Last time we found the probability of penetration is
the Coulomb barrier

\[ P = e^{-\sqrt{E_c}/E} \]

- Please note that the energy here is the energy in the center of mass frame.
- The nuclei need to borrow this energy \( E \) from the thermal environment. The probability of a successful energy loan is higher if the gas temperature \( kT \) is close to \( E \).

\[ e^{-E/kT} \]

So Fusion Probability = \( P(\text{energy loan}) \cdot P(\text{penetrating Coulomb Barrier}) \)

\[ e^{-E/kT} \cdot e^{-\sqrt{E_c}/E} \]

Fusion Probability

Has a maximum at

\[ \frac{dP}{dE} = \exp\left[ -\frac{E}{kT} - \sqrt{\frac{E_c}{E}} \right] \left( -\frac{1}{kT} + \frac{E_c^{1/2}}{2E^{3/2}} \right) = 0 \]

Zero if \( \frac{1}{kT} = \frac{E_c^{1/2}}{2E^{3/2}} \)

\[ E = \left( \frac{E_c (kT)^2}{4} \right)^{1/3} \]

\( E \approx E_c \) the energy where fusion is most likely.
• Reaction Cross Sections

Suppose we throw baseballs at a window of area $1 \text{ m}^2$. The cross section of the target $E$ is $1 \text{ m}^2$. But suppose we ask a slightly different question—namely what is the cross section for a collision that breaks the window. Now we know from experience that the window breaks once for every 5 times that we hit it with the ball. So the cross section is

$$\sigma = (1 \text{ m}^2) \cdot \text{(Probability of breaking)}$$

$$= 0.2 \text{ m}^2$$

By analogy, we expect the cross-section for a nuclear reaction to scale as an area representing the size of the nucleus times the probability of nuclear penetration.

We estimated the size of a nucleus from the energy of the pions mediating the strong force.

$$\Delta X = \frac{\hbar}{\sigma E} = \frac{\hbar}{\frac{135 \text{ MeV}}{1.5 \times 10^{-13} \text{ cm}}} = 1.5 \times 10^{-13} \text{ cm}.$$ 

Now nuclei grow at roughly constant density, so

$$r_{\text{nucl}} = 1.5 \times 10^{-13} \text{ A}^{1/3}$$

... but the cross section is actually much larger than $r_{\text{nucl}}^2$. 

The particle's $\lambda$ at low energies is much larger than $\lambda_{R\mu\nu}c$, and $\alpha M$ is quite important.

At a keV, the proton energy is

$$E = \frac{p^2}{2m_p},$$

and the proton wavelength is

$$\lambda = \frac{h}{p},$$

or

$$E = \frac{h^2}{2m_p \lambda^2}.$$

$$\lambda = \frac{6.63 \times 10^{-27} \text{ m·s}}{\sqrt{\frac{1.67 \times 10^{-24} \text{ g}}{1 \text{ keV}} \left(\frac{1 \text{ keV}}{10^3 \text{ eV}}\right) \left(\frac{16 \times 10^{-12} \text{ eV}}{\text{ eV}}\right)}} = 9 \times 10^{-11} \text{ cm} \left(\frac{1 \text{ keV}}{E}\right)^{1/2}$$

$\Rightarrow$ so the cross section is large at low energy.

Now in practice it is very difficult to measure fusion cross-sections at energies well below the Coulomb barrier, so we measure them at higher energies and use the $1/E$ factor to extrapolate to energies relevant to astrophysics.

$$\sigma(E) = \frac{S(E)}{E} e^{-\frac{\alpha M E}{E}}$$

The factor $S(E)$ contains the details of nuclear physics such as resonance energies, but it is generally a slowly varying function of $E$. 


So for strong reactions where \( E \gg E_0 \),

\[ \sigma(E) = \frac{s(E)}{E} \]

and \( s(E) = E \sigma(E) \)

\[ \approx (1 \text{ keV}) \left( \frac{\pi (9 	imes 10^{-15})^2}{9} \right) \approx 16 \times 10^{-22} \]

\[ \approx 2000 \text{ barns/keV} \]

\[ \text{\textbullet \ Reaction Rates} \]

Now suppose \( \sigma \) is the reaction cross section for nuclei of type A to fuse with nuclei of type B.

The average distance an A nucleus travels before fusing with B is

\[ l = \frac{1}{\sigma n_B} \text{ or } n_B \text{ RX per A nucleus is } \frac{1}{\sigma n_B} \]

So the reaction rate per unit volume becomes

\[ R = n_A n_B \langle \sigma v \rangle \]

where \( v \) is the relative speed between nuclei.

Let \( P(v_r) \, dv_r \) denote the probability that the relative speed is between \( v_r \) and \( v_r + dv_r \).

So the average value of the fusion cross section and relative nuclei speed is

\[ \langle \sigma v \rangle = \int_0^\infty \sigma v \, P(v) \, dv \]
\[ \langle n | v \rangle = \left( \frac{8}{3 \pi m v} \right)^{1/2} \left( \frac{L}{k T} \right)^{3/2} \int_0^\infty S(E) e^{-E/kT} e^{-\sqrt{2 E\epsilon_0}/kT} dE \]

Expanded the important part of the integrand about the energy where the reaction is most probable.

The integrand is largest when the exponents have their minima.

Let \( f(E) = e^{-E/kT - \sqrt{2 E\epsilon_0}/kT} \)

\[
\frac{df}{dE} = e^{-E/kT - \sqrt{2 E\epsilon_0}/kT} \left[ -1 + \frac{\sqrt{2 E\epsilon_0}}{2 E^{3/2}} \right] \Rightarrow 0
\]

\[
2 E_0^{3/2} = \sqrt{2 E\epsilon_0} kT
\]

\[
E_0^{3/2} = \sqrt{2 E\epsilon_0} kT
\]

\[
E_0 = \frac{E_0^{1/3} (kT)^{2/3}}{2^{2/3}}
\]

Let \( G(E) \) be the argument of the exponential.

\[
G(E) = -\frac{E}{kT} - \left( \frac{E_0}{E} \right)^{1/2}
\]

\[
G(E_0) = -\frac{E_0}{kT} - \left( \frac{E_0}{E_0} \right)^{1/2} = -\frac{E_0^{1/3} (kT)^{2/3}}{4^{1/3} kT} - \frac{\left( \frac{E_0}{E_0} \right)^{4/3} (kT)^{2/3}}{4^{1/3} kT} = -3 \left( \frac{E_0}{4 kT} \right)^{1/3}
\]

\[
\frac{1}{2} G''(E) (E - E_0)^2 = -3 \frac{E_0^{1/2}}{2 (E_0 (kT)^{2/3})^{2/3}} (E - E_0)^2 = \left( \frac{E - E_0}{\Delta} \right)^2
\]

\( \Delta = \)
The probability distribution for relative velocities ends up just being the MB distribution

$$P(v_r) dv_r = \left( \frac{m_r}{2\pi k T} \right)^{3/2} \exp \left( -\frac{m_r v_r^2}{2 k T} \right) dv_r.$$ 

so

$$\langle v_r \rangle = \left( \frac{m_r}{2\pi k T} \right)^{3/2} \int_0^\infty \sigma(E) v r e^{-m_r v_r^2 / 2kT} 4\pi v^2 dv$$

set \( E = \frac{1}{2} m_r v_r^2 \)
\( dE = m_r v_r dv_r \)

$$\langle v_r \rangle = 4\pi \left( \frac{m_r}{2\pi k T} \right)^{3/2} \int_0^\infty \sigma(E) \frac{2E}{m_r} e^{-E/kT} \frac{dE}{m_r v_r}$$

$$= 4\pi \left( \frac{m_r}{2\pi k T} \right)^{3/2} \int_0^\infty \sigma(E) \frac{2E}{m_r} \frac{1}{m_r} e^{-E/kT} dE$$

$$= 8\pi \left( \frac{1}{2\pi k T} \right)^{3/2} \int_0^\infty \frac{\sigma(E)}{E} e^{-N E / E} \frac{E}{E} e^{-E/kT} dE$$

$$= \frac{8\pi}{2\pi \sqrt{\pi}} \frac{1}{N m_r} \left( \frac{1}{k T} \right)^{3/2} \int_0^\infty \frac{\sigma(E)}{E} \exp \left[ -\frac{E}{kT} - \sqrt{\frac{E}{E}} \right] dE$$

Now the stuff in the exponent is the most important.
\( f(E) = \frac{E}{kT} + \sqrt{\frac{E_G}{E}} \)

\[ \frac{df}{dE} = 0 \Rightarrow E_0 = \left[ \frac{E_G}{\frac{4}{kT}} \right]^{\frac{1}{3}} \]

Now let's estimate the width of the energy window \( \Delta E \) where fusion takes place.

Taylor expand \( f(E) \) about \( E_0 \)

\[ \frac{df}{dE} = \frac{1}{kT} - \frac{1}{2} \frac{\sqrt{E_G}}{E^{3/2}} \]

\[ \frac{d^2f}{dE^2} = \frac{3}{4} \frac{\sqrt{E_G}}{E^{5/2}} \]

\[ f(E_0) = \frac{E_0}{kT} + \sqrt{\frac{E_G}{E_0}} \]

\[ = \left[ \frac{E_G}{\frac{4}{kT}} (kT)^2 \right]^{\frac{1}{3}} + \frac{E_G^{1/2}}{E_0^{1/6}} \frac{4^{1/6}}{(kT)^{1/3}} \]

\[ = 3 \left[ \frac{E_G}{(4kT)} \right]^{\frac{1}{3}} + \frac{3}{8} \frac{E_G^{1/2}}{E_0^{5/12}} (E-E_0)^2 \]

\[ f(E) = f(E_0) + \frac{df}{dE} \bigg|_{E_0} (E-E_0) + \frac{d^2f}{dE^2} \bigg|_{E_0} (E-E_0)^2 \]

\[ = 3 \left[ \frac{E_G}{(4kT)} \right]^{\frac{1}{3}} + 0 + \frac{3}{8} \frac{E_G^{1/2}}{E_0^{5/12}} (E-E_0)^2 \]
Nuclear Reaction Rates

\[
E = \frac{E_G - \sqrt{E_G}}{2} \cdot \frac{3/2}{E_G} \cdot \left(\frac{E_G}{kT}\right)^{3/2} \cdot \left(1 - \left(\frac{E - E_0}{E_0/2}\right)^2\right)^1 \cdot \exp \left(-\frac{E - E_0}{E_0/2}\right) \cdot \frac{\Delta^2}{4.8^2} \cdot \left[\frac{E_G}{kT}\right]^{5/6}
\]

where \( \frac{1}{\Delta^2} = \frac{3}{8} \cdot \frac{E_G^{1/2}}{E_G^{5/2}} \) and \( E_0 = \frac{E_G}{4} (kT)^2 \)

so the width of the fusion window is

\[
\frac{1}{\Delta^2} = \frac{3}{4.8^2} \cdot \frac{E_G^{5/2}}{(kT)^2} \cdot \left[\frac{E_G}{kT}\right]^{5/6}
\]

\[
\frac{1}{\Delta^2} = \frac{3}{2^{5/3}} \cdot \frac{1}{E_G^{1/3}} \cdot \left(\frac{E_G}{kT}\right)^{5/3}
\]

\[
\Delta = \frac{2^{5/3}}{3^{1/2}} \cdot \frac{E_G^{1/6}}{(kT)^{5/6}} \quad \text{or} \quad \Delta = \frac{4}{N^{1/3}} \cdot \left(\frac{E_0}{E_G}\right)^{1/2}
\]

This shows that fusion occurs at energies determined by the temperature of the gas and the Gamow energy of the Coulomb barrier.

\[
\frac{\Delta}{E_0} = \frac{4}{N^{1/3}} \cdot \sqrt{\frac{kT}{E_0}}
\]