we have studied H burning in some detail and found the relatively slow rate of fuel consumption completely determines stellar lifetimes.

Now a star spends almost its entire lifetime on the H-burning main sequence.

We will now address the evolution of a star off the main sequence at the end of its H-burning (core) phase. The bulk of the life-critical elements C and O are produced during this brief time.

- Evolution off the Main Sequence

Once ~10% of the H has burned, the core can no longer supply luminosity. The star contracts. (The virial theorem tells us that roughly half of the energy released is turned into heat in the star.)

\[ T < 10^8 \]

\[ P \]

\[^4\text{He}\] in the next fuel that can ignite,

\[ P + ^4\text{He} \rightarrow \]

does not help as there is no stable mass 5 isotope.
Production of $^8$Be

What about

$^4\text{He} + ^4\text{He} \rightarrow ^8\text{Be} + \gamma$ ?

Decays a $\tau$ for $2.6 \times 10^{-16}$ s $\rightarrow ^4\text{He} + ^4\text{He}$

Well, $^8$Be is unstable, but we can show that the concentration of $^8$Be is still significant in equilibrium.

The barrier energy for fusion of two $^4\text{He}$ nuclei is

$$E_b = (\frac{n \times Z_1 Z_2}{137})^2 \times 2 \times m_c c^2$$

$$= \left(\frac{11}{137}\right)^2 \times 4.4 \times 7 \times \frac{4(4)}{8} \times \frac{1.67 \times 10^{-24}}{3 \times 10^{-13} \text{cm}} \times \frac{1}{2}$$

$$= 31.6 \text{ MeV}$$

Now we know the energy where the two are most likely to both penetrate the Coulomb barrier and borrow enough energy from the gas to fuse is

$$E_0 = \left(\frac{E_b}{4}\right)^{1/3}$$

$$= \left[\frac{31.6 \times 10^{-24} \times (1.6 \times 10^{-12} \text{cm})\times (1.38 \times 10^{-16})}{4}\right]^{1/3}$$

$$= 8.3 \times 10^{-4} \text{ MeV} \left[\frac{1}{10^8}\right]^{2/3}$$

The $^8$Be ground state is $71.8 \text{ MeV}$ above $^4\text{He} + ^4\text{He}$, so once $T > 10^8 \text{K}$, the fusion to form $^8$Be begins to take place.
Now $^8\text{Be}$ decays with $2.6 \times 10^{-16}$ s, but the formation of $^8\text{Be}$ is fast enough relative to the decay to assume statistical equilibrium.

$$4\text{He} + 4\text{He} \rightleftharpoons ^8\text{Be}$$

$$\mu_4 + \mu_4 = \mu_8$$

$$\mu_4 = \frac{m_4 c^2 + kT \ln \left( \frac{n_4}{g_n a_4} \right)}{\hbar}$$

$$\mu_8 = \frac{m_8 c^2 + kT \ln \left( \frac{n_8}{g_n a_8} \right)}{\hbar}$$

$$2 \frac{m_4 c^2 - m_8 c^2}{\hbar T} = \ln \frac{n_8}{g_n a_8} - 2 \ln \left( \frac{n_4}{g_n a_4} \right)$$

$$\frac{n_8}{n_4} = \frac{(g_n a_4)^2}{g_n a_8} = e^{\frac{(2m_4 c^2 - m_8 c^2)}{kT}}$$

Recall $$n_8 = \left( \frac{2\pi m_8 kT}{\hbar^2} \right)^{3/2}$$

$$3 = 1 \quad \text{as} \quad J^m = 0 \quad \text{(spin=0)} \quad \text{for both}$$

$$\frac{\left( \frac{m_4^{3/2}}{m_8^{3/2}} \right)^2}{\left( \frac{m_4}{m_8} \right)^{3/2}} = e^{\frac{(2m_4 c^2 - m_8 c^2)}{kT}}$$

$$\frac{n_8}{n_4} = 2^{3/2} \left( \frac{\hbar^2}{2\pi m_4 kT} \right)^{3/2} e^{-(m_8 - 2m_4) c^2 / kT}$$
Now if \( n = 10^9 \text{ g/cm}^3 \) of pure \( ^4\text{He} \), then

\[
n_4 = \frac{10^9 \text{ g/cm}^3}{4 \times (1.67 \times 10^{-24} \text{ g})} = 1.5 \times 10^{27} \text{ cm}^{-3}
\]

and

\[
\frac{n_8}{n_4} = \left(1.5 \times 10^{27} \text{ cm}^{-3}\right)^{3/2} \left(\frac{(6.63 \times 10^{-27} \text{ cm}^2 \text{ s})^2}{2\pi^2 \times 4 \times (1.67 \times 10^{-24} \text{ g}) \times (1.38 \times 10^{-16} \text{ cm}) \times \frac{1}{10^8 \text{ K}}}
\right)^{3/2} = 91.8 \text{ MeV/keV}
\]

\[
\frac{n_8}{n_4} = 2.8 \times 10^{-6} e^{-10.61T} \times T_{10}^{-3/2} \left(\frac{1}{10^4}\right)
\]

At \( T = 2 \times 10^8 \text{ K} \), we get \( \frac{n_8}{n_4} = 5 \times 10^{-9} \) of the nuclei are \(^8\text{Be} \).

- **Production of \(^{12}\text{C} \)**

Now we can do something with the \(^8\text{Be} \), namely capture an \( \alpha \).

\[
\alpha + ^8\text{Be} \rightarrow ^{12}\text{C}^* + \gamma
\]

\[
^8\text{Be} + ^4\text{He} \rightarrow ^{12}\text{C}^* \rightarrow ^{12}\text{C}^0 + \gamma
\]

Energy difference is

\[
7.644 - 7.366 = 280 \text{ keV}
\]

\( ^{12}\text{C}^0 \) excited state \( J^P = 0^+ \)
Let's find the energy where $^8\text{Be}$ is likely to capture an $\alpha$ particle.

The Gamow energy for the $\alpha + ^8\text{Be} \rightarrow ^{12}\text{C} + \gamma + X$ reaction is:

$$E_G = \left( \frac{\pi}{2} \right)^2 \frac{2m_e c^2}{r^2} \frac{1}{a^4} \left( \frac{1}{137} \right)^2 \left( \frac{4}{3} \right)^2 \left( 1.67 \times 10^{-24} \text{ g} \right) \left( 8 \times 10^{10} \text{ cm s}^{-1} \right)^2$$

$$= 168 \text{ MeV}$$

$$E_0 = \left[ \frac{E_G (kT)^2}{4} \right]^{\frac{1}{3}}$$

$$= \left[ \frac{168 \times 10^6 (1.6 \times 10^{-12}) \left( 1.38 \times 10^{-12} \right)^2 \left( 10^8 \right)^2}{4} \right]^{\frac{1}{3}} \frac{1}{1.6 \times 10^{-12}}$$

$$= 146 \text{ keV} \quad T_5^{\frac{2}{3}}$$

$$= 2.31 \text{ at } T = 2 \times 10^8 \text{K}$$

Now the existence of red giants led Hayle to believe the reaction rate must be faster than expected. He suggested the existence of an excited state in the $^{12}\text{C}$ nucleus right around $E_0$.

This $7.644 \text{MeV}$ state was found.

It turns out the $^{12}\text{C}^{*}$ state can reach equilibrium, so the Saha formula gives us the abundances:

$^4\text{He} + ^4\text{He} \rightarrow ^8\text{Be}$

$^8\text{Be} + ^4\text{He} \rightarrow ^{12}\text{C}^{*}$
which we could just write as

\[ ^4\text{He} + ^4\text{He} + ^4\text{He} \xrightarrow{\alpha} ^{12}\text{C}^* \]

\[ 3 \, m_y = m_4 \, c^* \]

\[ 3 \, m_y \, c^2 + 379.5 \times 10^6 \, \text{eV} = m_{12} \, c^2 \]

so we can find the abundance of \(^{12}\text{C}^*\).

\[ \frac{3 \, m_y \, c^2 - m_{12} \, c^2}{J_e \, T} = \ln \frac{n_{12}^*}{n_{4, 4}} \frac{n_{4, 4}^3}{n_{4, 0}^3} \]

\[ n_4 = \left( \frac{2 \pi m_e \, k \, T}{\hbar^2} \right)^{\frac{3}{2}} \]

\[ \frac{n_{12}^*}{n_4^3} = \frac{n_{12}^*}{n_{4, 4}^3} \frac{n_{4, 4}^3}{n_{4, 0}^3} \quad e^{-\left( m_{12} - 3 \, m_y \right) c^2 / k \, T} \]

\[ \frac{n_{12}^*}{n_4^3} = 3^{3/2} \left( \frac{\hbar}{2 \pi m_e \, k \, T} \right)^3 e^{-\left( m_{12} - 3 \, m_y \right) c^2 / k \, T} \]

\[ \frac{n_{12}^*}{n_4} = \left( \frac{\rho}{4 \pi m_p} \right)^2 \quad 3^{3/2} \left( \frac{\hbar}{2 \pi m_e \, k \, T} \right)^3 e^{-\left( m_{12} - 3 \, m_y \right) c^2 / k \, T} \]

\[ \frac{n_{12}^*}{n_4} = 5.2 \times 10^{-10} \left( \frac{\rho}{10^5} \right)^2 \left( \frac{10^8}{T} \right)^3 e^{-44 / T_{8}} \]

This is the steady state population of \(^{12}\text{C}^*\).

We are still not done as \(^{12}\text{C}^* \rightarrow ^{12}\text{C} (8.5\ E\) with a mean time of \( T = \frac{T_{\text{rad}}}{1.79 \times 10^{-3}} \) S.
So $^{12}\text{C}^* \rightarrow ^{12}\text{C}$ (g.s.) is much slower than the decay of particles to $^4\text{He} + ^{3}\text{Be}$.

The rate of production of $^{12}\text{C}$ in ground state is thus

$$\frac{dN_{12}}{dt} = \frac{N_{12}^*}{\tau} = \frac{N_{12}^* \tau}{\frac{1}{1}}$$

which releases $7.65\text{ MeV}$.

So we get a total energy release

$$\varepsilon = \frac{1}{P} \frac{N_{12}^* \varepsilon}{\frac{1}{1}}$$

$$= \frac{N_{12}^* \varepsilon}{1} \frac{1}{4\pi \nu p \frac{1}{1}}$$

$$= 5.3 \times 10^{21} \text{ ergs/sec} \left( \frac{P}{10^5} \right)^2 \left( \frac{10^8}{T} \right)^3 e^{-44/T}$$

- Note the extreme temperature sensitivity!

- Other competing reactions

  $d + ^{12}\text{C} \rightarrow ^{16}\text{O} + \chi$

  But $E_G = (m_d m^2 \chi^2)^22m_w c^2$

  $= \left[ \frac{11}{137} (3)(6) \right]^2 \left( \frac{4}{4412} (1.67 \times 10^{-24}) (3 \times 10^{-8}) \right)^2$

  $= 4.26\text{ MeV}$

  So this tough
Evolution of the Main Sequence

<table>
<thead>
<tr>
<th>Reaction</th>
<th>$E_{\text{g}}$</th>
<th>$E_{\text{c}}$ ($2 \times 10^2$ K)</th>
<th>$\exp \left( -\frac{3}{2} \left( \frac{E_{\text{g}}}{T_\text{c}} \right)^{1/3} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha + ^{12}\text{C}$</td>
<td>4.24 MeV</td>
<td>31.5 keV</td>
<td>1.3 x 10^{-24}</td>
</tr>
<tr>
<td>$\alpha + ^{16}\text{O}$</td>
<td>3.04 MeV</td>
<td>39.0 keV</td>
<td>5.5 x 10^{-28}</td>
</tr>
<tr>
<td>$\alpha + ^{20}\text{Ne}$</td>
<td>1310 MeV</td>
<td>460 keV</td>
<td>1.5 x 10^{-44}</td>
</tr>
</tbody>
</table>

So the additional reactions are rather unlikely except for the first one. The resulting mixture of $^{12}\text{C} + ^{16}\text{O}$ is sensitive to $T_\text{c}/P$.

Think of $3\alpha$ as a dynamic equilibrium.

A reaction diagram illustrating $\alpha \leftrightarrow ^8\text{Be} \leftrightarrow ^{12}\text{C}$ with a slow leak from $^{12}\text{C}$ decaying to ground state.