CENTRAL FORCES & NEWTONIAN GRAVITY
Particle-Based Forces vs. Force Fields

- Particle-based **central** forces $\rightarrow \mathbf{F}_{1\text{ on }2}$ function of $\mathbf{r}_{12}$
  - 2 particles exert **attractive** / **repulsive** forces on each other
  - **In CM Frame:** Forces are toward / away from **origin**
  - **Example:** elastic forces

- Force fields $\rightarrow \mathbf{F}$ on particle is a function of position
  - Relative to some **origin**
  - Do fields come from “source particles”?
  - **Irrelevant** – examine **effect** of field separate from its **source**
  - “Test” particles assumed not to affect field $\rightarrow$ Newton 3$^{\text{rd}}$ Law?
  - **Examples:** centrifugal force, electric field
Reduced Mass

- What is time dependence of $\vec{r}_{12}$?

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 = \frac{\vec{F}_{1 on 2}}{m_2} - \frac{\vec{F}_{2 on 1}}{m_1}$$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 = \frac{\vec{F}_{1 on 2}}{m_2} + \frac{\vec{F}_{1 on 2}}{m_1}$$

$$\vec{r}_{12} = \vec{F}_{1 on 2} \left( \frac{m_1 + m_2}{m_1 m_2} \right)$$

Mathematically equivalent to Newton's 2\textsuperscript{nd} law for a single particle (with mass $\mu$) in a force field $\mathbf{F}(\mathbf{r})$

So solving the 2-body particle-based central force amounts to solving a 1-body field-based force (in the vector space of all possible $\mathbf{r}_{12}$ vectors)

“Reduced mass” ($\mu$)

Same as working in $m_1$ rest frame and including fictitious forces on $m_2$
Reduced Mass Example

• 2 masses on frictionless table connected by spring
  - At $t=0 \rightarrow$ spring is at equilibrium length $L$
  - “Flick” $m_1 \rightarrow$ initial velocity $v_0$
  - Calculate the motion of the CM
  - Calculate $\omega$ for the vibration of the spring in CM frame

- Both Newtonian and Lagrangian methods lead to same reduced mass
Central Force Motion

- Use inertial CM frame with \textit{z-axis} parallel to \( \mathbf{L}_{\text{system}} \)
  - And 2-D polar coordinates \((r, \theta)\) for xy-plane
  - \textbf{Goal:} solve for \( r(\theta), r(t), \text{and } \theta(t) \)

- Conserved quantities for the motion:
  
  \[
  L_z \equiv l = \mu \, r^2 \, \dot{\theta}
  \]

  \( \text{(Zero torque)} \)

  \[
  E = K + U = \frac{1}{2} \mu \left[ \dot{r}^2 + (r \, \dot{\theta})^2 \right] + U(r)
  \]

  \[
  \text{Plug in: } \quad E = K + U = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \mu \left( \frac{l}{\mu \, r} \right)^2 + U(r)
  \]

  \[
  E = \frac{1}{2} \mu \dot{r}^2 + \left[ \frac{l^2}{2 \, \mu \, r^2} + U(r) \right] \quad \text{U_{\text{effective}} (r)}
  \]

  Even though L-term is KE, it acts mathematically like PE because it depends only on \( r \)

  \( \text{Recall generalized coordinates} \)
The “Centrifugal Barrier”

- Consider 2-particle system with angular momentum $L$
  - As $r$ decreases, $\dot{\theta}$ must increase to conserve $L$
  - Infinite amount of energy required as $r \to 0$
  - Point particles can never touch unless $L=0$ exactly

“Centrifugal Barrier” – 1\textsuperscript{st} term in $U_{\text{effective}} = \frac{l^2}{2 \mu r^2} + U(r)$

- Mathematically identical to a repulsive force field on mass $\mu$

- Example: Binary star system
  - Stars have strong gravitational attraction
  - But centrifugal barrier keeps them apart
Newtonian Gravity

• Experiments in late 1700's (Cavendish and others):

\[ \vec{F}_{\text{gravity, 1 on 2}} = -G \frac{m_1 m_2}{r^2} \hat{r}_{1 \text{ to } 2} \]

“inverse-square law”
(confirmed idea proposed by Newton)

How might these experiments be designed?

• In this form, gravity is a conservative force:

\[ U(r) - U(r_0) = -\int_{r_0}^{r} \left( -G \frac{m_1 m_2}{r^2} \hat{r} \right) \cdot d\hat{r} = - \left( G \frac{m_1 m_2}{r} - G \frac{m_1 m_2}{r_0} \right) \]

• Convention for “zero point” of potential energy:

\[ U(\infty) = 0 \]
(Usually applies, but not always!)

With this convention:

\[ U(r) = -G \frac{m_1 m_2}{r} \]
Gravity Examples

• 3 point masses (each mass $m$) at infinite separation
  – Are then arranged into equilateral triangle of side length $L$
  – Calculate the change in gravitational PE

• Fixed spherical “shell” of mass $M$, radius $R$
  – Place a small “test mass” $m$ at any location in space
  – Solve for $U$ as a function of position of test mass
  – Interpret results inside and outside shell → “Shell Theorem”

• Drill hole through Earth along a diameter
  – Drop a rock into hole from surface
  – Calculate motion of rock (assume uniform Earth)
Circular Orbits

Gravity → attractive “potential well” (\( U \sim -\frac{1}{r} \))

Centrifugal barrier → repulsive (\( U \sim \frac{1}{r^2} \))

\[
U_{\text{effective}} = \frac{l^2}{2 \mu r^2} - G \frac{m_1 m_2}{r}
\]

Mathematically equivalent to 1-D motion with variable r

For small r:
centrifugal term dominates

For large r:
gravity term dominates

“Equilibrium” separation \( r_0 \):

\[
\frac{\partial}{\partial r} U_{\text{effective}} \bigg|_{r_0} = 0
\]

(corresponds to a circular orbit at radius \( r_0 \))

Examples:
1) Show that \( U_0 < 0 \) for all values of \( L \neq 0, m_1, m_2 \)

2) Show that \( U_{\text{gravity}} = -2\text{KE} \) for circular orbit
Elliptical / Hyperbolic Orbits

- $U_{\text{effective}}$ includes KE due to tangential ($\dot{\theta}$) motion
  - Can also have KE due to radial ($\dot{r}$) motion
  - Orbits take on shape of ellipse or hyperbola

3 possibilities for $E$ (the total energy):

1) $E = U_0$ → circular orbit
2) $0 > E > U_0$ → elliptical orbit
3) $E > 0$ → hyperbolic orbit

Example:
Imagine a slightly perturbed circular orbit ($E > U_0$, but barely)

Compare period of small radial oscillations to period of orbit.
Escape Velocity

• Orbits with E<0 → “bound” (ellipse)
  - 2 objects have minimum and maximum separation distance
  - Called perigee and apogee
  - For orbits around Sun: perihelion and aphelion

• Orbits with E>0 → no bounds on motion (hyperbola)
  - Objects will eventually reach infinite separation

• Escape velocity – velocity necessary to reach E=0
  - Example: Calculate $v_{\text{escape}}$ for object on surface of Earth
  - Example: Calculate $v_{\text{escape}}$ for object at $r = 2R_{\text{Earth}}$
  - Does direction of velocity affect your answer? (ignore air drag)
Kepler's Laws

• **Empirical laws** → from observing patterns in data
  - Rather than hypothesizing some underlying concept

• Johannes Kepler – Late 1500's *(before Newton!)*
  - Observed patterns in motion of planets around Sun:
    - 1) Elliptical orbits with Sun stationary at one focus
    - 2) “Equal areas in equal times”
    - 3) \((\text{Period})^2 = k (\text{major axis of ellipse})^3\) → same \(k\) for all planets

• **Example:** Prove 2\(^{nd}\) Law
  - Using conservation of angular momentum

**Note:** Kepler's Laws assume \(\mu \approx m\) – when is this true?
Gravity With Multiple Objects

- 3 or more objects → cannot reduce to 1-body problem
  - i.e. no such thing as “reduced mass” and single \( m \)
  - Equations of motion → no known “closed-form” solution
  - Must be solved by computer

- Often, 3\(^{rd}\) object interacts weakly with 2-body system
  - Causing perturbations in the 2-body orbit
  - Example: Effect of Mars gravity on Earth orbit around Sun

- “Gravitational slingshot”
  - Use Mars to increase energy of spacecraft orbit
Stability

- Thought experiment – gravity
  - Arrange equal point masses in a regular grid
  - Which extends out to infinity

- In equilibrium (similar to grid of masses and springs)
  - Is the equilibrium stable? (For springs – yes → oscillations)
  - Gravity → force gets weaker with larger separation
  - So the equilibrium is unstable!

- Distribution of observable matter in universe
  - Mostly concentrated into galaxies, solar systems, etc.
  - History of universe can be inferred from “clumping” of matter