Jamming occurs when an amorphous collection of particles spontaneously develops rigidity and supports weight like a solid instead of flowing like a liquid [1]. The transition takes place without static spatial ordering, but is accompanied by long range dynamical correlations arising from the collective motion of groups of particles [2–4]. In the case of sheared granular materials, the transition from jammed to flowing phases occurs as the applied stress \( \Sigma \) is increased or the packing fraction \( \phi \) is decreased, and there is a well defined packing fraction \( \phi_c \) at which the system jams in the limit of zero stress [5, 6]. Diverging correlation lengths are observed on each side of the transition, and are related to the average size of force chain networks for \( \phi < \phi_c \) [2, 7] and the average size of isostatic clusters for \( \phi > \phi_c \) [3, 8].

In the jammed state many macroscopic observables exhibit power law scalings in \((\phi - \phi_c)\) [5, 6] and the flowing state rheology changes dramatically as the packing fraction is increased above \( \phi_c \). Although the jamming transition controls both dynamic and static properties near \( \phi_c \), theories tend to focus mainly on the latter [8, 9]. In this Letter we theoretically explore the jamming transition by first considering flows with non-zero shear rate \( \dot{\gamma} \) and then taking the limit of \( \dot{\gamma} \to 0 \) to access the static case. This procedure leads to quantitative predictions for the flowing rheology that match observations in Ref. [10].

It also predicts the anomalous static scaling of the shear modulus measured in Ref. [5, 6] and a new scaling relation for the static yield strain, \( \gamma^* \propto (\phi - \phi_c)^{1/2} \), which provides a testable prediction of the theory.

Granular materials behave as peculiar liquids. This can be clearly demonstrated by measuring the stress tensor \( \Sigma \). Depending on various parameters, the system exhibits either Bagnold scaling \( \Sigma \propto \gamma^2 \), elastic-inertial scaling \( \Sigma \propto \gamma^0 \), or quasi-static scaling \( \Sigma \propto \gamma^1 \) (i.e. constant) [10]. The rheology of the flow is dependent on \( \phi \), with Bagnold scaling for \( \phi < \phi_c \), quasi-static scaling for \( \phi > \phi_c \), and elastic-inertial scaling in between. This is in marked contrast to Newtonian fluids where \( \Sigma \propto \gamma^0 \) and only the proportionality constant depends on \( \phi \). Although the phase diagram of granular shear flow has been extensively studied in simulations [10–12] and experiments [13], the origins of the rheological crossovers remain to be explained. Arriving at a solution to this problem requires a well developed theory for the stress tensor in dense granular flows.

The stress tensor ultimately depends on the amorphous microscopic arrangement of grains. Forces are transmitted via contacts between grains and a perturbation on one grain can have long range effects. Indeed, both experiments and simulations indicate that contact forces are correlated [2, 14] and tend to form quasi one dimensional filaments, or force chains, that permeate the material [15]. Since contact between grains is the only form of force transfer, \( \Sigma \) is fully determined by properties of the networks [16, 17]. Here we investigate the role of force chain networks in determining the value of the stress tensor by constructing a model of momentum transfer.

The Force Network Model (FNM) For Rigid Grains: The central concept of the FNM [17] is that the force \( F_{ij} \) between a pair of contacting grains \{i, j\} can be expressed as the sum of a collisional part \( F_{bc}^{ij} \) and an elastic part \( F_{el}^{ij} \). The collisional part is the force expected from collisions between pairs of grains in the absence of networks. It is proportional to the square of the relative velocity between the contacting grains i and j upon initial incidence and is predicted by kinetic theory to scale with \( \gamma^2 \) [18]. The elastic force is a consequence of the pressure induced by the network surrounding a pair of grains. Collisional forces from other contacts in the network are transferred to the pair from first nearest neighbors, second nearest neighbors, and so forth, all the way to the edge of the network, as illustrated in Fig. 1. The presence of multiple contacts leads to an elastic force \( F_{el}^{ij} \) between grains i and j, and this contribution can be expressed as a sum over all paths between the contact \{i, j\} and every other contact \{i', j'\} in the connected force network:

\[
F_{el}^{ij} = \sum_{\ell=1}^{\xi-1} \sum_{\{i', j'\}} G^{i'j'}(\ell) F_{bc}^{i'j'}(\ell).
\]  

In this equation, \( \ell \) is the path length (in grain diameters) between two contacts \{i, j\} and \{i', j'\}. The maximum path length is constrained by the average linear
size of physical networks $\xi$, which has been measured in simulations by considering correlations between grain forces [2, 17] and diverges as $\phi \to \phi_c$, as illustrated in Fig. 2. The value of $\xi$ increases with the local energy dissipation and the amount of friction. $F_{bc}(\ell)$ is the collisional force on the contact at the end of the path and the sum over $\{i', j'\}$ includes all collisional forces in the network, which may have multiple paths leading to the contact $\{i, j\}$. The function $G^{i'j'}(\ell)$ accounts for the fact that only a fraction of the collisional force is transferred at each link in the path, and no more than the collisional force can be transferred from each contact in the network. In constructing this equation we assume that grains are perfectly rigid, which ensures that the propagation of collisional forces is instantaneous.

By defining $N(\ell)$ as the total number of contacts separated by path length $\ell$, $G(\ell)$ as the average of $G^{i'j'}(\ell)$ over all paths of length $\ell$, and $F_{bc}(\ell)$ as the average collisional force for a path length $\ell$, Eqn. (1) reads

$$F_{sij}^{ij} = \sum_{\ell=1}^{\xi-1} N(\ell)G(\ell)F_{bc}(\ell).$$

This is the central equation of the FNM: its parameters are related to the network geometry and have all been measured in simulations [17].

The FNM is similar to other models that separate the stress tensor into elastic and collisional components [19], but introduces a new way to calculate elastic stresses. Elastic stresses arise from clusters (not necessarily percolating) of simultaneously contacting grains that transfer forces. These force networks are not static, but are constantly formed and destroyed by the shear flow. Since each of these rates is proportional to $\dot{\gamma}$, properties of the steady state networks are independent of $\dot{\gamma}$. Moreover, because the networks are deforming with the flow, the velocity profile is a smooth function [20] and does not depend on $\xi$. Because the average collisional force $F_{bc}$ is related to relative velocities of grains that come into contact, and the velocity profile is independent of the size of networks, $F_{bc}$ is also independent of $\xi$.

**Bagnold Scaling:** Eqn. (2) yields a constitutive relation for each component of the stress tensor that matches measurements from simulations for $\phi < \phi_c$ [17]. A major feature of the constitutive relations is that Bagnold scaling holds, i.e. $\Sigma \propto \dot{\gamma}^2$. This is because all of the network parameters in Eqn. (2) are independent of $\dot{\gamma}$ and the rheology is set by $F_{bc}$, as predicted by kinetic theory [18].

The form of the constitutive relations is especially simple for rigid granular materials slightly below $\phi_c$. At large packing fractions, each grain has many contacts and the entire collisional force from every contact is transferred to its nearest neighbors. Thus $N(1)G(1) = 1$, which implies that $N(\ell)G(\ell) = 1$ for all $\ell$ [17]. In this limit, it follows from Eqn. (2) that $\Sigma \sim F_{bc} \propto \xi \dot{\gamma}^2$. The FNM predicts that the stress tensor is proportional to the size of the force networks, independent of spatial dimension.

The Elastic-Inertial Regime: The FNM has been constructed for perfectly rigid grains by assuming that forces propagate instantaneously through networks. This assumption must be altered for realistic grains with a finite stiffness, where forces propagate at a finite speed $c$. Combined with the average lifetime of networks $\tau$, the speed of force propagation defines a length scale $c\tau$ that gives the maximum distance over which forces can correlate. If $\xi < c\tau$, then forces propagate through the entire physical network before it is destroyed. Therefore the assumptions of the rigid grain FNM hold and $\Sigma \propto \dot{\gamma}^2$, averaged over time scales larger than $\xi/c$. However, if $\xi > c\tau$, the collisional force from a single contact is not transferred to the entire network, but only over a distance $c\tau$. Since $\xi$ diverges at $\phi_c$, there is always a critical packing fraction $\phi_{hs}$ above which $\tau < \xi$, as illustrated in Fig. 2. For $\phi > \phi_{hs}$ the shear flow is in a regime where the size of the physical networks $\xi$ becomes irrelevant and must be replaced by the size of correlated networks $c\tau$.

In the case that $\xi > c\tau$, both length scales can be determined independently: $\xi$ from measuring the average physical size of networks and $c\tau$ from measuring correlations between grain forces. Since correlations do not extend over the entire physical network for $\xi > c\tau$, the values of $c\tau$ and $\xi$ are unrelated. The propagation speed $c$ increases with the stiffness of grains and depends on the network geometry, but not on the shear rate. The average network lifetime $\tau$ is proportional to $\dot{\gamma}^{-1}$, which is the rate that force networks are destroyed [10, 11]. Setting $\tau = \eta/\dot{\gamma}$, where $\eta$ is independent of shear rate but dependent on density, and substituting $\xi \to c\tau$ yields $\Sigma \propto \eta \dot{\gamma}^2$. This predicts a linear scaling with $\dot{\gamma}$, as was observed in Ref. [10] and named the elastic-inertial regime.
FIG. 2: A schematic illustration of three important length scales, plotted as a function of packing fraction $\phi$, and their relation to granular flow near $\phi_c$. $\xi$ gives the average physical size of force chain networks and diverges as $\phi \rightarrow \phi_c^-$. $l^*$ quantifies the length scale above which the system responds as an elastic solid and diverges as $\phi \rightarrow \phi_c^+$. $c\tau$ gives the maximum distance that forces, which propagate at a finite speed $c$, can become correlated. The macroscopic rheology (indicated beneath the graph) depends on the smallest length scale, and transitions occur at the intersections. $\phi_c$ is the packing fraction where the system jams in the limit of zero stress. Our predictions for $\phi_{hs}$ and $\phi_{qs}$ are given in Eqsns. (4, 5). The numerical values of the length scales depend on many parameters. Here we have used $\xi = (\phi_c - \phi)^{-3/2}$, $l^* = (\phi - \phi_c)^{-2}$ and $c\tau = 10 + 30\phi^3$ for visualization, but the crucial features are the intersections rather than the specific numerical values.

The onset of quasi-static flow: For $\phi > \phi_c$, a third length scale $l^*$ becomes relevant [8] that is related to the departure from the isostatic limit. Isostatically jammed materials are configured such that contact forces can be completely determined from the constraint that no particle moves [21]. This occurs when the coordination number $z$, equal to the average number of contacts per particle, approaches a critical value $z_c$ that depends on the spatial dimension. Contact forces are highly correlated in the isostatic state since breaking any single contact produces a cascade that alters the value of every other contact force and causes large rearrangements of grains.

Over length scales larger than $l^*$ the material behaves elastically, whereas over length scales smaller than $l^*$ the material behaves isostatically. For a system with coordination $z = z_c + \delta z$ it was predicted that $l^* \propto 1/\delta z$ [8], and this scaling has been verified in simulations [3]. The value of $l^*$ is also related to the packing fraction, since $\delta z \propto (\phi - \phi_c)^{1/2}$ [5, 6], and is plotted in Fig. 2.

The rheology of granular shear flow depends on the relative sizes of the correlated networks $c\tau$ and isostatic clusters $l^*$. In particular, when $c\tau < l^*$ forces propagate over regions where grain rearrangements are prevalent and inertial scalings are therefore important. However, when $c\tau > l^*$ forces propagate over distances that are large compared to the rearranging regions, redundant contacts stabilize the networks, and forces are no longer inertial. Thus, when $c\tau = l^*$, the rheology changes from the elastic-inertial behavior of $\Sigma \propto c\tau\dot{\gamma}$ for $c\tau \leq l^*$ to a quasi-static scaling of $\Sigma \propto c^2\eta^2/l^*$ for $c\tau \geq l^*$.

**FNM predictions at non-zero shear rate:** In the preceding sections a prediction for the stress tensor has been obtained using the FNM. The FNM was first derived in the context of perfectly rigid grains, and the effects of grain stiffness and system elasticity were incorporated by considering crossovers in characteristic length scales. The FNM prediction for the stress tensor is given by

$$\Sigma \propto \begin{cases} \xi^2, & \text{for } \xi \leq c\tau; \\ c\eta \dot{\gamma}, & \text{for } \xi \geq c\tau \text{ and } l^* \geq c\tau; \\ c^2\eta^2/l^*, & \text{for } \xi \geq c\tau \text{ and } l^* \leq c\tau. \end{cases} \quad (3)$$

The variables $\xi$, $c$, $\eta$, and $l^*$ depend on the viscoelastic grain properties and packing fraction, but not on $\dot{\gamma}$. A schematic of the length scales and flow regimes is shown in Fig. 2.

The transitions between flow regimes can also be expressed in terms of the critical packing fractions $\phi_{hs}$ and $\phi_{qs}$, as defined in Fig. 2. We focus here on the limit of stiff grains with $c\tau \gg 1$. According to Eqn. (3), the transition from elastic-inertial scaling to quasi-static scaling occurs when $l^* = c\tau$. Given that $\tau = \eta/\dot{\gamma}$ and $l^* \propto (\phi - \phi_c)^{-1/2}$,

$$\phi_{qs} - \phi_c \propto \left(\frac{\dot{\gamma}}{c\eta}\right)^{1/\psi}. \quad (5)$$

Equns. (4,5) fully determine the crossovers in rheology and are used to construct the flow-map in Fig. 3.

For constant $\dot{\gamma} < \dot{\gamma}_c$, corresponding to a horizontal slice through Fig. 3, the system exhibits Bagnold scaling for small $\dot{\gamma}$ and elastic-inertial scaling for large $\dot{\gamma}$. This unexpected behavior where Bagnold’s scaling, normally associated with “rapid” flows, actually occurs for small $\dot{\gamma}$ in flows with constant packing has been observed previously [10]. For constant $\dot{\gamma} > \dot{\gamma}_c$, the system exhibits quasi-static scaling for large $\dot{\gamma}$ and elastic-inertial scaling for small $\dot{\gamma}$. The emergence of quasi-static flow as the shear rate is reduced in dense materials is a feature that has been observed in experiments [13] and simulations [22]. Finally, for a flow with constant $\dot{\gamma} > 0$, the system passes through all three rheologies as the packing fraction is increased.

**FNM predictions at zero shear rate:** In addition to predicting the rheology of granular materials with $\dot{\gamma} > 0$, the
from Eqns. (4,5), using $\psi$ case [23], Eqn. (3) predicts that the yield pressure and the yield shear stress $s$ in the limit of $\dot{\gamma} \rightarrow 0$, forces propagate elastically and $c \propto \sqrt{G}$ [24]. Combined with $l^* \propto \delta z^{-1}$, Eqn. (3) predicts that

$$G \propto p/\delta z,$$

and

$$\gamma^* \propto \delta z.$$ (6)

The first scaling relates the shear modulus to the pressure and excess coordination. It matches measurements from simulation for all dimensions and interaction potentials [5]. The second relation predicts the shear strain at which the system yields. Although this has not been measured, the FNM predicts that yielding is a purely geometric phenomena, unaffected by interaction potential, and gives a testable expression for its scaling behavior.

**Conclusions:** We have taken a dynamic approach to the jamming transition using the FNM to predict the stress tensor in the flowing regime and then extrapolating our results to the jammed state. While this procedure was aimed at understanding athermal granular materials, a similar technique should apply in amorphous systems such as glasses that jam as the temperature is reduced. Since the isostatic length scale $l^*$ exists for all jammed amorphous materials, the challenge is to identify relevant correlation lengths in the flowing regime and relate them to properties of interest. Jamming may then be universally understood as the crossover of correlations from flowing length scales to the isostatic length scale.

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[24] Here we use the shear modulus because the bulk shear deformation occurs at constant volume.