Why the Higgs mechanism is NOT spontaneous symmetry breaking

The "Xiao-Gang Wen argument" for why gauge symmetries cannot be spontaneously broken goes like this: gauge symmetries are not actual symmetries, they are just a reflection of a redundancy in our description the system; two states related by a gauge transformation are actually the same physical state. Thus, a gauge symmetry is physically a "do-nothing transformation" and thus it does not make sense for it to be spontaneously broken.

This argument does seem like a bit of a cop-out, though – I could just declare any symmetry to be a "do-nothing transformation" by fiat if I wanted to. A more satisfying explanation is: *even if we interpret gauge symmetries as real symmetries, they can never be spontaneously broken.* This result is known as Elitzur's theorem, and it's quite easy to understand why it should be true. Let's focus on classical thermal systems – quantum systems at zero temperature map onto classical thermal systems in one higher space dimension so the argument should carry over.

First recall the hand-waving argument for why spontaneous symmetry breaking can take place in, say, the 2-D Ising model at finite temperature. The 2-D Ising model has two symmetry-breaking ground states: all $\uparrow$ and all $\downarrow$. But, if I want to get between them by local thermal fluctuations then I have to create a domain and grow it until it encompasses the whole system, which implies an extensive energy penalty due to the energy cost of the domain wall. Thus, at low temperatures transitions between the two ground states are exponentially suppressed in the system size and so the system gets stuck in either all $\uparrow$ or all $\downarrow$, so the symmetry is spontaneously broken. (The same argument shows why the *1-D* Ising model cannot have spontaneous symmetry breaking at finite temperature, because there is no extensive energy penalty to get from all $\uparrow$ to all $\downarrow$.)

On the other hand, since a gauge symmetry is a *local* symmetry, this argument breaks down. Any two symmetry-breaking ground states are related by a sequence of local gauge transformations, which (since they commute with the Hamiltonian) have exactly zero energy penalty. Thus, there is no energy barrier between different ground states, and the system will explore the entire space of ground states – so no symmetry-breaking. We expressed everything here in terms of classical thermal systems, but it will be important for later that the quantum version of no symmetry breaking is that the Hamiltonian must have a *unique* ground state (at least with appropriate boundary conditions), because degenerate ground states can always couple to each other through quantum fluctuations to create a superposition state with lower energy.

So now that we have established that the Higgs mechanism does not, and cannot, correspond to spontaneous symmetry breaking, let's take a look at what's really happening. For simplicity we will look at the simplest case, namely (quantum, $T = 0$) $\mathbb{Z}_2$ lattice gauge theory. This comprises two-dimensional quantum systems on all the vertices and links of some lattice. The ones on the vertices comprise the "matter field" and the ones on the links comprise the "gauge field". We denote the Pauli matrices on the links by σ^x_{ab}, etc. and on the vertices by τ^x_a, etc. The Hamiltonian is

$$H = -g \sum_{\langle a,b \rangle} \sigma^x_{ab} - \frac{1}{g} \sum_{\square} \sigma^z \sigma^z \sigma^z \sigma^z - \lambda \sum_a \tau^x_a - \frac{1}{\lambda} \sum_{\langle a,b \rangle} \tau^z_a \sigma^z_{ab} \tau^z_b \tag{1}$$

(the second-term is a sum of four-body σ^z interactions on plaquettes.) This Hamiltonian has a gauge symmetry $\tau^x_a \prod_{\langle a,b \rangle} \sigma^x_b$ for each vertex a.

One can map out the phase diagram of this Hamiltonian in detail, but here we will just want to focus on the "Higgs" phase, which occurs when g and λ are small so that the second and fourth terms dominate. We will take the limit $g \to 0$, claiming without proof that the g small but not zero case is qualitatively similar. In this limit the ground state must be a $+1$ eigenstate of the product of σ^z around every plaquette ("no-flux" condition). If the model is defined on a space with no non-contractible loops, this implies that we can write, for every "no-flux" configuration, $\sigma^z_{ab} = \widetilde{\sigma}^z_a \widetilde{\sigma}^z_b$ for some choice of $\{\widetilde{\sigma}^z_a\} = \pm 1$. Hence, all "no-flux" configurations can be made to satisfy $\sigma^z_{ab} = 1$ by an appropriate gauge transformation. Thus, under this gauge-fixing condition, the Hamiltonian reduces to the transverse-field quantum Ising model on the matter fields:

$$H_{gf} = -\lambda \sum_a \tau^x_a - \frac{1}{\lambda} \sum_{\langle a,b \rangle} \tau^z_a \tau^z_b \tag{2}$$

which we know will have a symmetry-breaking phase (i.e. a two-fold degenerate ground state) for small λ. This is the Higgs phase.

Q: But hang on, now, doesn't Elitzur's theorem say that gauge symmetries cannot *be spontaneously broken?*

A: Well, actually in fixing the gauge we used up the local part of the gauge symmetry, and the above Hamiltonian H_{gf} only has a $\mathbb{Z}_2$ *global* symmetry. Thus, it does not violate Elitzur's theorem for it to have spontaneous symmetry breaking.

Q: But what about the orginal Hamiltonian, H? It had a gauge symmetry, and it's equivalent to the new Hamiltonian H_{gf}, which has spontaneous symmetry-breaking, so the original Hamiltonian must have spontaneous symmetry-breaking too???!!!

A: You have to be very careful about the sense in which H and H_{gf} are equivalent, because the "gauge-fixing" transformation which relates them isn't unitary (since it's many-to-one). Still, if one thinks hard enough and uses the fact that H is invariant under the gauge symmetry, it is not hard to show that there is a correspondence between eigenstates of H and of H_{gf}. However, because the two degenerate ground states of H are related by a gauge transformation, they actually correspond only to a single unique ground state of H, in accordance with Elitzur's theorem. This unique ground state of H can be found in terms of the ground states of H_{gf} by symmetrizing them to make them gauge-invariant, i.e.

$$|\Psi\rangle_H = \sum_{\mathcal{G}} \mathcal{G} |\Psi\rangle_{H_{gf}}, \tag{3}$$

where the sum is over all possible gauge transformations $\mathcal{G}$ (since the two degenerate ground states are related by a gauge transformation, this gives the same $|\Psi\rangle_H$ regardless of which one you choose to be $|\Psi\rangle_{H_{gf}}$.)

So in summary, *the Higgs mechanism looks like spontaneous symmetry breaking if you fix a gauge, but it's an illusion. The actual ground state of a gauge theory is always unique and gauge-invariant.*