Basic physics determines relevant properties of stars.

We can ask what type of star is likely to behave like a highly relativistic fluid?

What type of particle, in the sun for example, is clearly relativistic?

- yes - but don't interact

photons!

General result for relativistic gas $P = \frac{1}{3} (\text{internal energy})$

$$P = \frac{1}{3} \frac{U}{T}$$

$$U = \frac{a}{T^4}$$

$$P = \frac{1}{3} a T^4$$

Consider the interior of the sun

$T \approx 6 \times 10^6 \text{K}$

$$\Rightarrow P_{\text{rad}} = \frac{1}{3} \left( \frac{2.56 \times 10^{15}}{15 \times 3 \times 10^8} \right) \left( 6 \times 10^6 \text{K} \right)^4$$

$$P_{\text{rad}} = 3.27 \times 10^{12} \text{ dyne/cm}^2 \sim 10^{-3} \text{ P}_c$$

which we see is quite small compared to the pressure needed for support against gravity

$$\langle P \rangle = -\frac{1}{3} \frac{E_{\text{GB}}}{V} \sim 8.75 \times 10^{14} \text{ dyne/cm}^2$$

+ So when is radiation pressure important in a star?

$$P_{\text{rad}} = \frac{1}{3} a T^4 \quad \text{vs.} \quad P_{\text{gas}} = \rho \frac{dE}{dV}$$

H.S.E. relates internal energy to binding energy

$$dE = \frac{GMm_p}{R}$$

per particle
\[
\frac{P_{\text{rad}}}{P_{\text{gas}}} = \frac{1}{8} a \frac{T^4}{\rho} \frac{m_p}{\rho c T} = \frac{a m_p}{T/3} \left( \frac{GM m_p}{R J} \right)^3 \approx \frac{a m_p}{T} \frac{R^3}{M} \left( \frac{GM m_p}{R J} \right)^3 \approx \frac{a m_p}{T} \frac{G^3 M^2}{\hbar^4}
\]

so \[\frac{P_{\text{rad}}}{P_{\text{gas}}} \propto M^2\]

and we can estimate the characteristic mass where radiation pressure becomes important.

\[M_{\text{rad}}^2 \approx \frac{2}{5} \frac{G^4}{m_p} (1) \]

\[\approx \frac{G^4}{m_p} \frac{15 c^6}{8 \pi^5 \hbar^4} \]

\[\approx \frac{G^4}{m_p} \frac{c^3 h^3}{G^3 m_p^4} \]

\[(\frac{M_{\text{rad}}}{m_p})^2 \approx \frac{c^3 h^3}{G^3 m_p^6} \approx \frac{(3 \times 10^8 \text{cm/s})^3 (6.63 \times 10^{-27} \text{erg s})^3}{(6.67 \times 10^{-8} \text{cm}^3 \text{g}^{-1} \text{s}^{-2}) (1.67 \times 10^{-29})^2 \times 10^{-32} \text{g}} = 10^{-32} \text{g} \]

\[M_{\text{rad}} \approx 10^{-32} \text{g} \Rightarrow M_{\text{rad}} > 10^{32} (1.7 \times 10^{-29}) / 2 \times 10^{32} \text{M}_\odot = 10 \text{M}_\odot \]

\[\Rightarrow \text{A stars much more massive than this are highly unstable.} \]

This is the first time in this class that we have used the basic physics to define a characteristic mass. And we obtained a stellar value – how cool!

We will learn several ways in which basic physics determines fundamental stellar masses in this class.
The utility of the adiabatic index $\gamma$ as a parameterization of stability leads us to consider simple equations of states like

$$PV^\gamma = \text{const}.$$ 

or $P = \text{const}$

$$\frac{1}{V^\gamma}.$$ 

or $P = k\rho^\gamma$.

Let's build a star.

§5.2 Use simple models to insight into most basic ideas of stellar structure.

Complete analysis requires calculations of considerable complexity. Find numerical solution to a coupled set of differential equations.

+ 4 Equations of Stellar Structure

+ 4 boundary conditions $m(C) = 0$, $l(C) = 0$, $P(T, \rho)$

During most of its existence, a star is close to H.S.E.

$$\frac{dP}{dr} = -\frac{G m(r) \rho(r)}{r^2} \quad (1) \quad \text{H.S.E}$$

$$m(r) = \int_0^r 4\pi r^2 \rho(r) \, dr \quad (2) \quad \text{Mass Conservation}$$

The internal temperature gradient is just sufficient to maintain the power flux. Energy transport is by radiative diffusion:

$$\frac{dT}{dr} = -\frac{3}{4\alpha c} \frac{k(r) \rho(r)}{[T(r)]^3} \frac{L(r)}{4\pi r^2} \quad (3)$$

and

$$\frac{dL}{dr} = 4\pi r^2 \varepsilon(r) \quad (4)$$

power generated per unit volume.
Can find static structure if pressure, opacity, and power can be related to $p$ and $T$.

Now $P = P(p, T) \Rightarrow \frac{dP}{dr} = \frac{k}{r^{n+1}}$ Polytrope model of index $n$.

Later $K = K(p, T) \Rightarrow$ Thermic scattering and...

$E = E(p, T) \Rightarrow$ $H$ burning and...

Must obey $H.S.E \ (1)$ and mass conservation $E_2$:

$$\frac{dP}{dr} = -\frac{GM(r)p_c(r)}{r^2} \quad \text{and} \quad \frac{dm}{dr} = 4\pi r^2 p$$

Divide by $p$ and multiply by $r^2$

$$\frac{r^2}{p} \frac{dP}{dr} = -GM(r)$$

Take $d/dr$

$$\frac{d}{dr} \left( \frac{r^2}{p} \frac{dP}{dr} \right) = -4\pi Gp$$

Look familiar?

Recall Poisson's Eqn.

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{2\pi G}{dr} \right) = \nabla^2 \Phi$$

$$\nabla^2 \Phi = +4\pi Gp$$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{2\pi G}{dr} \right) = -g = -\frac{dr}{dr} \quad \text{we'll - not quite, so let's see}$$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dP}{dr} \right) = -4\pi Gp$$

$$\nabla^2 \Phi = -\left(-4\pi Gp\right)$$

Polytropic Model

$$\frac{1}{r^2} \frac{d}{dr} \left[ \frac{r^2}{p} \frac{dP}{dr} \left( \frac{k}{p} p^{n+1}/n \right) \right] = -4\pi Gp$$
Polytropic models — played a role in development of stellar structure theory.

Can find a unique solution by imposing boundary conditions such as $p = p_c$ and $\frac{dp}{dr} = 0$ at $r = 0$.

$$-\frac{dp}{dr} = -G \frac{m(r)p(r)}{r^2} \Rightarrow -G \frac{p_c}{r^2} \Rightarrow 0$$

$\Rightarrow G \frac{p_c^2}{2} \Rightarrow 0$ as $r \to 0$

Similarly $\frac{dp}{dr}$ is small as $r \to R$ at the surface.

$$\frac{dp}{dr} = -G \frac{m(R)p(R)}{r^2} \Rightarrow 0$$

Not a complete description of a star!

What have we neglected?

3) Nuclear power generation

Let $L(r)$ be the power generated inside a sphere of radius $r$.

Then

$$\frac{dL}{dr} = 4\pi r^2 E(r)$$

where $E(r)$ is the nuclear power density of radius $r$.

4) will show that the energy is transported by radiative diffusion.

Define opacity of the gas

$$\kappa = \kappa(p, T) \quad \text{area / unit mass}$$

as its interaction cross section with the radiation.

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa(r)p(r) L(r)}{[\kappa(r)]^3 4\pi r^2}$$

$\Rightarrow$ 4 Equations of Stellar Structure
Stellar Models

How to Build a Star

1. Guess \( P(r) \) or \( p(r) \)

2. Integrate \( \frac{dm}{dr} \) to get \( P(r) \)
   
   Integrate \( \frac{dp}{dr} \) to get \( P \)

3. Get \( T(r) \) from e.o.s. \( P = \rho T \) to estimate power flow \( L(r) \)

4. Compare power flow to that found by integrating the nuclear power density.

... and Iterate until they match.

\( \Rightarrow \) So you need a good guess for the initial \( P(r) \) profile

Know \( \frac{dp}{dr} \bigg|_{\infty} < 0 \) and \( \frac{dp}{dr} \bigg|_{r} > 0 \)

and the Pressure Gradient < 0 to counter gravity

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Clayton model — Use in HUV

\[
\frac{dP}{dr} = -\frac{4\pi}{3} \frac{Gm^2}{c} e^{-r/a^2}
\]

\( \Rightarrow \) so a related to minimum in \( dp/dr \) steepest gradient

\[
P = \frac{2\pi}{3} Gm^2 a^2 \left[ e^{-r/a^2} - e^{-2r/a^2} \right]
\]

\( \Rightarrow \) reasonable model for the sun

\( \Rightarrow \) central pressure \( p_c \) determines minimum + maximum stellar masses.