Heat Transport
1) Random motions of particles
2) Bulk flow

Consider the $T, p, P$ gradients in a star.

Now suppose we place a bubble in the star at position $x$, and heat it a lot. The bubble is lighter than the surrounding air, so it will rise.

In some stars, radiative diffusion cannot transport the heat flux fast enough. The star becomes unstable to convection, and the convective currents transport heat even more efficiently than radiative diffusion.

Let's see when a star becomes convective.

As long as the bubble rises at a speed $\geq$ the thermal sound speed, we can assume the gas in the bubble responds adiabatically to the change in environment with height.

\[ \text{i.e. } dQ = dE + PdV \Rightarrow 0 \]

And the bubble moves on a timescale slower than \[ t_{cp} \approx \frac{h}{c_s} \]

The time for pressure disturbances to be communicated across an atmosphere scale height.

Recall Lec. 2: \[ h = \frac{2T}{g m} \]
In some stars, radiative diffusion cannot transport the heat flux fast enough. The star becomes unstable to convection, and the convective currents transport heat even more efficiently than radiative diffusion.

Let's see how a star becomes convective.

Consider the $T$, $P$, $P$ gradients in a star.

Now suppose we place a bubble in the star at $x_1$ and heat it a bit. The bubble is lighter than the surrounding air, so it will rise.

Will the bubble rise back or continue to rise at $x_2$?

\[
h = \sqrt{\frac{\beta (T)}{g}} = \sqrt{\frac{1.38 \times 10^{-16} \text{erg/g} \frac{T}{K}}{1.67 \times 10^{-14} \text{g/cm}^2 \text{K}^{-2}}}) \approx 183 \text{ km} \approx 18 \times 10^7 \text{ cm}
\]

Well-it will continue to rise if it is lighter than the surrounding air at $x_2$.

As long as the bubble rises slowly at $v \ll \sqrt{g}$, we can assume the gas in the bubble responds adiabatically to changes in environment with height.
Bubble:
\[ dQ = dE + PdV = 0 \]
\[ \Rightarrow P_1 V_1^2 = P_2 V_2^2 \quad \text{(as we derived earlier)} \]

So the density change in the bubble must be

\[ \frac{\Delta P}{P} = \frac{1}{8} \frac{\Delta P}{P} \]

We need to compare the change in the bubble density to that in the density of the star between \( x_1 \) and \( x_2 \).

The star is an ideal gas in a gravitational field

\[ \text{Star:} \quad P = \frac{\rho m}{m} \frac{kT}{m} \]

\[ \frac{\Delta P}{P} = \frac{\Delta P}{P} - \frac{\Delta T}{T} \]

Convection is possible if \( \frac{\Delta P}{P} \) is greater than \( \frac{\Delta P}{P} \) of the star.

Assuming the bubble moves slowly, we can set

\[ \frac{\Delta P}{P} \text{ of bubble} = \frac{\Delta P}{P} \text{ of star} \]

because the bubble has time to respond to pressure change on surface.

\[ \frac{\Delta P}{P} < \frac{\Delta P}{P} - \frac{\Delta T}{T} \]

\[ \frac{\Delta T}{T} < \left( 1 - \frac{1}{8} \right) \frac{\Delta P}{P} \quad \text{For Convection} \]
Since \( p = \frac{2pT}{m} \)

\[
\frac{\Delta p}{p} = \frac{\Delta p}{p} + \frac{\Delta T}{T}
\]

\[
\frac{\Delta p}{p} - \frac{\Delta p}{p} < \left(1 - \frac{1}{8} \right) \frac{\Delta p}{p}
\]

\[
- \frac{\Delta p}{p} < - \frac{1}{8} \frac{\Delta p}{p}
\]

\[
\frac{\Delta p}{p} > \frac{1}{8} \frac{\Delta p}{p}
\]

For convection

\[
\frac{dlnp}{dT} > \frac{1}{8}
\]

Note:

When \( \zeta \) is low, the bubble has more ways to absorb heat.

If \( p_{\text{bubble}} \) is set by boundary condition and \( p = p_T \),

then we want \( T_{\text{bub}} \) to stay high in order to keep \( p \) low.

If \( \frac{dT}{dt} \) is high, then bubble can exchange energy without lowering the temperature much.
Convection

So the bubble is always in equilibrium with its surroundings.

For adiabatic expansion, we showed that

\[ P_1 V_1^\gamma = P_2 V_2^\gamma \]

or

\[ \frac{P_1}{P_1^\gamma} = \frac{P_2}{P_2^\gamma} \]

So the density of bubble at \( x_2 \) decreases

\[ \rho_2^\gamma = \frac{P_2}{P_1^\gamma} < \rho_1^\gamma \]

since the pressure gradient is negative.

The key question is whether the density of the bubble is less than the density of its new surroundings. If not, it will sink.

For instability, we need

\[ P_{\text{bubble}} < P_{2,*} \]  \hspace{1cm} \text{INSTABILITY} \]

The density in the star at \( r_2 \) is

\[ \rho_{r_2} = \rho_{r_2} + \Delta r \frac{d\rho}{dr} \bigg|_{\text{star}} \]

and the pressure in the star at \( r_2 \) is

\[ P_{2,*} = P_{1,*} + \Delta r \frac{dP}{dr} \bigg|_{\text{star}}. \]
Convection

But what is the density of the bubble at $r_2$?

The bubble density depends on the bubble pressure at $r_2$. But we said

$$P_{r_2, \text{bubble}} = P_{r_2, \ast}.$$  

So we can write

$$P_{r, \text{bubble}} = P_{r, \text{bubble}} + \Delta r \frac{dP}{dr}$$

where $dP = \frac{P}{P} \frac{dP}{dr}$ since $P = KP_\ast$.

$$P_{r, \text{bubble}} = P_{r, \text{bubble}} + \Delta r \frac{P_{r, \text{bubble}}}{P} \frac{dP}{dr}$$

$$= P_{r, \text{bubble}} \left[ 1 + \Delta r \frac{d\ln P}{dr} \right].$$

So the condition for convection to set in is

$$P \left[ 1 + \Delta r \frac{d\ln P}{dr} \right] = P_{r, \ast} + \Delta r \frac{dP}{dr}$$

$$P_{r, \ast} - P_{r, \ast} = \Delta r \frac{dP}{dr} - P_{r, \ast} \Delta r \frac{d\ln P}{dr}.$$

To convect, we want $P - P_{r, \ast} < 0$ when the bubble moves up (i.e., $\Delta r > 0$).

$$\frac{dP}{dr} \bigg|_{\ast} = P_{r, \ast} \frac{d\ln P}{dr} \bigg|_{\ast}$$
Convection

\[ \frac{d \ln \rho}{dr} > \frac{1}{\sigma} \frac{d \ln \rho}{dr} \]

**Criteria for Instability**

\[ \left. \frac{d \ln \rho}{d \ln \rho \text{ star}} \right| > \frac{1}{\sigma} \]

Consider what this means.

For adiabatic changes \( P = KP^\gamma \), and \( dP = \gamma KP^{\gamma-1} dP \)

\[ \frac{dP}{P} = \gamma dP \]

or

\[ \frac{d \ln \rho}{d \ln \rho} = \frac{1}{\sigma} \]

So for convection we need the change in density with pressure to be greater than the adiabatic value (in the star).

Now let's find the consequences for the temperature gradient.

We have

\[ \frac{1}{\rho} \frac{d \rho}{dr} > \frac{1}{\sigma} \frac{1}{\rho} \frac{d \rho}{dr} \]

for instability.

From H.S.E., we can describe the \( P \) gradient as

\[ \frac{d \rho}{dr} = -\rho(r) g \]

So

\[ \frac{1}{\rho} \frac{d \rho}{dr} > -\frac{\rho g}{\sigma P} \]

For ideal gas \( P = \rho \frac{kT}{\mu m_H} \)

\[ \rho = \frac{\mu m_H}{kT} P \]
\[
\frac{1}{\rho} \frac{d\rho}{dr} = \frac{1}{P} \left[ \frac{\mu m_H P}{\rho_T} \frac{dP}{dr} - \frac{\mu m_H P}{\rho_T} \frac{dT}{dr} \right]
\]
\[
= \frac{\mu m_H P}{\mu m_H P} \left[ \frac{\mu m_H P}{\rho_T} \frac{dP}{dr} - \frac{\mu m_H P}{\rho_T + T} \frac{dT}{dr} \right]
\]
\[
= \frac{1}{P} \frac{dP}{dr} - \frac{1}{T} \frac{dT}{dr}
\]

So we get
\[
\frac{1}{P} \frac{dP}{dr} - \frac{1}{T} \frac{dT}{dr} > -\frac{\rho g}{\gamma P}
\]

or
\[
\frac{1}{T} \frac{dT}{dr} < \frac{1}{P} \frac{dP}{dr} + \frac{\rho g}{\gamma P}
\]

since \(\frac{dP}{dr} = -\rho g\)

we have
\[
\frac{1}{T} \frac{dT}{dr} < \frac{1}{P} \left( \rho g + \frac{1}{\gamma P} \right)
\]

\[
\frac{1}{T} \frac{dT}{dr} < \frac{\rho g}{P} \left( \frac{1}{\gamma} - 1 \right)
\]

or since \(dT/dr\) will be negative, we can write
\[
-\frac{1}{T} \frac{dT}{dr} > \frac{\rho g}{P} \left( \frac{1}{\gamma} - 1 \right)
\]

check signs: Typically \(\gamma = \frac{5}{3}\), so \(\frac{5}{3} - 1 = \frac{2}{3}\).

since \(\rho g = -\frac{dP}{dr}\), we could write
\[
-\frac{1}{T} \frac{dT}{dr} > -\frac{1}{P} \frac{dP}{dr} \left( \frac{\gamma - 1}{\gamma} \right)
\]
\[
\frac{1}{T} \frac{dT}{dr} < \frac{1}{P} \frac{dP}{dr} \left( \frac{\gamma - 1}{\gamma} \right)
\]

Now if we take \(\frac{dP}{dr}\) to the other side we are multiplying by a positive number!
Convection

\[
\frac{d\ln T}{d\ln P} = \frac{2}{5} \quad \text{For Instability}
\]

So as long as \( T \propto P^{2/5} \) or stronger, the star becomes convective.

\[
\frac{dT}{dP} = \frac{2}{5}\frac{dT}{dP}
\]

NOTE: The temp and press gradients are both negative!

\( \Rightarrow \) Convection requires the temperature of the star to fall off rapidly with height.

\[
\frac{dT}{dx} < \left( \frac{2}{5} \right) \frac{T}{P} \frac{dP}{dx}
\]

How rapidly depends on the ability of the gas particles to absorb heat. If they absorb heat very efficiently, the critical gradient is less steep. (i.e. small \( \gamma \))

Recall that \( \gamma = \frac{C_p}{C_v} = \frac{kT + \nu}{\nu} = \frac{kT + \frac{1}{2}kT (\#\text{dof})}{\frac{1}{2}kT (\#\text{dof})} \)

Of the gas particles can rotate and vibrate in addition to the 3 translational degrees of freedom, then

\[
\gamma = 1 + \frac{5}{2} \frac{1}{5/2} = \frac{7}{5}
\]

and \( 1 - \frac{1}{\gamma} \) is smaller

\[
\frac{dT}{dx} \to 0 \text{ as } \gamma \to 1 \text{ or as d.o.f. } \to \infty
\]

\( \Rightarrow \) ionization zones where heat can be absorbed by the removing e-s from atoms are prone to convection.
Convection

Why is convection favored in ionization zones?

Equations ⇔ law &
Physical intuition says –

(a) Opacity is large in ionization zone because we have the contribution from b-f. When $k$ is large, the temperature gradient must be steep to transport heat at a fixed rate.

$$ F \propto \frac{1}{k} \frac{dT}{dr} $$

(b) The temperature gradient needed to convect is not so steep because a rising pocket of gas does not cool so much since recombination provides some energy to expand the gas, cools from recombination.

We also had

$$ -\frac{1}{T} \frac{dT}{dr} > \frac{\rho g}{\rho} (\frac{\gamma - 1}{\gamma}) $$

For convection

Convection can be important when the gravitational acceleration is low;

⇒ central core of stars

Large amounts of energy flow through a region where the acceleration due to gravity is low; the pressure falls off gradually, and a rising pocket of gas is more likely to remain buoyant because it need not expand much!
Consider the $T-P$ relation for a stellar atmosphere with constant flux and opacity dominated by $e^-$ scattering.

\[ F = -\frac{1}{3} \frac{ac}{kP} \frac{d}{dr} (aT^4) \]

If the flux is constant

\[ -\frac{1}{3} \frac{ac}{kP} \frac{d}{dr} (aT^4) = \text{const.} \]

Need to write this in terms of $dP$ to get a $P$ differential.

From H.S.E.

\[ \frac{1}{P} \frac{dP}{dr} = -g \]

we have

\[ -\frac{1}{3} \frac{ac}{k} (-\frac{g}{dP}) d(aT^4) = \text{const.} \]

\[ \frac{1}{3} \frac{acg}{k} \frac{d(aT^4)}{dP} = \text{const.} \]

(Assume $g = \text{const.}$ in a atmosphere)

So

\[ \frac{d(aT^4)}{dP} = \text{const.} \]

\[ S \frac{dT^4}{dP} = S \frac{dP}{dP} \]

\[ T^4 \propto P \]

\[ T \propto P^{\frac{1}{4}} \]

\[ T(R) = 0 \text{ and } P(R) = 0 \]

\[ dT \propto P^{-\frac{3}{4}} dP \]

\[ \frac{dT}{T} \times \frac{1}{4} \frac{dP}{P} \]

Compare to convection criterion. Star convects if

\[ \left| \frac{d\ln T}{d\ln P} \right| > \frac{2-1}{3} = \frac{1}{5} \approx 0.2 \]

\[ 0.2 < 0.4 \quad \text{So star does not convect.} \]