and \( p \) is the probability of a given link having an error. From the partition function, we can define the free energy to be minimized as \( F[J, \eta] = \ln Z[J, \eta] \). The minimization procedure produces the following result (not proven here): The chain of minimal energy with the same boundary as the actual error chain will be in the same homology class as the actual error chain with probability one in the infinite
volume limit. So we select among all the chains that have the same boundary as E the chain $E_{min}$ which has the lowest energy and highest probability. This step would require the use of a classical computer to figure out which gates need to recover from error, and we assume this process takes polynomial time in the number of qubits in the system. We see here how the error recovery process depends crucially on the ability to determine the homology class of the errors.

4 Conclusion

This paper has introduced the problem of decoherence in quantum information, defined a general class of codes called stabilizer codes that could be used to deal with the effects of quantum errors, and finally outlined a specific model of this type called the Toric Code model. One of the great advantages of this model is that it only requires measurements of check operators on qubits located near each other, which makes for an easier physical realization of the model. Another benefit of this model is the ability to implement topological quantum error correction codes, which can provide excellent identification and correction of error syndroms during a quantum computation. The ultimate goal for these models is to predict a quantum information processing scheme that can maintain the required accuracy threshold to make quantum computers physically realizable, and there is promising evidence that suggests they will be able to accomplish this.

References