The simplest version of the p-p cycle is:

\[ p + p \rightarrow d + e^- + \nu_e \] \hspace{2cm} \text{Weak Interaction}

\[ p + d \rightarrow ^3\text{He} + \gamma \] \hspace{2cm} \text{Electromagnetic interaction}

\[ ^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + 2p \] \hspace{2cm} \text{Strong nuclear interaction}

Proceeds via the rate-limiting step of the weak interaction.

Now if we apply this rate to the Sun

\[ L_0 = E_{pp} M_0 \]

\[ E_{pp} = 3 \times 10^{30} \text{ ergs} \]

\[ \frac{5}{2000 \text{ keV-barn}} \frac{0}{T_7^{1/3}} \frac{0}{e^{-15.69/T_7^{1/3}}} \]

\[ \rho = \frac{M_0}{4 \pi R_0^3} \text{ Msun} \] \hspace{2cm} \text{Not average density is higher, but anyway... Just use } M_1 R_0^3

\[ S_{\nu,e} = 2 \times 10^{-25} \]

\[ L_0 = 4 \times 10^{38} \left( \frac{M}{M_0} \right)^2 \left( \frac{R}{R_0} \right)^3 \frac{1}{T_7^{1/3}} e^{-15.69/T_7^{1/3}} \]

\[ \text{Found } T_7 \sim 1.9 \times 10^7 \]

This is somewhat higher than the real value (1.5 \times 10^7 K).

Indeed, the \( T^4 \) dependence of the reaction rate for the p-p chain fails to explain the steep (\( L < 10^{35} \)) luminosity increase in more massive stars. A more temperature-dependent mechanism is needed to explain the luminosities, i.e., such as a reaction network governed by a high Coulomb barrier. But heavy elements are only present in low abundance.
It was Hans Bethe who raised the question whether there could be a catalytic cycle involving the heavier elements.

So let's see how high the atomic number would need to be.

\[ \ln x = \frac{s}{s_s} e^{-3\left(\frac{E_g}{14\ h}\right)^{1/3}} \]

\[ E_g = \left(\frac{\pi}{2}\right)^2 \left(\frac{2m_e c^2}{E_g}\right) \]

**PP**

\[ 10^{-25} \ e^{-3\left(\frac{E_g,pp}{14\ h}\right)^{1/3}} = e^{\frac{25}{3}\ln 10 + 3\left(\frac{E_g,pp}{14\ h}\right)^{1/3}} \]

\[ 25\ln 10 + 3\left(\frac{E_g,pp}{14\ h}\right)^{1/3} = 3\left(\frac{E_g,pp}{14\ h}\right)^{1/3} \]

\[ 25\ln 10 \left(\frac{1}{4\ h}\right)^{1/3} + E_g,pp^{1/3} = E_g,pp^{1/3} \]

\[ E_g,pp = \left[19.2 \left(\frac{14\ h}{E_g}\right)^{1/3} + E_g,pp^{1/3}\right]^3 \]

\[ = \left[19.2 \left(\frac{138 \times 10^6 \text{eV}}{1.6 \times 10^{-13} \text{eV}}\right)^{1/3} + 494^{1/3}\right]^3 \]

\[ = \left[29^{1/3} + 7.7\right]^3 \times 10^6 \text{eV} \]

So we want \( E_g \approx 50 \text{ MeV} \)

\[ 50 \text{ MeV} = \left(\frac{\pi}{2}\right)^2 \left(\frac{2m_e c^2}{E_g}\right) \]

\[ z = \frac{\sqrt{50 \times 10^6 \text{eV}}}{2 \sqrt{(439 \times 10^6 \text{eV})}} = 137 \approx 7 \]
This result tells us that the competition between weak and strong interactions gives us a big factor in the exponent as we can afford the extra Coulomb tunneling.

\[ E_c \approx \left( \frac{29 T_7^{1/3} + 7.9}{\alpha^2} \right)^3 \]

\[ 6 \alpha^2 \frac{2m_p c^2}{11} \]

\[ \frac{2}{n^2} \frac{(29)^3}{(7.9)^2 (939 \text{ MeV})^{10}} \left[ \frac{1}{T_7^{1/3}} + 0.27 \right]^3 \]

\[ = 24.7 \left( T_7^{1/3} + 0.27 \right)^3 \]

\[ z_c = 5 \left( T_7^{1/3} + 0.27 \right)^{3/2} \]

<table>
<thead>
<tr>
<th>( T_7 )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.15</td>
</tr>
<tr>
<td>0.5</td>
<td>5.5</td>
</tr>
</tbody>
</table>

But the lower main sequence is much colder than this as the L\(_{\text{PP}}\) are so much lower.

\[ L \uparrow \]

\[ L_{\text{CNO}} \quad L_{\text{PP}} \]

\[ T_c \]
The other way to see this is when things are very $T$ sensitive since

$$L \propto e^{-3(E_e/4kT)^{1/3}}$$

$$dL \propto e^{-3(E_e/4kT)^{1/3}} \left(\frac{E_e}{4k}\right)^{1/3} \frac{1}{T^{4/3}} dT$$

Divide by $L$

$$\frac{dL}{L} \propto \left(\frac{E_e}{4kT}\right)^{1/3} \frac{dT}{T}$$

$$\frac{d\ln L}{d\ln T} = \left(\frac{E_e}{4kT}\right)^{1/3} \sim \left(\frac{494 \text{ keV}}{3.95}\right)^{1/3} \approx 5.23$$

But for $\mathbf{p + ^14N}$

$$E_g = \left[\pi a (1)(c)\right]^2 \frac{2 \left(1)(c)\right)}{1+14} m_p c^2$$

$$= 0.98 m_p c^2$$

$$E_g = 45 \text{ MeV}$$

So

$$\frac{d\ln L}{d\ln T} \propto \left(\frac{45 \times 10^3 \text{ keV}}{3.45 \text{ keV}}\right)^{1/3} = 23.5$$

which is much steeper.

So the $\mathbf{p + ^{14}N}$ will win at higher $T$.

What about $\mathbf{p + ^{12}C}$

$$E_g = \left[\pi a (1)(c)\right]^2 \frac{2 \left(1)(c)\right)}{1+12} (939 \text{ MeV})$$

$$= 32.8 \text{ MeV}$$

So

$$\frac{d\ln L}{d\ln T} \propto \left(\frac{32.8 \times 10^3 \text{ keV}}{3.45}\right)^{1/3} = 21.2$$
The CN Cycle

Hans Bethe and Weizsäcker both showed in 1938 that there is a CN cycle that works this way using C and N in the star as catalysts.

The cycle has no beginning or end, but it is often thought of as starting with p fusing with $^{12}\text{C}$.

\[
P + \frac{12}{6}\text{C} \rightarrow \frac{13}{7}\text{N} + \gamma \quad s_0 = 1.5 \text{ keV-Barns}
\]

\[
\left( \frac{13}{7}\text{N} \rightarrow \frac{13}{6}\text{C} + e^+ + \nu_e \quad (t = 670 \text{ sec.}) \right)
\]

\[
P + \frac{13}{7}\text{N} \rightarrow \frac{14}{8}\text{N} + \gamma \quad s_0 = 5.5 \text{ keV-Barns}
\]

\[
P + \frac{14}{8}\text{N} \rightarrow \frac{15}{10}\text{O} + \gamma \quad s_0 = 3.3 \text{ keV-Barns}
\]

\[
\left( \frac{15}{10}\text{O} \rightarrow \frac{15}{7}\text{N} + e^+ + \nu_e \quad (t = 178 \text{ sec.)} \right)
\]

\[
\left. \frac{1}{2}p + \frac{15}{7}\text{N} \rightarrow \frac{12}{6}\text{C} + \frac{4}{2}\text{He} \quad s_0 = 70 \text{ MeV-Barns} \right.
\]

\[\text{Net Effect: } 4p \rightarrow \frac{4}{2}\text{He} + 2e^+ + 2\nu_e + 23.8 \text{ MeV}\]

The slowest reaction is $p + ^{14}\text{N}$. (This comes from consideration of Coulomb barriers and the $s$ factors.)

The penalty of capturing onto high Z elements is made up by the strong reactions.

Still need weak reactions, but get around those by creating unstable elements. These yield long-lived species, allowing plenty of time for the reactions we want.