INTERFEROMETRIC VISIBILITY OF A SCINTILLATING SOURCE

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ABSTRACT

I calculate the distribution of the product of complex factors drawn from a bivariate Gaussian distribution. The interferometric visibility of a scintillating point source follows this distribution. I find the distributions of the real part, imaginary part, and amplitude of the product. I investigate effects of a small, partially resolved source on the distribution. These effects depend on whether the source is elongated, and if so, on its orientation relative to the interferometer baseline. A single sample of interferometric visibility, for any source, is also the product of complex factors drawn from a bivariate Gaussian distribution.

Subject headings: scattering, techniques: interferometric

1. INTRODUCTION

1.1. Gaussian Noise from Astrophysical Sources

Correlation of complex numbers is fundamental to astrophysics. One polarization of electromagnetic radiation is commonly represented as a complex number, corresponding to the amplitude and phase of the electric field. Series of complex values then represent the time evolution, or spectral variation, of the electric field. Electric fields from all known astrophysical sources are noiselike, in the sense that the value at any time is drawn from a circular Gaussian distribution centered at the origin in the complex plane. Similarly, the complex value of the spectrum at a given frequency is drawn from a zero-mean circular Gaussian distribution. Auto- or cross-correlation of the complex values yields useful observables.

Because the underlying statistics of astrophysical electric fields are Gaussian, the variances and covariances of the resulting series describe the fields completely. The important covariances may be among different times, frequencies, locations, or polarizations. For example, the intensity $\mathcal{I}_A$ at a location A is the mean square modulus of the electric field E_A: $\mathcal{I}_A = \langle E_A E_A^* \rangle_E$. Here, the subscripted angular brackets $\langle .. \rangle_E$ indicate a statistical average over a statistically-identical ensemble of electric fields, drawn from the complex Gaussian distribution of E_A. Because the electric field is drawn from a zero-mean circular Gaussian distribution, $\mathcal{I}_A$ is the variance of E_A. In contrast, the interferometric visibility $\mathcal{C}_{AB}$ is the statistically-averaged product of electric fields at two different locations: $\mathcal{C}_{AB} = \langle E_A E_B^* \rangle_E$. Thus, the intensity is the variance of the distribution of E_A, and the visibility is the covariance of E_A and E_B. Changes of the signal with time or frequency can be represented as changes in the variance or covariance of the underlying distribution of random, complex values.

In practice, finite observational averages are used to approximate the statistical averages. Usually, these averages draw on many individual measurements of the electric fields, correlated and then averaged to form useful observables. The number of samples in an observational average is usually large. In this case, the central limit theorem implies that the distribution of observed values approaches a Gaussian distribution about the ensemble-average value. However, in some cases the observational average may contain only a few terms. For example, some pulsars show rapid variations with frequency and time, so that the observer can measure only a few samples of the electric field before the parent distribution changes. In §4 of this paper, I find the distribution for the extreme case of a single sample. This distribution is that of the product of a pair of complex values, drawn from a bivariate Gaussian distribution.

1.2. Interferometric Visibility of a Strongly Scattered Source

A problem mathematically identical to that of correlation of Gaussian noise is the interferometric visibility of a point source in strong scintillation. In the thin-screen geometry traditional for scintillation problems, a source radiates electromagnetic waves onto a screen, where they suffer a random phase changes. The waves then propagate onward to the plane of the observer, who combines waves that have traveled along different paths, and forms observables by correlating the electric fields. Figure 1 shows the geometry. The problem is a prototypical example of Kirchoff diffraction. The scattering is said to be strong if the contributions from different paths differ in phase by many times 2π. The scattered field is referred to as the speckle pattern. The speckle pattern changes with time as the screen evolves and as the relative positions of source, screen, and observer shift. The speckle pattern also changes with frequency because the geometric phases are proportional to frequency. (The geometric phase is that introduced by travel distance.) If the observations are made over sufficiently short times and narrow bandwidths to avoid smearing together of different samples of the scattered field, they are in the "speckle limit."

Although the electric field of the source is noiselike, and so varies randomly with time and frequency as discussed in §1.1, those random variations often converge quite rapidly relative to changes in the speckle pattern. Therefore, I assume in this paper (with the exception of §4) that the intensity, visibility, and other observable properties of the speckle pattern are determined with arbitrary accuracy. In other words, I assume that all observations yield statistical averages over an ensemble of electric fields $\langle ... \rangle_E$. Differences among observations then reflect only differences among particular realizations of the scattering screen. I will use the unsubscripted angular brackets $\langle ... \rangle$ to denote the statistical average over many speckle patterns. Commonly, observations within a single speckle approach the statistical average over electric fields $\langle ... \rangle_E$, whereas observations over many such speckles are required to approach the statistical average over screens $\langle ... \rangle$.

The effect of scattering is to multiply the electric field from one point on the source, measured at a single point in the observer plane, by a complex gain (Cornwell, Anantharamaiah, & Narayan 1989). This complex gain is a sum over random phasors, each corresponding to a different path linking points on the source and in the observer plane. In strong scattering, the observer receives radiation from many paths, with lengths differing by many wavelengths. The central limit theorem then implies that the total complex gain is drawn from a Gaussian distribution. An observer who measures the variance or covariance of electric fields, and exactly determines the statistically-averaged intensity or visibility in the speckle limit, thus measures a product of complex gains from propagation, multiplied by a statistically-averaged product of electric fields.

For a point source, the observables reflect only the electric field of the source and its moments, times the complex gains introduced by propagation. In both theory and observations, the effects of the noiselike electric field of the source are easily separated from those of propagation. Consequently, it is convenient to assume that the source's electric field has constant amplitude and zero phase if we are concerned only with the gains resulting from propagation. When the source has finite extent, different source points may radiate with different amplitude and phase, and the situation becomes more complicated. In the simplest case, amplitude and phase vary completely independently at different points on the source, and the source is said to be completely incoherent. Introduction of a coherence function for the source, as discussed in §3 below, accommodates this situation (Goodman 1985).

The visibility, the fundamental observable of interferometry, is the product of the two electric fields at the ends of the interferometer baseline. As these locations are moved farther apart, the electric fields at the two locations become less correlated, because propagation introduces different phases. The field at each location is drawn from a zero-mean circular Gaussian distribution, as noted above. The fields at the two locations are drawn from a joint distribution with some specified degree of covariance. Thus, the distribution of visibility is that of the product of 2 factors drawn from a bivariate zero-mean Gaussian distribution. I calculate the distribution of the resulting product, the interferometric visibility, in §2.

In principle, the scattering screen acts as a corrupt lens, to produce an image of the source, convolved with the response to a point source, in the observer plane. If the source is incoherent, as usually assumed, this convolution holds for the intensity (Goodman 1968). For a point source, the distribution of intensity in the speckle pattern is exponential. If the scattering material, viewed as a lens, resolves the source, then speckle patterns from different parts of the source overlap, and the distribution of intensity departs from an exponential. This fact can be used to determine the size of the source, when the location of the scatterer is known (Cohen et al. 1967; Salpeter 1967; Gwinn et al. 1998).

A small extended source affects the distribution by superposing many, slightly offset speckle patterns. One can model the speckle pattern as the result of a phasor sum, and find how structure of a small source modifies that pattern. The result is the convolution of 3 distributions, with appropriate weights. I calculate this distribution for the real and imaginary parts of the visibility in §3.

2. VISIBILITY FOR A POINT SOURCE

2.1. Background

Consider the electric field in the observer plane of a scattered point source. We ignore the intrinsic phase and amplitude variations of the noiselike electric field of the source, and suppose that the source radiates with constant amplitude and zero phase, as discussed above. At each point in the observer plane, the electric field can be represented as the phasor sum of contributions from many different paths, with each path having a different complex gain. The phases introduced in propagating along these paths are random, so that this phasor sum has the character of a random walk. In strong scattering, the phases are distributed uniformly around 2π. The central limit theorem then implies that the sum is drawn from a Gaussian distribution centered on the origin.

The intensity at a point $\mathbf{p}_A$ in the observer plane is the square modulus of the electric field: $\mathcal{I}_A = E_A E_A^*$. Because the electric field is drawn from a zero-mean Gaussian distribution, the intensity is drawn from an exponential distribution, as is well known (Goodman 1985). The interferometric visibility $\mathcal{C}_{AB}$ is the product of electric fields at points $\mathbf{p}_B$ and $\mathbf{p}_B$:

$$\mathcal{C}_{AB} = E_A E_B^*. \tag{1}$$

Here the interferometer baseline is $\mathbf{b} = \mathbf{p}_B - \mathbf{p}_A$. The statistically-averaged visibility V, normalized by the mean intensity, is given by the well-known expression (Mercier 1962; Rickett 1977):

$$\langle V(\mathbf{b}) \rangle = \frac{\langle \mathcal{C}_{AB} \rangle}{\langle \mathcal{I} \rangle} = \exp\left\{ -\tfrac{1}{2} D_{\phi o}(\mathbf{b}) \right\}, \tag{2}$$

where the phase structure function in the observer plane $D_{\phi o}$ is defined by:

$$D_{\phi o}(\mathbf{b}) = \langle (\phi(\mathbf{p}_B) - \phi(\mathbf{p}_A))^2 \rangle. \tag{3}$$

In these expressions, the angular brackets $\langle ... \rangle$ denote a statistical average over an ensemble of statistically-identical scattering screens. The phase $\phi(\mathbf{p})$ is the total phase introduced between the source and observer plane, by the screen and propagation. The statistically-averaged intensity is the same at all locations in the observer plane: $\langle \mathcal{I} \rangle \equiv \langle \mathcal{I}_A \rangle = \langle \mathcal{I}_B \rangle$. Note that the average visibility, for a point source, is purely real, even though the visibility is complex. This reflects the fact that the average difference in phase of E_A and E_B is zero. Commonly, the phase structure function varies approximately as the square of the interferometer baseline $\mathbf{b}$, and can be associated with the angular broadening θ as:

$$D_\phi(\mathbf{b}) = \left(\frac{\pi}{\sqrt{2\ln 2}}\frac{\theta_H|\mathbf{b}|}{\lambda}\right)^2 . \tag{4}$$

(Gwinn et al. (1988) present a more general expression). Here, θ_H is the angular full width at half maximum of the broadened image. It is related to the angular variance θ by $\theta_H = \sqrt{8\ln 2}\theta$. The angular broadening characterizes the "scattering disk", the region of scattering material from which the observer receives radiation, with diameter approximately $\theta_H D$. The observing wavelength is λ. Gwinn et al. (1998, §6) give expressions for the mean square normalized visibility, $\langle VV^*\rangle$ and $\langle VV\rangle$. For a point source, these are purely real as well. In this case, they give the mean square real and imaginary parts: $\langle \mathrm{Re}[V]^2\rangle = \frac{1}{2}(\langle VV^*\rangle + \langle VV\rangle)$ and $\langle \mathrm{Im}[V]^2\rangle = \frac{1}{2}(\langle VV^*\rangle - \langle VV\rangle)$.

2.2. Distribution of Visibility

The interferometric visibility is the product of the electric fields at the locations of two stations, $\mathbf{p}_A$, $\mathbf{p}_B$. In strong scattering each of the electric fields is drawn from a circular Gaussian distribution in the complex plane, with zero mean. Thus, the complex electric field E at either station is drawn from the Gaussian distribution:

$$P(E) = \frac{1}{2\pi\sigma}\exp\left\{-\frac{1}{2}\frac{|E|^2}{\sigma^2}\right\}, \tag{5}$$

where $\sigma^2 = \langle |E|^2\rangle = \langle \mathcal{I}_A\rangle = \langle \mathcal{I}_B\rangle$. Note that this distribution is uniform in the phase of E. For $\mathbf{b} \neq 0$, the distributions have the normalized cross-correlation coefficient, or more simply the "correlation" (Thompson, Moran, & Swenson 1986, p. 214):

$$\rho = \frac{\langle E_A E_B^*\rangle}{\sqrt{\langle |E_A|^2\rangle\langle |E_B|^2\rangle}} = \frac{\langle E_A E_B^*\rangle}{\langle \mathcal{I}\rangle} = \langle V(\mathbf{b})\rangle, \tag{6}$$

where $\mathbf{b} = \mathbf{p}_B - \mathbf{p}_A$. Comparison with Eq. 2 shows that the correlation ρ is the normalized mean visibility $\langle V(\mathbf{b})\rangle$. The correlation, as used here, differs slightly from the usage of

Thompson et al., who statistically average only over samples of noiselike electric field, $\langle ... \rangle_E$, and do not consider scintillation.

The electric fields at each location are drawn from zero-mean circular Gaussian distributions in the complex plane. Therefore,

$$\begin{aligned} \langle A_r \rangle &= \langle A_i \rangle = 0 \\ \langle A_r A_r \rangle &= \langle A_i A_i \rangle = \tfrac{1}{2} \\ \langle A_r A_i \rangle &= 0, \end{aligned} \tag{7}$$

where I use the notation $A_r = \mathrm{Re}[E_A]/\sigma$, $A_i = \mathrm{Im}[E_A]/\sigma$. The statistical averages, denoted by angular brackets $\langle ... \rangle$, are over an ensemble of statistically-identical scattering screens. Noise is ignored: variation of the electric field arises purely from scintillation. Facts analogous to Equation 7 characterize the distribution of $B_r = \mathrm{Re}[E_B]/\sigma$ and $B_i = \mathrm{Im}[E_B]/\sigma$. Because ρ is statistically averaged over scintillations, and we assume a point source, ρ is real, as Eq. 2 indicates. The covariances of the components are:

$$\begin{aligned} \langle A_r B_r \rangle &= \langle A_i B_i \rangle = \tfrac{1}{2}\rho \\ \langle A_r B_i \rangle &= \langle A_i B_r \rangle = 0. \end{aligned} \tag{8}$$

These equations also hold for correlation of any noiselike signals (Gwinn et al. 2000a).

Although Eqs. 2 and 6 give the mean visibility, individual samples of the visibility will be distributed about the mean. The distribution of visibility is that of products of elements drawn from a Gaussian joint probability distribution. Its projection onto either complex element is a zero-mean circular Gaussian distribution in the complex plane (Eq. 7), but the elements are correlated (Eq. 8). The Appendix gives an expression for the parent distribution of elements, and presents a calculation of the distribution of products. This distribution of products has the form:

$$P(V) = \frac{2}{2\pi} \frac{1}{(1-\rho^2)} K_0\left(\tfrac{2}{(1-\rho^2)} |V|\right) \exp\left(\tfrac{2\rho}{(1-\rho^2)} \mathrm{Re}[V]\right). \tag{9}$$

In this expression, V is complex, because the electric fields E_A, E_B are complex and different (except at $\mathbf{b} = 0$, where they are identical). The function K_0 is the modified Bessel function of the second kind, of order 0. The correlation $\rho = \langle V \rangle$ is real, and is defined by Equation 6. The mean intensity is set to unity: $\mathcal{I}_0 = \langle \mathcal{I} \rangle = 1$. The mean visibility is $\langle V \rangle = \rho$, and lies on the real axis. The distribution is singular at $V = 0$. The variance of the real part about the mean is $\langle \mathrm{Re}[V]^2 \rangle - \langle \mathrm{Re}[V] \rangle^2 = \frac{1}{2} + \frac{1}{2}\rho^2$, and the variance of the imaginary part is $\langle \mathrm{Im}[V]^2 \rangle = \frac{1}{2} - \frac{1}{2}\rho^2$. Gwinn et al. (1998) derived an approximation to this distribution for moderate correlation, when ρ is not much less than 1.

Note that Eq. 9 can be generalized to mean intensity of $\mathcal{I}_0 \neq 1$ by substituting $V_1 = V/\mathcal{I}_0$ for V on the right-hand side, and multiplying the right-hand side by the normalization factor $1/\mathcal{I}_0^2$. The mean intensity appears twice in the normalization factor because the distribution is a function of both real and imaginary parts of V_1, both of which scale with the mean intensity.

Figure 2 shows several examples of the distribution of visibility V, for different values of correlation ρ. If the baseline is short, then the electric fields at the two locations are nearly identical, $\rho \approx 1$, and the visibility $\mathcal{C}_{AB}$ is nearly equal to the intensity $\mathcal{I}$. The distribution then approaches that of the intensity: a declining exponential along the positive real axis. For baselines of medium length, the distribution of intensity is approximately an exponential along the real axis, times a Gaussian distribution in the imaginary direction at each point, with the standard deviation of the Gaussian distribution proportional to the square root of the real part (Gwinn et al. 1998). For extremely long baselines, $\rho \to 0$ and the distribution becomes nearly circularly symmetric in the complex plane. The exponential in Equation 9 approaches unity, and the distribution is given by the modified Bessel function K_0 of the magnitude of V. Real and imaginary parts follow identical distributions in this long-baseline limit.

2.3. Real and Imaginary Parts of Visibility

The distributions of the real part, imaginary part, and amplitude of the visibility can be calculated as projections of the distribution in Equation 9 onto the real and imaginary axes, by integration over the perpendicular component. The Appendix presents the results. The distribution of the real part of V takes the convenient form:

$$P(V_r) = \exp\left\{2\frac{\rho V_r - |V_r|}{1-\rho^2}\right\}, \tag{10}$$

where I introduce the notation $V_r = \mathrm{Re}[V]$. Again I have taken $\langle \mathcal{I} \rangle = 1$. This distribution is an exponential declining away from the origin, along both positive and negative directions. The difference in exponential scales in the two directions is equal to the correlation ρ. The derivative of the distribution is discontinuous at $V_r = 0$. For short baselines, $\rho = 1$, the negative-going exponential vanishes and one recovers the declining exponential along the positive real axis expected for intensity. For long baselines, $\rho \to 0$, and the distribution is symmetric about the origin. Gwinn et al. (2000b) found that results of observations of the Vela pulsar, on long Earth-based baselines and still longer baselines from Earth to the orbiting HALCA radio observatory, followed distributions in good agreement with these predictions, with exponential declines for both positive and negative real part, and with the exponential scales becoming nearly identical for long baselines.

The distribution of the imaginary part is likewise exponential, declining away from the origin in both directions. However, the distribution is symmetric about the origin, with an exponential scale that increases as correlation ρ decreases. The distribution is:

$$P(V_i) = \frac{1}{\sqrt{1-\rho^2}} \exp\left\{-2\frac{|V_i|}{\sqrt{1-\rho^2}}\right\}, \tag{11}$$

where $V_i = \mathrm{Im}[V]$. Again I take $\langle \mathcal{I} \rangle = 1$. Note that in the limit $\rho \to 0$, the distributions for V_r and V_i become identical.

The distribution for amplitude is somewhat broader than an exponential distribution with the same mean. As shown in the Appendix, it has the form:

$$P(|V|) = \frac{4|V|}{(1-\rho^2)} K_0\left(\tfrac{2}{(1-\rho^2)}|V|\right) I_0\left(\tfrac{-2\rho}{(1-\rho^2)}|V|\right), \tag{12}$$

where again $\langle \mathcal{I} \rangle = 1$. Here, I_0 is the modified Bessel function of the first kind, of order 0. Figure 3 shows distributions of the real and imaginary parts, and of the amplitude, for $\rho = 0.8$.

3. VISIBILITY FOR AN EXTENDED SCINTILLATING SOURCE

In this section I discuss the distribution of interferometric visibility for a small, extended, scintillating source, in strong scattering. I assume that the source is fully spatially incoherent.

I assume that the scattering material is concentrated into a thin screen between source and observer. I represent effects of scattering by a Kirchoff integral, and I approximate the Kirchoff integral as a sum over points of stationary phase (Gwinn et al. 1998). These points are those where the gradients of the phase introduced by the scattering screen precisely cancel the gradients of geometric phase. The electric field at the observer is then (Gwinn et al. 1998, Eq. 4):

$$E(\mathbf{p}) = \sum_\imath \frac{1}{H_\imath} e^{i\phi_\imath} e^{-i(k/D)\mathbf{p}\cdot\mathbf{x}_\imath} \int_{\rm source} d\mathbf{s}\, E(\mathbf{s}) e^{i(k/R)\mathbf{x}_\imath\cdot\mathbf{s}}. \tag{13}$$

In this expression, $\mathbf{p} = (p_\xi, p_\eta)$ is the location of the observer. The points of stationary phase lie at points $\mathbf{x}_\imath = (\xi_\imath, \eta_\imath)$ on the screen, and have phase $\phi_\imath$ and weight $H_\imath$. The phases $\phi_\imath$ include both screen and geometric phase. The coordinate at the source plane is $\mathbf{s} = (s_\xi, s_\eta)$. The wavenumber is $k = 2\pi/\lambda$. The distance from screen to observer is D and from screen to source is R. Figure 1 shows the geometry. In strong scattering, the scattering disk contains many points of stationary phase, and their distribution on the screen, weighted by $1/|H_\imath|^2$, is approximately Gaussian (Gwinn et al. 1998).

Useful observables involve averages of products of the electric field. In particular, observers measure the variance and covariance, the second moments of the electric field. The spatial coherence of the source determines whether electric fields from different locations on the source can combine to contribute to these observational averages. Here, I assume that the source is fully spatially incoherent. In the language of randomly-varying electric fields, §1.1, products of electric field from different points on the source average to zero. Thus, the statistically-averaged product of electric fields at two points on the source $\mathbf{s}_1$, $\mathbf{s}_2$ is described by a delta-function coherence function (Goodman 1985):

$$\langle E(\mathbf{s}_1) E^*(\mathbf{s}_2)\rangle_E = \mathcal{I}(\mathbf{s}_1)\delta(\mathbf{s}_1 - \mathbf{s}_2). \tag{14}$$

The distribution of intensity in the source plane is $\mathcal{I}(\mathbf{s})$. Again, note that the statistical average denoted by the subscripted angular brackets $\langle ...\rangle_E$ is formed over an ensemble of realizations of the electric field of the source, rather than over an ensemble of statistically-identical scattering screens. Because of the delta-function $\delta(\mathbf{s}_1 - \mathbf{s}_2)$, only electric fields from the same point on the source contribute to the statistically-averaged second moments, even though these fields may have propagated along vastly different paths. A generalization of this picture, to arbitrary coherence functions, underlies the theory of propagation of coherence functions (Goodman 1985).

The interferometric visibility on baseline $\mathbf{b}$ is the product of two copies of Eq. 13, with different coordinates at the source $\mathbf{s}_1$, $\mathbf{s}_2$, each integrated over the source. The coherence relation Eq. 14 can be used to eliminate one of these two coordinates, yielding (Gwinn et al. 1998, Eq. 11):

$$\mathcal{C}_{AB}(\mathbf{b}) = \left(\sum_\imath \frac{e^{i\phi_\imath}}{H_\imath} e^{-i(k/D)\mathbf{b}\cdot\mathbf{x}_\imath}\right)\left(\sum_\jmath \frac{e^{-i\phi_\jmath}}{H_\jmath}\right)\int_{\text{source}} d\mathbf{s}\, e^{i(k/R)(\mathbf{x}_\imath - \mathbf{x}_\jmath)\cdot\mathbf{s}}\mathcal{I}(\mathbf{s}). \tag{15}$$

Note that Eq. 15 holds for the interferometric visibility within a single speckle, statistically averaged over an ensemble of electric fields. In other words, it represents the visibility for a single realization of the screen, but an ensemble average of electric field.

If the source is an elliptical Gaussian, the intensity distribution in the source plane is

$$\mathcal{I}(\mathbf{s}) = \mathcal{I}_{s0} \frac{1}{2\pi\sigma_\xi\sigma_\eta} e^{-\frac{1}{2}(s_\xi^2/\sigma_\xi^2 + s_\eta^2/\sigma_\eta^2)}. \tag{16}$$

Here, σ_ξ and σ_η are the major and minor axes of the source, and s_η and s_ξ are the corresponding coordinates in the source plane. The intensity of the source is $\mathcal{I}_{s0}$. The point of stationary phase $\mathbf{x}_\imath$ has corresponding coordinates $(\xi_\imath, \eta_\imath)$, and the baseline is likewise projected into components along the two axes b_ξ, b_η. The visibility takes the form

$$\mathcal{C}_{AB}(\mathbf{b}) = \left(\sum_\imath G_\imath e^{-i(k/D)\mathbf{b}\cdot\mathbf{x}_\imath} \right) \left(\sum_\jmath G_\jmath \right) e^{(k/R)^2 \{\xi_\imath \xi_\jmath \sigma_\xi^2 + \eta_\imath \eta_\jmath \sigma_\eta^2\}}, \tag{17}$$

where I adopt the combined phase and weight parameters $G_\imath$ for the points of stationary phase:

$$G_\imath = \frac{e^{i\phi_\imath}}{H_\imath} \sqrt{\mathcal{I}_{s0}}\, e^{-\frac{1}{2}(k/R)^2 \{\xi_\imath^2 \sigma_\xi^2 + \eta_\imath^2 \sigma_\eta^2\}}. \tag{18}$$

I assume that the scattering is isotropic. If the scattering is not isotropic, it is most convenient to restate the problem as the equivalent isotropic problem. Without loss of generality, one can take η as the direction of the minor axis of the scattering disk. If the ratio of major to minor axes of the scattering disk is a, the scaling $\eta \to a\eta$, $\xi \to \xi$ restores a circularly symmetric distribution of points of stationary phase. Inspection of Eqs. 17 and 18 shows that scaling lengths in the source plane as $s_\eta \to s_\eta/a$ and $s_\xi \to s_\xi$, and in the observer plane as $b_\eta \to b_\eta/a$ and $b_\xi \to b_\xi$, yields the same observables, but under the assumption of an isotropic scattering disk.

If the source is small compared with the linear resolution of the scattering disk viewed as a lens, then $\sigma << \lambda/\theta(D/R)$. For a non-circular source, this corresponds to the conditions $(k/R)\xi_\imath\sigma_\xi << 1$ and $(k/R)\eta_\imath\sigma_\eta << 1$. In this limit Equation 17 can be expanded to find:

$$\mathcal{C}_{AB}(\mathbf{b}) \approx E_{A0}E^*_{B0} + E_{A1\xi}E^*_{B1\xi} + E_{A1\eta}E^*_{B1\eta}, \tag{19}$$

where

$$\begin{aligned} E_{A0} &= \sum_\imath G_\imath e^{-i(k/D)\mathbf{b}\cdot\mathbf{x}_\imath}, \\ E_{A1\xi} &= \sum_\imath G_\imath e^{-i(k/D)\mathbf{b}\cdot\mathbf{x}_\imath}(k/R)\xi_\imath\sigma_\xi, \\ E_{A1\eta} &= \sum_\imath G_\imath e^{-i(k/D)\mathbf{b}\cdot\mathbf{x}_\imath}(k/R)\eta_\imath\sigma_\eta. \end{aligned} \tag{20}$$

The E_B are defined analogously, but with $\mathbf{b} = 0$.

Because the $\mathbf{x}_\imath$ run over many Fresnel zones in strong scattering, and $\phi_\imath$ varies greatly within each zone, E_{A0}, $E_{A1\xi}$, and $E_{A1\eta}$ have the statistics of independent random walks (Gwinn et al. 1998). Their sums are drawn from independent zero-mean Gaussian distributions. However, E_{A0} is correlated with E_{B0}, $E_{A1\xi}$ with $E_{B1\xi}$, and $E_{A1\eta}$ with $E_{B1\eta}$. Equation 9 gives the distribution of each of the 3 products in terms of the variances and covariances of the factors. Note that I here appeal to the stationary phase approximation, and to the fact that the scattering disk contains many stationary-phase points. Codona et al. (1986) discuss the required approximations in more detail, and calculate some of the neglected terms.

I introduce the notations

$$\begin{aligned} \mathcal{C}_0 &= \langle E_{A0}E^*_{B0}\rangle \\ \mathcal{C}_{1\xi} &= \langle E_{A1\xi}E^*_{B1\xi}\rangle \\ \mathcal{C}_{1\eta} &= \langle E_{A1\eta}E^*_{B1\eta}\rangle. \end{aligned} \tag{21}$$

The imaginary parts of the $\mathcal{C}$'s are antisymmetric about the origin and average to zero for a Gaussian source, or any source with an axis of symmetry along s_ξ or s_η. The complex exponentials in Eq. 20 then become cosines. I express the statistical average as an integral over the distribution of points of stationary phase:

$$\mathcal{C}_0 = \langle \sum_\imath |G_\imath|^2 \cos(\frac{k}{D}\mathbf{b}\cdot\mathbf{x}_\imath)\rangle = \mathcal{I}_0 \int_{\rm screen} d\mathbf{x}\, P(\mathbf{x}) \cos(\frac{k}{D}\mathbf{b}\cdot\mathbf{x}), \tag{22}$$

where $P(\mathbf{x}_\imath)$ is the distribution on the screen of the points of stationary phase $\mathbf{x}_\imath$, weighted by the square modulus of the $G_\imath$. Equivalently, $P(\mathbf{x})$ is the distribution of intensity for a point source, statistically averaged over an ensemble of scattering screens. The distribution $P(\mathbf{x})$ is approximately Gaussian (Gwinn et al. 1998). I assume that it is a circular Gaussian distribution, because anisotropic scattering can be treated by appropriately scaling coordinates in the source and observer plane, as argued above.

If $P(\mathbf{x})$, the distribution of points of stationary phase, is represented as a circular Gaussian distribution with standard deviation θD, so that

$$P(\mathbf{x}) = \frac{1}{2\pi(\theta D)^2} e^{-\frac{1}{2}(|\mathbf{x}|/\theta D)^2}, \tag{23}$$

then (Gradshteyn & Ryzhik 1994, 3.896.4):

$$\begin{aligned} \mathcal{C}_0 &= \frac{\mathcal{I}_0}{(2\pi\theta D)} \int_{-\infty}^{\infty} d\xi\, e^{-\frac{1}{2}\xi^2/(\theta D)^2} \cos\left(\frac{k}{D} b_\xi \xi\right) \int_{-\infty}^{\infty} d\eta\, e^{-\frac{1}{2}\eta^2/(\theta D)^2} \cos\left(\frac{k}{D} b_\eta \eta\right) \\ &= \mathcal{I}_0 \exp\left\{-\tfrac{1}{2}k^2\theta^2|b|^2\right\} \end{aligned} \tag{24}$$

The parameter $\mathcal{I}_0$ is the value of $\mathcal{C}_0$ at $\mathbf{b} = 0$; it is related to, although not equal to, the mean intensity. Similarly (Gradshteyn & Ryzhik 1994, 3.952.4):

$$\begin{aligned} \mathcal{C}_{1\xi} &= \frac{\mathcal{I}_0}{(2\pi\theta D)} \int_{-\infty}^{\infty} d\xi\, e^{-\frac{1}{2}\xi^2/(\theta D)^2} \cos\left(\frac{k}{D} b_\xi \xi\right) \left(\frac{k}{R}\xi\sigma_\xi\right)^2 \\ &\quad \times \int_{-\infty}^{\infty} d\eta\, e^{-\frac{1}{2}\eta^2/(\theta D)^2} \cos\left(\frac{k}{D} b_\eta \eta\right) \\ &= \mathcal{I}_0 \exp\left\{-\tfrac{1}{2}k^2\theta^2|b|^2\right\} \left(1-(b_\xi k\theta)^2\right) (kM\theta\sigma_\xi)^2 . \end{aligned} \tag{25}$$

Here I define $M = D/R$, a parameter equivalent to the magnification of the scattering disk treated as a lens. Similarly,

$$\mathcal{C}_{1\eta} = \mathcal{I}_0 \exp\left\{-\tfrac{1}{2}k^2\theta^2|b|^2\right\} \left(1-(b_\eta k\theta)^2\right) (kM\theta\sigma_\eta)^2 . \tag{26}$$

The $\mathcal{C}$'s reflect both source structure and the direction and length of the baseline. Source structure can be expected to have the greatest effect when $\mathcal{C}_{1\xi}$ or $\mathcal{C}_{1\eta}$ are greatest; this is when the source size, given by σ_ξ or σ_η, is largest, and when the baseline is perpendicular to the major axis of the source.

The variances of the E's must be independent of position in the observer plane. These variances can be expressed as intensity-like quantities:

$$\begin{aligned} \mathcal{I}_0 &\equiv \langle E_{A0}\ E^*_{A0}\ \rangle = \langle E_{B0}\ E^*_{B0}\ \rangle \\ \mathcal{I}_{1\xi} &\equiv \langle E_{A1\xi} E^*_{A1\xi}\rangle = \langle E_{B1\xi} E^*_{B1\xi}\rangle = \mathcal{I}_0 (kM\theta\sigma_\xi)^2 \\ \mathcal{I}_{1\eta} &\equiv \langle E_{A1\eta} E^*_{A1\eta}\rangle = \langle E_{B1\eta} E^*_{B1\eta}\rangle = \mathcal{I}_0 (kM\theta\sigma_\eta)^2. \end{aligned} \tag{27}$$

Note that the $\mathcal{I}$'s contain the same dependences on source structure as the $\mathcal{C}$'s, but do not depend on baseline length. I define the normalized correlations

$$\begin{aligned} \tau &\equiv \frac{\langle E_{A0}\ E^*_{B0}\ \rangle}{\sqrt{\langle |E_{A0}\ |^2\rangle \langle |E_{B0}\ |^2\rangle}} = \frac{\mathcal{C}_0}{\mathcal{I}_0} &&= \exp\{-\tfrac{1}{2}(k\theta|b|)^2\} \\ \mu &\equiv \frac{\langle E_{A1\xi} E^*_{B1\xi}\rangle}{\sqrt{\langle |E_{A1\xi}|^2\rangle \langle |E_{B1\xi}|^2\rangle}} = \frac{\mathcal{C}_{1\xi}}{\mathcal{I}_{1\xi}} &&= \left(1-(b_\xi k\theta)^2\right)\exp\{-\tfrac{1}{2}(k\theta|b|)^2\} \\ \nu &\equiv \frac{\langle E_{A1\eta} E^*_{B1\eta}\rangle}{\sqrt{\langle |E_{A1\eta}|^2\rangle \langle |E_{B1\eta}|^2\rangle}} = \frac{\mathcal{C}_{1\eta}}{\mathcal{I}_{1\eta}} &&= \left(1-(b_\eta k\theta)^2\right)\exp\{-\tfrac{1}{2}(k\theta|b|)^2\} \end{aligned} \tag{28}$$

The normalized correlations τ, μ, and ν are independent of source structure. They reflect only the direction and length of the baseline.

Each of the 3 products $E_{A0}E^*_{B0}$, $E_{A1\xi}E^*_{B1\xi}$, and $E_{A1\eta}E^*_{B1\eta}$, follows the distribution of Eq. 9, that of the product of factors drawn from correlated Gaussian distributions. Eq. 19 shows that the distribution of visibility for an extended source is that of the sum of these 3 products. The distribution of that sum is that of the convolution of 3 distributions of the form of Eq. 9. For each of these 3 distributions, the $\mathcal{I}$'s play the role of the mean intensity (set to 1 in Eq. 9), and the normalized correlations τ, μ, and ν play the role of covariance τ in that equation.

The convolution of 3 copies of Equation 10 yields the distribution of the real part of the visibility of an extended source. This is somewhat easier to calculate than the joint distribution of real and imaginary parts. The parameters of the 3 distributions are given by the $\mathcal{I}$'s and τ, μ, and ν, as defined by Equations 21 and 28. The distribution of V_r is then found to be:

$$\begin{aligned} P(V_r) \;=\;& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (29) \\ & \frac{\mathcal{I}_0^3 (S+\tau)^4}{\left((\mathcal{I}_0 (S+\tau) - \mathcal{I}_{1\xi}\,\mu)^2 - \mathcal{I}_{1\xi}^2\right)\left((\mathcal{I}_0 (S+\tau) - \mathcal{I}_{1\eta}\,\nu)^2 - \mathcal{I}_{1\eta}^2\right)} \exp\{\tfrac{-2(S-\tau)V_r}{\mathcal{I}_0(1-\tau^2)}\} \\ & + \frac{\mathcal{I}_{1\xi}^3 (S+\mu)^4}{\left((\mathcal{I}_{1\xi} (S+\mu) - \mathcal{I}_{1\eta}\,\nu)^2 - \mathcal{I}_{1\eta}^2\right)\left((\mathcal{I}_{1\xi} (S+\mu) - \mathcal{I}_0\,\tau)^2 - \mathcal{I}_0^2\right)} \exp\{\tfrac{-2(S-\mu)V_r}{\mathcal{I}_{1\xi}(1-\mu^2)}\} \\ & + \frac{\mathcal{I}_{1\eta}^3 (S+\nu)^4}{\left((\mathcal{I}_{1\eta} (S+\nu) - \mathcal{I}_0\,\tau)^2 - \mathcal{I}_0^2\right)\left((\mathcal{I}_{1\eta} (S+\nu) - \mathcal{I}_{1\xi}\,\mu)^2 - \mathcal{I}_{1\xi}^2\right)} \exp\{\tfrac{-2(S-\nu)V_r}{\mathcal{I}_{1\eta}(1-\nu^2)}\} \end{aligned}$$

where S is the sign of V_r: $S = 1$ for $V_r > 0$, $S = -1$ for $V_r < 0$. Note that $P(V_r)$ is the sum of 3 exponentials, with different scales for positive or negative V_r. The exponential scales of the 3 are those of the distributions of the products $E_{A0}E_{B0}^*$, $E_{A1\xi}E_{B1\xi}^*$, and $E_{A1\eta}E_{B1\eta}^*$. The distribution $P(V_r)$ is continuous at $V_r = 0$, and its first through fourth derivatives are continuous there. This continuity results from the convolution of the 3 component distributions of the form of Eq. 10, which are continuous, but with discontinuous derivative, at $V_r = 0$. Each convolution increases the degree of continuity at $V_r = 0$ by 2. The exponential scales for positive and negative V_r, the continuity of 4 derivatives, and the normalization completely characterize $P(V_r)$.

The distribution of the imaginary part of the visibility of an extended source is the convolution of 3 copies of Equation 11, with the parameters of the 3 distributions given by Equations 21 and 28. The distribution of $V_i = \mathrm{Im}[V]$ has the form:

$$\begin{aligned} P(V_i) \;=\;& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (30) \\ & \frac{\mathcal{I}_0^3 (1-\tau^2)^{3/2}}{\left(\mathcal{I}_0^2(1-\tau^2) - \mathcal{I}_{1\xi}^2(1-\mu^2)\right)\left(\mathcal{I}_0^2(1-\tau^2) - \mathcal{I}_{1\eta}^2(1-\nu^2)\right)} \exp\left\{\frac{-2|V_i|}{\mathcal{I}_0\sqrt{1-\tau^2}}\right\} \end{aligned}$$

$$+\frac{\mathcal{I}_{1\xi}^3(1-\mu^2)^{3/2}}{\left(\mathcal{I}_{1\xi}^2(1-\mu^2)-\mathcal{I}_{1\eta}^2(1-\nu^2)\right)\left(\mathcal{I}_{1\xi}^2(1-\mu^2)-\mathcal{I}_0^2(1-\tau^2)\right)}\exp\left\{\frac{-2|V_i|}{\mathcal{I}_{1\xi}\sqrt{1-\mu^2}}\right\}$$
$$+\frac{\mathcal{I}_{1\eta}^3(1-\nu^2)^{3/2}}{\left(\mathcal{I}_{1\eta}^2(1-\nu^2)-\mathcal{I}_0^2(1-\tau^2)\right)\left(\mathcal{I}_{1\eta}^2(1-\nu^2)-\mathcal{I}_{1\xi}^2(1-\mu^2)\right)}\exp\left\{\frac{-2|V_i|}{\mathcal{I}_{1\eta}\sqrt{1-\nu^2}}\right\}.$$

Like $P(V_r)$ in Eq. 29 above, $P(V_i)$ is the sum of 3 exponentials, with the scales for positive and negative V_i those of the 3 convolved distributions. At $V_i = 0$, $P(V_i)$ and its first through fourth derivatives are continuous. Together with the requirement of unit normalization, the exponential scales and continuity of derivatives completely specify $P(V_i)$. In the limit of $\tau = \mu = \nu = 0$, the distributions for $P(V_r)$ and $P(V_i)$ become identical, as expected. Figures 4 and 5 show sample distributions of the real and imaginary parts of the visibility, for sample sources with different orientations relative to the baseline.

The statistically-averaged mean visibility is

$$\langle V_r\rangle = \tau\mathcal{I}_0 + \mu\mathcal{I}_{1\xi} + \nu\mathcal{I}_{1\eta}, \tag{31}$$

and the variance of the real part of the visibility about that mean is

$$\langle V_r^2\rangle - (\langle V_r\rangle)^2 = \tfrac{1}{2}\left(\mathcal{I}_0^2(1+\tau^2) + \mathcal{I}_{1\xi}^2(1+\mu^2) + \mathcal{I}_{1\eta}^2(1+\nu^2)\right). \tag{32}$$

The mean imaginary part of the visibility is $\langle V_i\rangle = 0$, and the variance of the imaginary part of the visibility is

$$\langle V_i^2\rangle = \tfrac{1}{2}\left(\mathcal{I}_0^2(1-\tau^2) + \mathcal{I}_{1\xi}^2(1-\mu^2) + \mathcal{I}_{1\eta}^2(1-\nu^2)\right). \tag{33}$$

These results are in accord with the exact results for $\langle VV^*\rangle$ and $\langle VV\rangle$ of Gwinn et al. (1998, Eqs. 48 and 52), in the limit of small source size as described by the approximation of Eqs. 19 and 20. The dependence of the interferometric visibility on source structure, as parameterized by the definitions of the $\mathcal{C}$'s (Eqs. 24 through 26) or of τ, μ, and ν (Eq. 28), do not appear in the exact expressions for the mean square visibility. The contributions to the second moments are canceled by terms of higher order in $kM\theta\sigma$. However, they do affect the distributions for the real and imaginary parts, as Eqs. 29 and 30 indicate.

In the limit $\tau, \mu, \nu \to 1$, the distribution of $P(V_r)$ becomes that of the intensity of a small, resolved source, the weighted sum of three exponentials (Gwinn et al. 1998, Eq. 30). The mean intensity, observed at either A or B, is

$$\langle \mathcal{I} \rangle = \mathcal{I}_0 + \mathcal{I}_{1\xi} + \mathcal{I}_{1\eta}, \tag{34}$$

and the variance of the intensity about the mean is

$$\langle \mathcal{I}^2 \rangle - \langle \mathcal{I} \rangle^2 = \mathcal{I}_0^2 + \mathcal{I}_{1\xi}^2 + \mathcal{I}_{1\eta}^2. \tag{35}$$

(Note that these expressions correct an error in Eq. 31 of Gwinn et al. (1998)).

4. ONE SAMPLE OF INTERFEROMETRIC VISIBILITY

Correlation of noiselike signals yields products of elements drawn from correlated Gaussian distributions. Equation 9 describes the distribution of these products. Commonly astrophysical signals are sampled (that is, they are averaged over short time intervals), and then quantized (the electric field is converted from a continuous quantity to one of a set of discrete values, using some transfer function) (Thompson, Moran, & Swenson 1986; Jenet & Anderson 1998a; Gwinn et al. 2000a). These discrete values are then correlated. The resulting correlations are almost invariably integrated over many samples. Equation 9 gives the distribution for a single sample, in the absence of quantization. Integration over N samples yields a distribution that is the convolution of N distributions of the form of Equation 9. When N is large the resulting distribution approaches a Gaussian. Equation 9 may be useful for understanding correlation observations of strong sources that vary on extremely short timescales, such as some pulsars (Hankins 1996; Jenet et al. 1998b).

5. SUMMARY

I have presented the distribution of interferometric visibility for a scintillating point source. This distribution is that of the product of 2 complex factors drawn from correlated zero-mean Gaussian distributions, with specified correlation ρ. This distribution is a declining exponential along the positive real axis for perfectly correlated distributions ($\rho = 1$); and is the modified Bessel function of the second kind of order zero, K_0, of the absolute value of the visibility, for completely uncorrelated distributions ($\rho = 0$). For moderately correlated distributions ($0 < \rho < 1$), the distribution of visibility is approximately a declining exponential along the positive real axis, with the imaginary part approximately following a Gaussian distribution, with standard deviation proportional to the square root of the real part. Equation 9 gives the distribution and Figure 2 shows examples.

The distribution of the real part of the visibility, for a scintillating point source, follows declining exponentials along the positive and negative axes, as Equation 10 shows. The exponential along the negative real axis is steeper than that along the positive axis. The difference in exponential scales is proportional to the correlation ρ. The distribution is purely positive for identical signals ($\rho = 1$), and is symmetric about the origin for uncorrelated signals ($\rho = 0$). The distribution of imaginary part likewise declines exponentially along both positive and negative axes, with equal scales in the two directions, as Equation 11 shows. The distribution of imaginary part is a delta function for perfect correlation ($\rho = 1$), narrow for strong correlation ($\rho \approx 1$), and broadens with decreasing ρ. The distribution of amplitude of the visibility is given by the product of two modified Bessel functions of order zero, as Equation 12 expresses. It is approximately exponential at large amplitude, and goes to 0 at zero amplitude. Figure 3 shows sample distributions of real and imaginary parts, and amplitude, for $\rho = 0.8$.

If the source is extended, but small compared with the resolution of the scattering disk treated as a lens, the distribution of visibility is the convolution of 3 distributions of the form for a point source, with specific scales, correlations, and weights. The form of the distribution depends on whether the source is elongated, and whether it is oriented parallel or perpendicular to the interferometer baseline. Equation 29 describes the distribution for the visibility of the real part, and Figures 4 and 5 show examples for point and extended sources with different orientations relative to the baseline.

I point out that the distribution of a single sample of interferometric visibility is precisely that of the interferometric visibility of a scintillating point source. It is the distribution of products of elements drawn from correlated complex distributions. Commonly, results of correlation of many samples are averaged together. The resulting distribution is then the convolution of equally many distributions, and thus rapidly approaches a Gaussian distribution.

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A. PRODUCT OF FACTORS DRAWN FROM CORRELATED GAUSSIAN DISTRIBUTIONS

We consider the product of 2 correlated, complex quantities E_A, E_B. We assume that each quantity is drawn from a zero-mean circular Gaussian in the complex plane, with unit variance. The variances of the real and imaginary parts of E_A fully characterize their distribution:

$$\begin{aligned} &\langle A_r\rangle = \langle A_i\rangle = 0 \\ &\langle A_r A_r\rangle = \langle A_i A_i\rangle = \tfrac{1}{2} \\ &\langle A_r A_i\rangle = 0, \end{aligned} \tag{A1}$$

where $A_r = \mathrm{Re}[E_A]/\sigma$, $A_i = \mathrm{Im}[E_A]/\sigma$. Similar facts characterize the distribution of $B_r = \mathrm{Re}[E_B]/\sigma$ and $B_i = \mathrm{Im}[E_B]/\sigma$. The covariances of the components are:

$$\begin{aligned} &\langle A_r B_r\rangle = \langle A_i B_i\rangle = \tfrac{1}{2}\rho \\ &\langle A_r B_i\rangle = \langle A_i B_r\rangle = 0. \end{aligned} \tag{A2}$$

Both E_A and E_B are drawn from the bivariate complex distribution

$$\begin{aligned} P(E_A, E_B) = \; & \tfrac{1}{\pi^2}\tfrac{1}{1-\rho^2} \quad \exp\left\{-\frac{1}{1-\rho^2}(A_r^2 - 2\rho A_r B_r + B_r^2)\right\} \\ & \times \exp\left\{-\frac{1}{1-\rho^2}(A_i^2 - 2\rho A_i B_i + B_i^2)\right\}. \end{aligned} \tag{A3}$$

The complex visibility, the quantity measured by interferometry, is $V = E_A E_B^*/\sigma^2$. In particular,

$$\begin{aligned} \mathrm{Re}[V] &= \{A_r B_r + A_i B_i\} \\ \mathrm{Im}[V] &= \{A_i B_r - A_r B_i\}. \end{aligned} \tag{A4}$$

One can calculate the distribution of correlated flux density by integrating over the bivariate distribution of electric field. Define $V_r = \mathrm{Re}[V]$ and $V_i = \mathrm{Im}[V]$. Then,

$$\begin{aligned} P(V_r, V_i) &= \int_{-\infty}^{\infty} dA_r \int_{-\infty}^{\infty} dA_i \int_{-\infty}^{\infty} dB_r \int_{-\infty}^{\infty} dB_i \frac{1}{\pi^2} \frac{1}{(1-\rho^2)} \\ &\times \exp\left\{ -\frac{1}{1-\rho^2}(A_r^2 - 2\rho A_r B_r + B_r^2) \right\} \exp\left\{ -\frac{1}{1-\rho^2}(A_i^2 - 2\rho A_i B_i + B_i^2) \right\} \\ &\times \delta(A_r B_r + A_i B_i - V_r)\, \delta(A_i B_r - A_r B_i - V_i). \end{aligned} \tag{A5}$$

where δ is the Dirac delta function.

I use the facts $V_r = A_r B_r + A_i B_i$ and $|V|^2 = (V_r^2 + V_i^2) = (A_r^2 + A_i^2)(B_r^2 + B_i^2)$ to express the integrand in terms of A_r and A_i alone. The delta functions can then be re-expressed by using the rule $\delta(ax) = \frac{1}{a}\delta(x)$, for constant a and integration variable x, and by applying the re-expressed first delta function to simplify the second. Then,

$$\begin{aligned} P(V_r, V_i) &= \int_{-\infty}^{\infty} dA_r \int_{-\infty}^{\infty} dA_i \int_{-\infty}^{\infty} dB_r \int_{-\infty}^{\infty} dB_i \, \frac{1}{\pi^2} \frac{1}{(1-\rho^2)} \\ &\times \exp\left\{ -\frac{1}{(1-\rho^2)} \left(A_r^2 + A_i^2 + \frac{(V_r^2 + V_i^2)}{A_r^2 + A_i^2} - 2\rho V_r \right) \right\} \\ &\times \frac{1}{A_i}\, \delta\left(B_i - \frac{V_r - A_r B_r}{A_i} \right) \frac{A_i}{(A_r^2 + A_i^2)} \delta\left(B_r - \frac{V_i A_i - V_r A_r}{A_r^2 + A_i^2} \right). \end{aligned} \tag{A6}$$

Now trivially integrate over B_i and B_r to obtain:

$$P(V_r, V_i) = \int_0^{\infty} A dA \int_0^{2\pi} d\theta \frac{1}{\pi^2} \frac{1}{(1-\rho^2)} \frac{1}{A^2} \exp\left\{ -\frac{1}{(1-\rho^2)} \left(A^2 + \frac{(V_r^2 + V_i^2)}{A^2} - 2\rho V_r \right) \right\} \tag{A7}$$

where I have defined A and θ by $A_r = A\cos\theta$, $A_i = A\sin\theta$. Integrate over θ, also trivially. Then make the substitution $u = A^2$, and integrate over u to find (Gradshteyn & Ryzhik 1994):

$$P(V_r, V_i) = \frac{2}{\pi} \frac{1}{(1-\rho^2)} K_0\left(\tfrac{2}{(1-\rho^2)} |V| \right) \exp\left(\tfrac{2\rho}{(1-\rho^2)} V_r \right). \tag{A8}$$

Here K_0 is the modified Bessel function of the second kind, of order 0. Recall that $|V| = \sqrt{V_r^2 + V_i^2}$.

The distribution of the amplitude of visibility is found by integrating Equation A8 over complex phase, $\phi = \tan^{-1}(V_r/V_i)$. Note that $dV_r\, dV_i = |V|\, d|V|\, d\phi$, and find:

$$P(|V|) = \frac{4|V|}{(1-\rho^2)} K_0\left(\tfrac{2}{(1-\rho^2)}|V|\right) I_0\left(\tfrac{-2\rho}{(1-\rho^2)}|V|\right). \tag{A9}$$

Here, I_0 is the modified Bessel function of the first kind, of order 0.

The distribution of the real part, V_r, is given by:

$$P(V_r) = \int_{-\infty}^{\infty} dV_i P(V_r) = \exp\left\{2\frac{\rho V_r - |V_r|}{1-\rho^2}\right\}. \tag{A10}$$

This result is obtained most easily by integrating Equation A7 first over V_i, and then over A and θ. Similarly, integration of Equation A7 over V_r, A, and θ yields the distribution of V_i:

$$P(V_i) = \frac{1}{\sqrt{1-\rho^2}} \exp\left\{-2\frac{|V_i|}{\sqrt{1-\rho^2}}\right\}. \tag{A11}$$

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Fig. 1.— Geometry for scattering in a thin screen with a random index of refraction. Radiation from **s** at the source travels along **r** to **x** on the screen. At the screen, the phase of the radiation changes by $\Phi(\mathbf{x})$, and it then propagates along **d** to **p** in the observer plane. The separation of source and screen is **R**, and that of screen and observer plane is **D**.

Fig. 2.— Distribution of interferometric visibility on different-length baselines. Upper: On short baselines ($\rho = 0.99$), the distribution of visibility is nearly that of intensity: an exponential along the real axis. Middle: On medium baselines ($\rho = 0.8$), the distribution widens in the imaginary direction. Lower: On long baselines ($\rho = 0$), the distribution is isotropic in the complex plane.

Fig. 3.— Distributions for products of factors drawn from correlated zero-mean Gaussian distributions, with normalized correlation $\rho = 0.8$, showing distributions of amplitude (top), real part (middle), and imaginary part (lower). The condensed distribution of the imaginary part requires a different vertical scale for that plot.

Fig. 4.— Distribution of visibility for sources with different sizes and orientations. Upper panel: imaginary part. Lower panel: real part. Solid line: point source; dotted line: major axis of source parallel to interferometer baseline; dashed line: major axis perpendicular to baseline. Sources have the same mean flux density, $\langle \mathcal{I} \rangle = 1$. The baseline lengths are given by the parameters $(k\theta b) = (k\theta b_\xi) = 1$, and $(k\theta b_\eta) = 0$; and the source structure by the parameters $kM\theta\sigma = 0.7$ along the major axis of the source, and $kM\theta\sigma = 0$ along the minor axis.

Fig. 5.— Distribution of visibility for sources with different sizes and orientations. Upper panel: imaginary part. Lower panel: real part. Solid line: point source; dotted line: major axis of source parallel to interferometer baseline; dashed line: major axis perpendicular to baseline. Sources have the same mean visibility, $\langle \mathcal{C} \rangle = 1$. Source and baseline parameters are as in Figure 4.