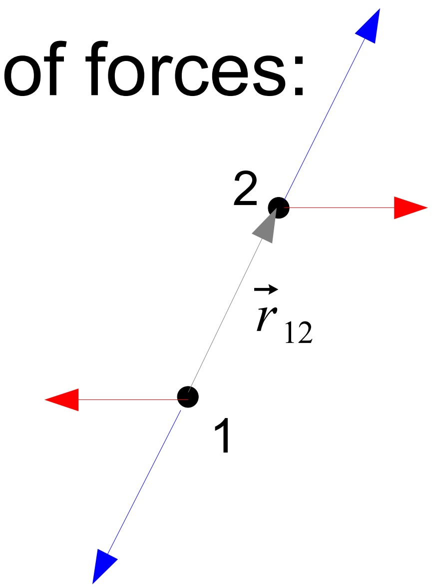
ROTATIONAL DYNAMICS

Newton's 3rd Law – A Closer Look

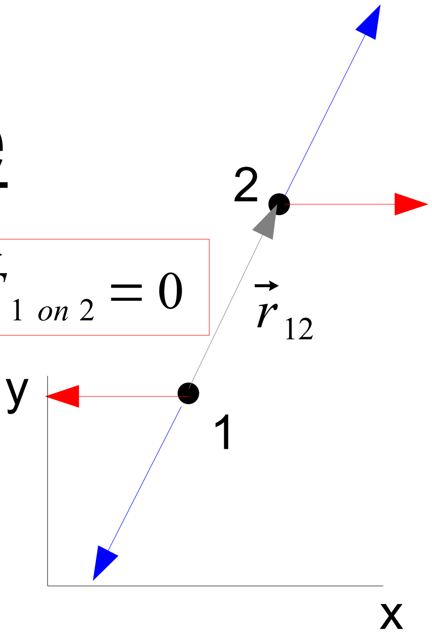
- Consider 2 particles with action/reaction pair of forces:
  - Newton's 3rd Law $\rightarrow$ $\vec{F}_{1\ on\ 2} = -\vec{F}_{2\ on\ 1}$
  - Puts no restrictions on the direction of $\vec{F}_{1\ on\ 2}$



- Symmetry Considerations
  - Attempt to write a formula for the force $\vec{F}_{1\ on\ 2}$
  - No "universal" xyz directions $\rightarrow$ What can $\vec{F}_{1\ on\ 2}$ depend on?
  - Relative position vector $\vec{r}_{12}$ and relative velocity vector $\vec{v}_{12}$ only

- If $\vec{F}_{1\ on\ 2}$ depends only on $\vec{r}_{12}$ mathematically:
  - Direction of $\vec{r}_{12}$ $\rightarrow$ only "defined" direction in space
  - $\vec{F}_{1\ on\ 2}$ must point in direction of $\vec{r}_{12}$ (or opposite direction)
  - Forces of this type are called "central" forces

Central Forces and Torque



- Mathematical definition of central force: $\boxed{\vec{r}_{12} \times \vec{F}_{1\ on\ 2} = 0}$
  - True in every reference frame!

- Calculating in a particular reference frame S:

$$\vec{r}_{12} \times \vec{F}_{1\ on\ 2} = 0$$

$$\left(\vec{r}_2 - \vec{r}_1\right) \times \vec{F}_{1\ on\ 2} = 0$$

$$\left(\vec{r}_2 \times \vec{F}_{1\ on\ 2}\right) - \left(\vec{r}_1 \times \vec{F}_{1\ on\ 2}\right) = 0$$

$$\left(\vec{r}_2 \times \vec{F}_{1\ on\ 2}\right) + \left(\vec{r}_1 \times \vec{F}_{2\ on\ 1}\right) = 0$$

The quantity $\vec{r}_i \times \vec{F}_i$ is called the "torque" ($\vec{\tau}_i$) on the i^{th} particle

Internal central forces produce zero net torque on a system

External forces and non-central internal forces can exert non-zero net torque on a system

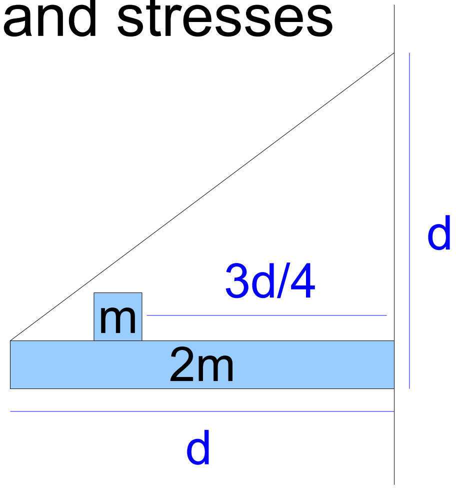
- Examples:
  - Central forces: gravity, electric
  - Non-central force: magnetic (depends on position and velocity)

Static Equilibrium

- "Equilibrium" → $\mathbf{F}_{net} = 0$ → $\mathbf{a}_{CM} = 0$

- "Static Equilibrium" → $\mathbf{a}_i = 0$ for every particle
  - Examples: buildings, bridges → (not perfectly static!)
  - Requires: $\vec{F}_{net\ ,\ external} = 0$ and $\vec{\tau}_{net\ ,\ external} = \sum_i \left(\vec{r}_i \times \vec{F}_i\right) = 0$
  - Useful for calculating structural loads and stresses

- Examples:
  - Shelf → calculate tension in cable
  - Calculate force of wall on plank



  - Door of width w and height h → draw direction of each F_{hinge}

Angular Momentum

- Consider the net torque on a system of particles

$$\vec{\tau}_{net} = \sum_i \left(\vec{r}_i \times \vec{F}_i\right) \longrightarrow \vec{\tau}_{net} = \sum_i \left(\vec{r}_i \times \frac{d\,\vec{p}_i}{dt}\right) \longrightarrow \vec{\tau}_{net} = \frac{d}{dt}\left(\sum_i \left(\vec{r}_i \times \vec{p}_i\right)\right)$$

  - $\vec{r}_i \times \vec{p}_i$ is referred to as “angular momentum”($\vec{L}_i$) of ith particle
  - Internal, central forces exert zero net torque
  - Net torque must be provided by external forces:

$$\boxed{\vec{\tau}_{net\,,\,external} = \frac{d\,\vec{L}_{total}}{dt}}$$

(Similar to Newton's 2nd Law)

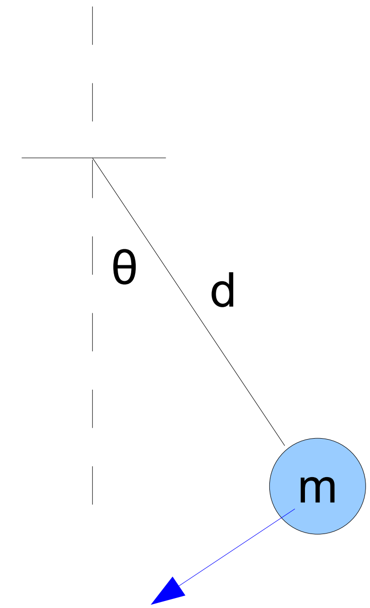
  - If zero net external torque → angular momentum is conserved

- Both $\vec{\tau}$ and $\vec{L}$ depend on choice of origin
  - Unlike force and momentum (only depend on xyz directions)
  - However, the equation above is true in all reference frames

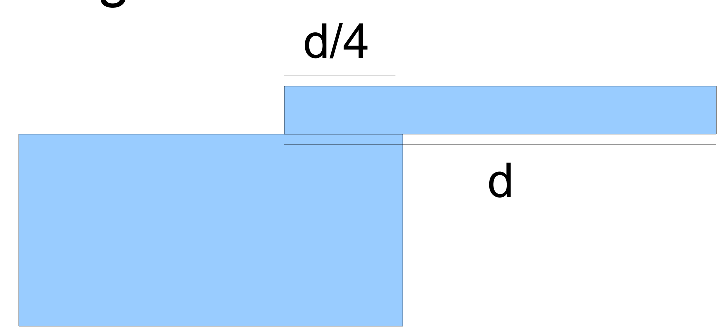
Examples

- Pendulum at some instant (angle θ, speed v)
  - Using top of string as origin:
  - Calculate torque and angular momentum
  - Plug in to $T_{net} = dL/dt$

    Repeat, using mass's lowest point as origin



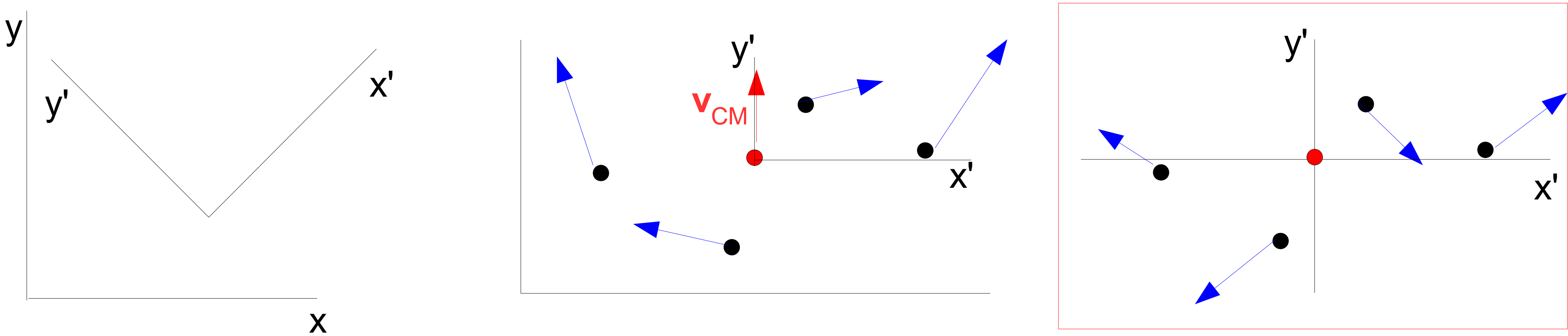
- Wooden board falls off table
  - Mass m, starting from rest
  - Using edge of table as origin:
  - Calculate T_{net} and $a_{\text{right edge of board}}$ at t=0
  - (Assume board stays rigid → v proportional to r)



- Why does a helicopter need a tail rotor?

CM Frame of a Physical System

- Empty space → homogeneous and isotropic
  - All points are physically the same → no point is “special”
  - All directions physically the same → no direction is “special”



- System of particles → breaks this symmetry
  - A “special” reference frame can be defined for the system:
  - Origin = CM of system (called the “CM frame”)
  - x'y'z' axes → defined by particle positions (“principal axes”)
  - Angular velocity vector $\vec{\Omega}_{CM}$ → defined by particle velocities

Angular Momentum in CM Frame

$$\vec{p}_{system} = M_{system}\,\vec{v}_{CM}$$

$$KE_{system} = \frac{1}{2} M_{system}\, v_{CM}^{2} + \sum_{i=1}^{N} \frac{1}{2} m_i \left(v'_i\right)^2$$

- For a system of particles measured in the CM frame:
  - Total momentum must be zero, but total KE can be non-zero!

- Angular Momentum (calculated in an inertial frame S):

$$\vec{L}_{system} = \sum_i \left[\vec{r}_i \times \vec{p}_i\right] = \sum_i \left[\left(\vec{r}_{CM} + \vec{r}_i{}'\right) \times m_i \left(\vec{v}_{CM} + \vec{v}_i{}'\right)\right]$$

$$\vec{L}_{system} = \sum_i m_i \left[\left(\vec{r}_{CM} \times \vec{v}_{CM}\right) + \left(\vec{r}_i{}' \times \vec{v}_{CM}\right) + \left(\vec{r}_{CM} \times \vec{v}_i{}'\right) + \left(\vec{r}_i{}' \times \vec{v}_i{}'\right)\right]$$

$$\sum_i m_i \vec{r}_i{}' = 0 \qquad \sum_i m_i \vec{v}_i{}' = 0$$

$$\boxed{\vec{L}_{system} = \vec{L}_{CM} + \sum_i m_i \left(\vec{r}_i{}' \times \vec{v}_i{}'\right)}$$

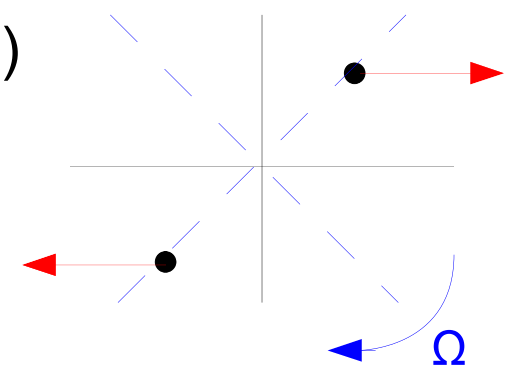
"spin" angular momentum

"orbital" angular momentum (zero in CM frame)

Rotating CM Frame

- “Special” reference frame for a system of particles:
  - Must allow for rotating axes to account for angular momentum

- Let S be an inertial (non-rotating) CM frame
  - Let S' be a rotating CM frame with matrix R(t)
  - Recall: $\vec{v}\,' = R\,\vec{v} - \left[\vec{\Omega}\times\left(R\,\vec{r}\right)\right]$



$$\vec{L}_{system}\,' = \sum_i m_i\left(\vec{r}_i\,'\times\vec{v}_i\,'\right) = 0$$

For some “special” $\vec{\Omega}$

$$\vec{L}_{system}\,' = \sum_i m_i\left[R\,\vec{r}_i\times\left(R\,\vec{v}_i - \vec{\Omega}\times\left(R\,\vec{r}_i\right)\right)\right] = 0$$

$$\vec{L}_{system}\,' = \sum_i m_i\left[\left(R\,\vec{r}_i\right)\times\left(R\,\vec{v}_i\right)\right] - \sum_i m_i\left[\left(R\,\vec{r}_i\right)\times\left(\vec{\Omega}\times\left(R\,\vec{r}_i\right)\right)\right] = 0$$

$$\vec{L}_{system}\,' = R\left[\vec{L}_{spin\ ,\ inertial} - \sum_i m_i\left[\vec{r}_i\times\left(\vec{\Omega}\times\vec{r}_i\right)\right]\right] = 0$$

$$\boxed{\vec{L}_{spin\ ,\ inertial} = \sum_i m_i\left[\vec{r}_i\times\left(\vec{\Omega}\times\vec{r}_i\right)\right]} \longrightarrow$$

This can be used to calculate the “special” $\vec{\Omega}$ for a given system (which makes L' = 0)

Inertia Tensor

$$\vec{L}_{spin\,,\,inertial} = \sum_i m_i \left[\vec{r}_i \times \left(\vec{\Omega} \times \vec{r}_i\right)\right]$$

$$\vec{L}_{spin\,,\,inertial} = -\sum_i m_i \left[\vec{r}_i \times \left(\vec{r}_i \times \vec{\Omega}\right)\right]$$

$$\vec{L}_{spin\,,\,inertial} = -\sum_i m_i \left[\begin{pmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{pmatrix}\begin{pmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{pmatrix}\begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix}\right]$$

$$\vec{L}_{spin\,,\,inertial} = \left[\sum_i m_i \begin{pmatrix} \left(y_i^2 + z_i^2\right) & -x_i\, y_i & -z_i\, x_i \\ -x_i\, y_i & \left(x_i^2 + z_i^2\right) & -y_i\, z_i \\ -z_i\, x_i & -y_i\, z_i & \left(x_i^2 + y_i^2\right) \end{pmatrix}\begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix}\right] \equiv \tilde{I}\, \vec{\Omega}$$

"Inertia Tensor" – fully describes the distribution of mass in a system

Diagonal elements are called "moments of inertia"

Off-diagonal elements are called "products of inertia"

<u>Reference frame for a system of particles is almost complete:</u>

1) origin → CM
2) angular velocity → using **L** and **I**
3) need to find "principal axes"

Inertia Tensor for Mass Distributions

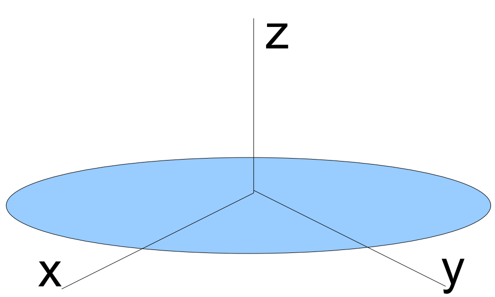
- For a system of particles: $$\tilde{I} = \sum_i m_i \begin{pmatrix} \left(y_i^2 + z_i^2\right) & -x_i\ y_i & -z_i\ x_i \\ -x_i\ y_i & \left(x_i^2 + z_i^2\right) & -y_i\ z_i \\ -z_i\ x_i & -y_i\ z_i & \left(x_i^2 + y_i^2\right) \end{pmatrix}$$

- Inertia tensor extends naturally to mass distributions:

$$\tilde{I} = \int \rho\left(x\ ,\ y\ ,\ z\right) dxdydz \begin{pmatrix} \left(y^2 + z^2\right) & -x\ y & -z\ x \\ -x\ y & \left(x^2 + z^2\right) & -y\ z \\ -z\ x & -y\ z & \left(x^2 + y^2\right) \end{pmatrix}$$

- <u>Example</u>: Calculate **I** for a flat disk in the xy-plane

$$\rho\left(x\ ,\ y\ ,\ z\right) = \delta\left(z\right) \begin{cases} \dfrac{M}{\pi\ R^2} & \left(x^2 + y^2\right) < R \\ 0 & elsewhere \end{cases}$$

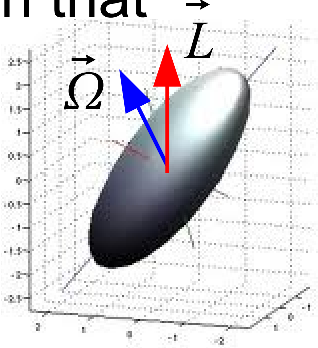


$$\tilde{I} = \begin{pmatrix} \dfrac{1}{4}\ M\ R^2 & 0 & 0 \\ 0 & \dfrac{1}{4}\ M\ R^2 & 0 \\ 0 & 0 & \dfrac{1}{2}\ M\ R^2 \end{pmatrix}$$

Principal Axes

$$\tilde{I} = \sum_i m_i \begin{pmatrix} (y_i^2 + z_i^2) & -x_i\, y_i & -z_i\, x_i \\ -x_i\, y_i & (x_i^2 + z_i^2) & -y_i\, z_i \\ -z_i\, x_i & -y_i\, z_i & (x_i^2 + y_i^2) \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

- Elements of inertia tensor depend on choice of xyz axes
  - Is there a “special” set of xyz axes for a given system?
  - Yes! Possible (but difficult) to find “principal axes” such that products of inertia are zero → in this reference frame:



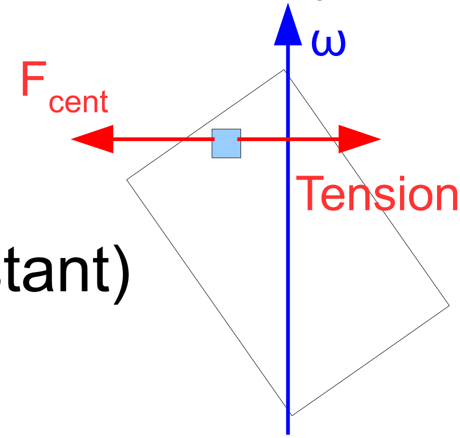
$$\tilde{I} = \sum_i m_i \begin{pmatrix} (y_i^2 + z_i^2) & 0 & 0 \\ 0 & (x_i^2 + z_i^2) & 0 \\ 0 & 0 & (x_i^2 + y_i^2) \end{pmatrix} \longrightarrow \boxed{\begin{aligned} L_x &= I_{xx}\, \Omega_x \\ L_y &= I_{yy}\, \Omega_y \\ L_z &= I_{zz}\, \Omega_z \end{aligned}}$$

Note: $\vec{L}$ and $\vec{\Omega}$ can point in different directions!

- For a system which is symmetric about an axis:
  - The symmetry axis is one of the principal axes of the system

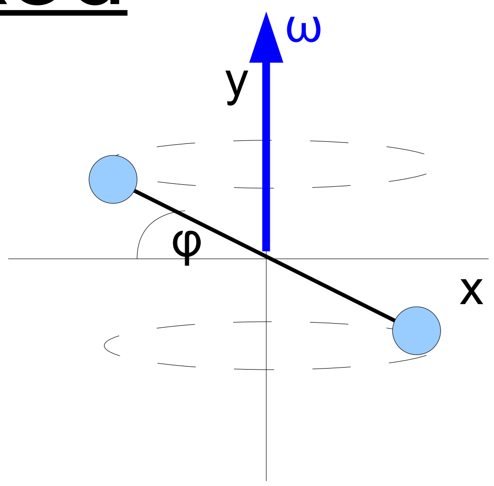
Rigid Body Rotation

- Physics definition of "rigid body"
  - System of particles which maintains its shape (no deformation)
  - i.e. velocity of particles in CM frame comes from rotation only
  - Notation: $\boldsymbol{\omega}$ = angular velocity of rigid body (inertial CM frame)



- View from rotating CM frame
  - Every particle stands still in equilibrium (**I** is constant)
  - Centrifugal forces balanced by internal stresses
- Rigid body model for solid objects → reasonably good
  - In reality, solids can 1) deform and 2) dissipate energy as heat
  - Rigid body model → not applicable to fluids, orbits, stars, etc.

Example: Rotating Skew Rod



- Rigid massless rod (length 2d):
  - Rotates as shown with mass m at each end
  - At the instant shown:
  - Calculate $\mathbf{L}_{system}$, using 2 different methods:
  - 1) $\vec{r}_i \times \vec{p}_i$, and 2) calculate the inertia tensor
  - Notice **L** and **ω** don't point same direction!
- Is an external torque needed to sustain this motion?
  - If so, calculate it
- Use symmetry to guess the principal axes
  - Verify guess by calculating inertia tensor in principal frame

Fixed-Axis Rotation

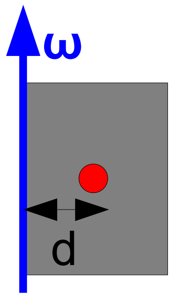
- Common practice → hold **ω** constant (not **L**) → $\frac{d\vec{L}}{dt} \neq 0$
  - If **L** and **ω** not parallel → "axle" must exert an external torque!

- Example: Rotating ceiling fan
  - If mass is symmetrically distributed:
  - **L** and **ω** parallel → fan turns smoothly
  - If mass distribution has asymmetry (poor alignment, etc.):
  - **L** and **ω** not parallel → fan wobbles

- High rotation speeds → wheels, lathes, etc.
  - Mounting must be able to exert external torque
  - Enough to handle a "tolerable" amount of asymmetry

Parallel-Axis Theorem

- Often: fixed axis does not pass through CM of system
  - Example: door hinge – how to calculate $\mathbf{L}_{\mathbf{door}}$?
  - Recall $\vec{L}_{system} = M_{total}\, \vec{v}_{CM} \times \vec{r}_{CM} + \left(\tilde{I}_{principal}\right) \vec{\omega}$
  - For rigid body rotation: $\vec{v}_{CM} = \vec{\omega} \times \vec{r}_{CM}$



Plug in:

$$\vec{L}_{system} = M_{total} \left(\vec{\omega} \times \vec{r}_{CM}\right) \times \vec{r}_{CM} + \left(\tilde{I}_{principal}\right) \vec{\omega}$$

$$\vec{L}_{system} = M_{total}\, \vec{r}_{CM} \times \left(\vec{r}_{CM} \times \vec{\omega}\right) + \left(\tilde{I}_{principal}\right) \vec{\omega}$$

$$\vec{L}_{system} = \left(\tilde{I}_{orbital}\right) \vec{\omega} + \left(\tilde{I}_{principal}\right) \vec{\omega}$$

- Parallel-Axis Theorem:
  - Let d be the perpendicular distance from CM to fixed axis

$$I_{orbital\,,\,axis} = M\, d^2 \longrightarrow \boxed{I_{axis} = I_{principal} + M\, d^2}$$

$$I_{door\ about\ hinges} = \left(\frac{1}{12} M_{door} (2\, d)^2 + M_{door}\, d^2\right) = \frac{1}{3} M_{door} (2\, d)^2$$

Rotational KE

- For a rotating rigid body (in inertial frame) :

$$KE_{system} = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \left(\vec{\omega} \times \vec{r}_i\right) \cdot \left(\vec{\omega} \times \vec{r}_i\right)$$

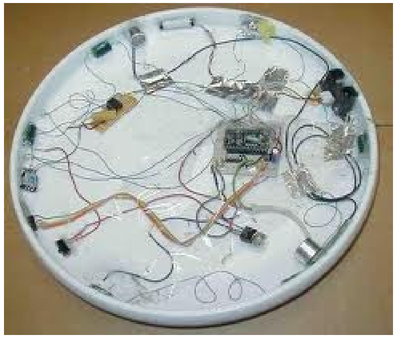
$$KE_{system} = \frac{1}{2} \vec{\omega} \cdot \left[\sum_i m_i \left(\vec{r}_i \times \left(\vec{\omega} \times \vec{r}_i\right)\right)\right]$$ (Using vector identity)

$$\boxed{KE_{system} = \frac{1}{2} \vec{\omega} \cdot \left(\tilde{I}\, \vec{\omega}\right) = \frac{1}{2} \vec{\omega} \cdot \vec{L}}$$ (maximum angle between $\vec{\omega}$ and $\vec{L}$ is 90º)

(For non-rigid bodies, must also include motion of particles toward/away from CM)

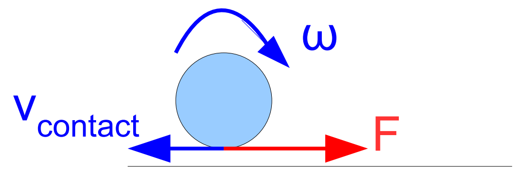
- In principal axis frame: $$\boxed{KE_{principal} = \frac{1}{2} I_{xx}\, \omega_x^2 + \frac{1}{2} I_{yy}\, \omega_y^2 + \frac{1}{2} I_{zz}\, \omega_z^2}$$

- Example: "average" frisbee toss
  - Estimate $KE_{translation}$ and $KE_{rotation}$ in Joules



Rolling Without Slipping

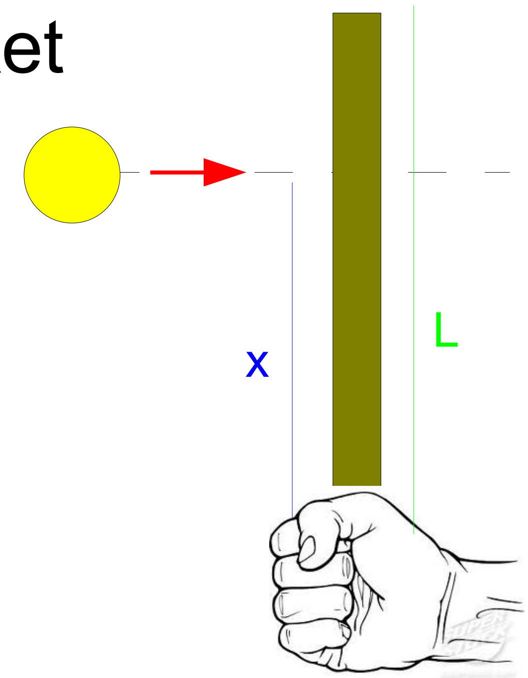
- System's $\vec{v}_{CM}$ and $\vec{\omega}$: independent of each other
  - No relation between translational / rotational motion in general
  - However, by using a force → the 2 can be coupled
- Example: friction



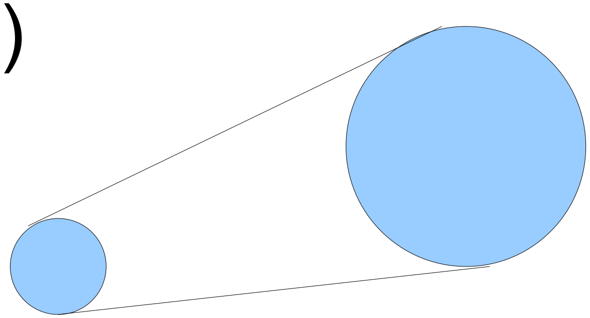
  - Drop a ball spinning at angular velocity ω on the floor
  - Relative velocity of ball's surface / floor causes kinetic friction
  - This force has 2 effects:
  - 1) pushes the ball to the right (affecting $\vec{v}_{CM}$)
  - 2) exerts a CCW torque on the ball (affecting $\vec{\omega}$)
- Rolling without slipping → condition where $v_{CM} = R\,\omega$
  - $v_{contact}$=0 → fixed-axis rotation about contact pt. (static friction)

Examples

- "Sweet spot" of baseball bat or tennis racket
  - Goal: Minimize effect of ball impact on hands
  - Assume quick impulse → F_{hand} has no effect
  - If collision is elastic, calculate x such that:
  - $v_{\text{end of bat}} = 0$ (closest to hands) due to collision

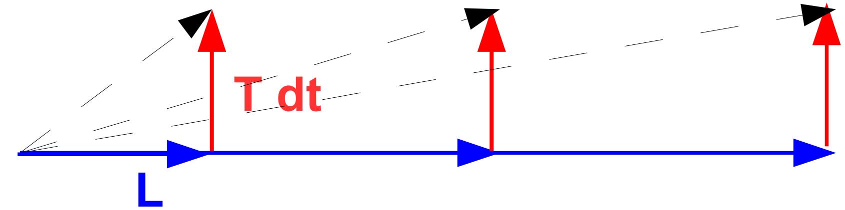


- Belt around two rotors (m_1, r_1 and m_2, r_2)
  - Belt (mass m_b) moves at speed v
  - Calculate the total KE of the system
  - What happens to belt as v gets large?



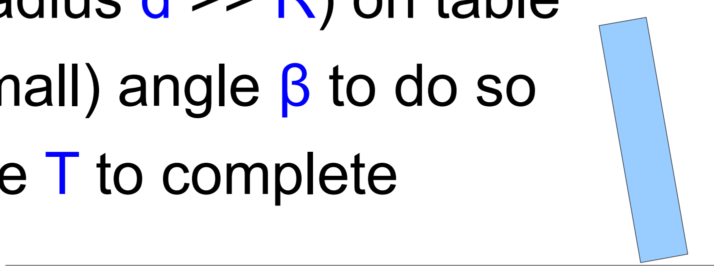
“Stability” of Rotating Objects

- Consider a rotating object which is deflected:
  - The larger $L_{initial}$ gets:
  - The smaller the deflection angle



- Example: riding a bicycle
  - At low speed → leaning leads to falling over
  - At high speed → same amount of torque has less impact

- Coin (mass M, radius R) rolls without slipping on table
  - Traces out circular path (of radius $d >> R$) on table
  - Coin must lean inward by (small) angle β to do so
  - Calculate β if circle takes time T to complete



Video:

Euler's Equations

- <u>Linear Momentum</u>: $\vec{p}_{system} = m\,\vec{v}_{CM}$ and is conserved
  - Since m is a scalar and is constant $\rightarrow$ $\mathbf{v}_{CM}$ is constant
- <u>Angular Momentum</u>: $\vec{L}_{spin\,,\,inertial} = \tilde{I}\,\vec{\omega}$ and is conserved
  - **I** $\rightarrow$ <u>not</u> a simple scalar and <u>can</u> vary with time
  - No such thing as "conservation of **ω**" $\rightarrow$ even with no torque!

<u>Chain rule:</u>

$$\vec{\tau} = \frac{d\,\vec{L}_{spin\,,\,inertial}}{dt} = \left(\frac{d\,\tilde{I}}{dt}\right)\vec{\omega} + \tilde{I}\left(\frac{d\,\vec{\omega}}{dt}\right)$$

$$\boxed{\vec{\tau} = \left(\vec{\omega}\times\tilde{I}\right)\vec{\omega} + \tilde{I}\left(\frac{d\,\vec{\omega}}{dt}\right)}$$

(similar to **F** = m d**v**/dt)

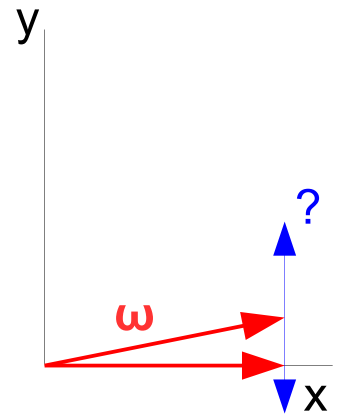
To evaluate d **I** /dt, recall:
In "rotating principal axis CM frame" $\rightarrow$ **I'** is constant for a rigid body

"Euler's Equations"

Notice $\vec{\omega}$ can change even if torque is zero!

$$\frac{d\,\tilde{I}\,'}{dt} \equiv 0 = \frac{d\,\tilde{I}}{dt} - \vec{\omega}\times\tilde{I}$$

Stable/Unstable Rotations



- Rigid-body rotation about any <u>principal</u> axis:
  - Every particle in equilibrium (as viewed in rotating CM frame)
  - Is this equilibrium stable? If $\vec{\omega}$ has small off-axis component:
  - Do Euler's equations predict it will grow / shrink / neither?

- Consider rotating principal-axis system with $\left(\omega_y\ ,\ \omega_z\right) \ll \omega_x$
  - Euler's equations (to 1st order in ω_y and ω_z) become:

$$I_{xx} \frac{d\ \omega_x}{dt} = 0$$ (No external torque)

$$I_{yy} \frac{d\ \omega_y}{dt} + \left(I_{xx} - I_{zz}\right) \omega_x\ \omega_z = 0$$

$$I_{zz} \frac{d\ \omega_z}{dt} + \left(I_{yy} - I_{xx}\right) \omega_x\ \omega_y = 0$$

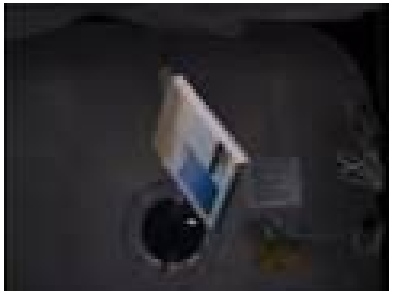
$$\frac{d^2\ \omega_y}{dt^2} - \left(\frac{\left(I_{xx} - I_{zz}\right)\left(I_{yy} - I_{xx}\right)}{I_{yy}\ I_{zz}}\ \omega_x^{\ 2} \right) \omega_y = 0$$

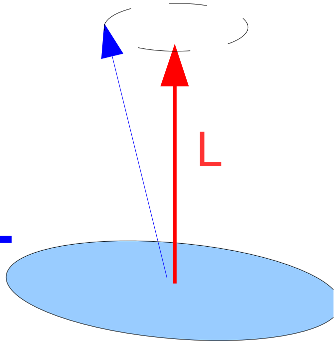
“C” = Constant

<u>C < 0</u> : rotation about x-axis is stable
<u>C > 0</u> : rotation about x-axis is unstable
Largest & smallest moments of inertia: stable axis
Intermediate moment of inertia: unstable axis

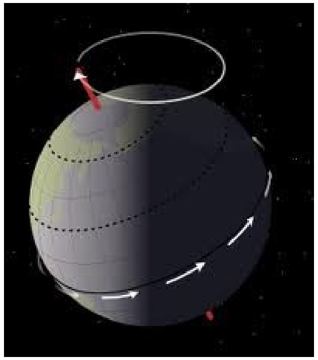
Precession

- In general → motion of **ω** very complicated
  - Euler's Equations – coupled nonlinear DE's
  - http://www.youtube.com/watch?v=GgVpOorcKqc
  - Terminology: "Precession" and "Nutation"
  - Precession → motion of **ω** simpler to describe than nutation





- Torque-free precession
  - **L** has large stable-axis component → **ω** "circles" **L**
  - Example: spinning coin tossed in air "wobbles"



- Torque-induced precession
  - Causes **L** and **ω** to rotate around fixed axis
  - Example: Precession of Earth axis due to Moon/Sun

Gyroscopes

- Devices for studying / utilizing precession and nutation
  - Basic setup: single axis with low friction & good alignment
  - Can be used to produce strong **L** due to "spin":
  - http://www.youtube.com/watch?v=hVKz9G3YXiw

- Commonly used in sensor systems
  - Can be used to measure orientation, latitude, acceleration, etc.

- Example: "Uniform Precession"
  - http://www.youtube.com/watch?v=8H98BgRzpOM&feature=related
  - Wheel has mass M, radius R, string attached distance d from center
  - Horizontal wheel spins at angular velocity ω_{spin}, calculate $\omega_{precession}$

<u>“State” of a System</u>

- “Microscopic state” of a system of particles $\longrightarrow \quad m_i\ ,\ \vec{r}_i\ ,\ \vec{v}_i$
  - Given the system's current state and forces between particles:
  - Use Newton's Laws to predict future motion of <u>each</u> particle

- “Macroscopic state” of a system → $\mathbf{P}_{total}$, KE, PE, **L**
  - Quantities associated with the system as a whole
  - Conservation Laws – predict the future motion of the system

- Experiments in early-mid 1900's
  - Showed that elementary particles (like e^-) have **L** and energy
  - So they act more like <u>systems</u> than particles!
  - Quantum Mechanics explains how this can be possible