Physics 115C
Homework 4

Problem 1

(a) In the Heisenberg picture, the dynamical equation is the Heisenberg equation of motion: for any operator Q_H, we have

$$\frac{dQ_H}{dt} = \frac{1}{i\hbar}[Q_H, H] + \frac{\partial Q_H}{\partial t}$$

where the partial derivative is defined as

$$\frac{\partial Q_H}{\partial t} \equiv e^{iHt/\hbar}\frac{\partial Q_S}{\partial t}e^{-iHt/\hbar}$$

where Q_S is the Schrödinger operator. If we're interested in the evolution of the lowering operator of the simple harmonic oscillator, we let $Q = a$, and we get

$$\frac{da_H}{dt} = \frac{1}{i\hbar}[a_H, H]$$

To evaluate the commutator, let's express the Hamiltonian in terms of the Heisebnerg raising and lowering operators. To do so, recall that in the Schrödinger picture, the Hamiltonian for the simple harmonic oscillator was

$$H = \hbar\omega\left(a_S^\dagger a_S + \frac{1}{2}\right)$$

Now, let's multiply the Hamiltonian by unity in the form

$$1 = e^{iHt/\hbar}e^{-iHt/\hbar}$$

Using the fact that the Hamiltonian commutes with any function of itself, we get

$$\begin{aligned}
H &= 1 \cdot H \\
&= e^{iHt/\hbar}e^{-iHt/\hbar}H \\
&= e^{iHt/\hbar}He^{-iHt/\hbar} \\
&= \hbar\omega e^{iHt/\hbar}\left(a_S^\dagger a_S + \frac{1}{2}\right)e^{-iHt/\hbar} \\
&= \hbar\omega\left(e^{iHt/\hbar}a_S^\dagger a_S e^{-iHt/\hbar} + \frac{1}{2}\right)
\end{aligned}$$

Cleverly inserting another factor of unity between the ladder operators, we get

$$
\begin{aligned}
H &= \hbar\omega \left(e^{iHt/\hbar} a_S^\dagger e^{-iHt/\hbar} e^{iHt/\hbar} a_S e^{-iHt/\hbar} + \frac{1}{2} \right) \\
&= \hbar\omega \left(a_H^\dagger(t) a_H(t) + \frac{1}{2} \right)
\end{aligned}
$$

where we recognized the relationship between the Schrödinger and Heisenberg operators

$$
a_H(t) = e^{iHt/\hbar} a_S e^{-iHt/\hbar}
$$

and likewise for $a^\dagger$. The next thing we'll need are the commutation relations for a_H and $a_H^\dagger$. In the Schrödinger picture, we know that

$$
[a_S, a_S^\dagger] = 1
$$

Multiplying on the left by $e^{iHt/\hbar}$ and the right by $e^{-iHt/\hbar}$, we get

$$
\begin{aligned}
e^{iHt/\hbar} [a_S, a_S^\dagger] e^{-iHt/\hbar} &= e^{iHt/\hbar} e^{-iHt/\hbar} \\
e^{iHt/\hbar} \left(a_S a_S^\dagger - a_S^\dagger a_S \right) e^{-iHt/\hbar} &= 1
\end{aligned}
$$

Again inserting unity between the operators, we get

$$
\begin{aligned}
\left(e^{iHt/\hbar} a_S e^{-iHt/\hbar} e^{iHt/\hbar} a_S^\dagger e^{-iHt/\hbar} - e^{iHt/\hbar} a_S^\dagger e^{-iHt/\hbar} e^{iHt/\hbar} a_S e^{-iHt/\hbar} \right) &= 1 \\
\left(a_H(t) a_H^\dagger(t) - a_H^\dagger(t) a_H(t) \right) &= 1
\end{aligned}
$$

Thus we obtain the so-called *equal-time commutation relation*

$$
[a_H(t), a_H^\dagger(t)] = 1
$$

(it is crucial that the times be equal, for otherwise our arguments above wouldn't have worked!). We are now ready to calculate the explicit time-evolution of a_H. We had the equation of motion

$$
\frac{da_H}{dt} = \frac{1}{i\hbar}[a_H, H]
$$

Writing

$$
H = \hbar\omega \left(a_H^\dagger(t) a_H(t) + \frac{1}{2} \right)
$$

and using our equal-time commutation relations, we find that

$$
\begin{aligned}
[H, a_H] &= \hbar\omega \left[a_H(t), a_H^\dagger(t) a_H(t) + \frac{1}{2} \right] \\
&= \hbar\omega \left[a_H(t), a_H^\dagger(t) a_H(t) \right] \\
&= \hbar\omega \left(a_H^\dagger(t) \left[a_H(t), a_H(t) \right] + \left[a_H(t), a_H^\dagger(t) \right] a_H(t) \right) \\
&= \hbar\omega a_H(t)
\end{aligned}
$$

(The first commutator in the second-to-last line vanishes trivially, since $a_H(t)$ commutes with itself). Thus the Heisenberg equation of motion gives

$$\begin{aligned}\frac{da_H}{dt} &= \frac{1}{i\hbar}\left(\hbar\omega a_H(t)\right) \\ &= -i\omega a_H(t)\end{aligned}$$

This is a simple differential equation: its solution is

$$a_H(t) = a_H(0)e^{-i\omega t}$$

To get the initial condition $a_H(0)$, we go back to the definition of the Heisenberg operator:

$$a_H(t) = e^{iHt/\hbar}a_S e^{-iHt/\hbar}$$

We see easily then that

$$a_H(0) = a_S$$

and therefore we have the simple relationship

$$\boxed{a_H(t) = a_S e^{-i\omega t}}$$

(b) To get the corresponding time-evolution of $a_H^\dagger$, we could simply take the adjoint of $a_H(t)$, but let's work it out explicitly for practice. This time, we have

$$\begin{aligned}\frac{da_H^\dagger}{dt} &= \frac{1}{i\hbar}\left[a_H^\dagger, H\right] \\ &= \frac{1}{i\hbar}\hbar\omega\left[a_H^\dagger(t), a_H^\dagger(t)a_H(t) + \frac{1}{2}\right] \\ &= -i\omega\left[a_H^\dagger(t), a_H^\dagger(t)a_H(t)\right] \\ &= -i\omega\left(a_H^\dagger(t)\left[a_H^\dagger(t), a_H(t)\right] + \left[a_H^\dagger(t), a_H^\dagger(t)\right]a_H(t)\right) \\ &= i\omega a_H^\dagger(t)\end{aligned}$$

Thus we find that

$$a_H^\dagger(t) = a_H^\dagger(0)e^{i\omega t}$$

or

$$\boxed{a_H^\dagger(t) = a_S^\dagger e^{i\omega t}}$$

as expected.

(c) That the Hamiltonian is time-independent in the Heisenberg picture follows trivially from our work in part (a) above, where we showed that the Hamiltonian is in fact the same in both pictures (since it commutes with both $e^{iHt/\hbar}$ and $e^{-iHt/\hbar}$). However, we can show it explicitly using the results we just found:

$$\begin{aligned} H &= \hbar\omega\left(a_H^\dagger(t)a_H(t) + \frac{1}{2}\right) \\ &= \hbar\omega\left(a_S^\dagger e^{i\omega t}a_S e^{-i\omega t} + \frac{1}{2}\right) \\ &= \hbar\omega\left(a_S^\dagger a_S + \frac{1}{2}\right) \end{aligned}$$

So indeed, the individual time dependencies in a_H and $a_H^\dagger$ cancel themselves out to give a time-independent Hamiltonian.

Problem 2

(a) The Biot-Savart law states that in magnetostatics, the magnetic field created by an infinitesimal current element $\boldsymbol{I d\ell}$ at position $\boldsymbol{r'}$ is

$$\boldsymbol{dB}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{I d\ell} \times (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3}$$

Now, this formula is only true in magnetostatics (i.e. for a steady line current), so technically, it does not apply to our case. The reason is that for a changing current, we need to take into account retardation: since information can't travel faster than the speed of light, it takes time for the information about the changing current to propagate from $\boldsymbol{r'}$ to $\boldsymbol{r}$. However, assuming that this effect can be neglected (i.e. if we work in the nonrelativistic regime), then for a moving point charge we can approximate the moving current element as $\boldsymbol{I d\ell} = q\boldsymbol{v}$, and we find that the magnetic field created by a point charge moving with velocity $\boldsymbol{v}$ at position $\boldsymbol{r'}$ is

$$\boldsymbol{B}(\boldsymbol{r}) \approx \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v} \times (\boldsymbol{r} - \boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \qquad \text{(when } v/c \ll 1\text{)}$$

(Incidentally, even if we wanted the full relativistic formula, we couldn't get it from the given information. To get the exact answer, we'd need to know the trajectory of the particles, but all we're given are their position and velocity at some instant, so the nonrelativistic formulas are all we can get). Using this formula, we can calculate the magnetic field each particle feels due to the other. First, we have

$$\begin{aligned}
\boldsymbol{v_1} &= v_1\,\hat{x} \\
\boldsymbol{v_2} &= v_2\,\hat{y} \\
\boldsymbol{r_1} - \boldsymbol{r_2} &= d\,\hat{y}
\end{aligned}$$

The magnetic field that particle 1 feels due to particle 2 is thus

$$\begin{aligned}
\boldsymbol{B}_{12} &= \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}_2 \times (\boldsymbol{r}_1 - \boldsymbol{r}_2)}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} \\
&= \frac{\mu_0}{4\pi}\frac{qv_2}{d^2}\hat{y} \times \hat{y} \\
&= 0
\end{aligned}$$

Does this make sense? Sure! Particle 2 effectively creates a current pointing up along the y-axis, so the magnetic field lines (by the right-hand rule) should be going around the y-axis. But particle 1 is right on the y-axis, where there is no magnetic field from particle 2.

Next, the magnetic field that particle 2 feels due to particle 1 is

$$\begin{aligned}\boldsymbol{B}_{21} &= \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}_1 \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|^3} \\ &= \frac{\mu_0}{4\pi}\frac{qv_1}{d^2}\hat{x} \times (-\hat{y}) \\ &= -\frac{\mu_0}{4\pi}\frac{qv_1}{d^2}\hat{z}\end{aligned}$$

This makes sense too: particle 1 looks like a current traveling along the x-axis, so the magnetic field lines go around the x-axis, and so along the negative y-axis (where particle 2 is), the magnetic field should be pointing in the negative z-direction, as we found.

Finally, we need to calculate the electric field between the charges. This one's easy, since in the nonrelativistic regime, we just use Coulomb's law:

$$\begin{aligned}\boldsymbol{E}_{12} &= \frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\hat{y} \\ \boldsymbol{E}_{21} &= -\frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\hat{y}\end{aligned}$$

To find the force on each particle, we just use the Loretz force law:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

The force on particle 1 is then

$$\begin{aligned}\boldsymbol{F}_1 &= q\left(\boldsymbol{E}_{12} + \boldsymbol{v}_1 \times \boldsymbol{B}_{12}\right) \\ &= q\left(\frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\hat{y} + 0\right)\end{aligned}$$

$$\boxed{\boldsymbol{F}_1 = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{y}}$$

Next,

$$\begin{aligned}\boldsymbol{F}_2 &= q\left(\boldsymbol{E}_{21} + \boldsymbol{v}_2 \times \boldsymbol{B}_{21}\right) \\ &= q\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{d^2}\hat{y} - \frac{\mu_0}{4\pi}\frac{qv_1v_2}{d^2}\hat{y} \times \hat{z}\right)\end{aligned}$$

$$\boxed{\boldsymbol{F}_2 = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{y} - \frac{\mu_0}{4\pi}\frac{q^2v_1v_2}{d^2}\hat{x}}$$

Clearly, these forces are in violation of Newton's third law: the elecric forces between the particles are indeed equal and opposite, but the magnetic forces are not.

(b) In special relativity, kinetic energy (actually, total energy, but they just differ by a constant) is the time component of the momentum four-vector. The spatial components are (no surprise) the spatial momentum. Thus the momentum four-vector is

$$p^\mu = (E/c, \boldsymbol{p})$$

where the index μ takes the values 0, 1, 2, 3 (or t, x, y, z, if you prefer). The extra factor of c is simply needed so that all the components of p^μ have units of momentum. Likewise, the scalar potential ϕ is the time component of a four-vector whose spatial components are the vector potential $\boldsymbol{A}$. The four-vector potential is thus

$$A^\mu = (\phi/c, \boldsymbol{A})$$

Again, the factor of c is necessary to give all the components units of $\boldsymbol{A}$. At this point, we recognize that the conservation equation

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + q\phi\right) = 0$$

is simply the time-component of the four-vector conservation equation

$$\frac{d}{d\tau}\left(p^\mu + qA^\mu\right) = 0$$

(I switched from time coordinate t to proper time τ to emphasize the covariant nature of the equation; in the nonrelativistic limit, $\tau \approx t$ and therefore $d/d\tau \approx d/dt$). All we have to do now is extract the spatial components of the conservation equation to get

$$\frac{d}{d\tau}\left(\boldsymbol{p} + q\boldsymbol{A}\right) = 0$$

In the nonrelativistic limit, $p \approx m\boldsymbol{v}$, $d/d\tau \approx d/dt$, and thus we get

$$\boxed{\frac{d}{dt}\left(m\boldsymbol{v} + q\boldsymbol{A}\right) = 0}$$

Of course, this wasn't at all a proof; we simply motivated the above equation based on relativistic covariance. However, it is relatively simple to prove it from the Hamiltonian formalism. Recall that the (classical) Hamiltonian for a particle moving in an electromagnetic field is

$$H = \frac{\boldsymbol{p}_{\text{canon}}^2}{2m} + q\phi$$

where $\boldsymbol{p}_{\text{canon}}$ is the canonical momentum:

$$\boldsymbol{p}_{\text{canon}} = m\boldsymbol{v} + q\boldsymbol{A}$$

The Hamilton-Jacobi equations of motion tell us that

$$\begin{aligned}\frac{d\boldsymbol{p}_{\text{canon}}}{dt} &= -\frac{\partial H}{\partial \boldsymbol{x}} \\ &= -\nabla\phi \\ \frac{d}{dt}\left(m\boldsymbol{v} + q\boldsymbol{A}\right) &= -\nabla\phi\end{aligned}$$

In the presence of no external electric fields (as is required for conservation of momentum), $\nabla\phi = 0$, and we obtain the desired conservation equation.

(c) Our job is to evaluate the expression

$$\frac{d}{dt}\left(m\boldsymbol{v} + q\boldsymbol{A}\right)$$

for each particle. Individually, these will not necessarily be equal to zero, because each particle feels an external electrical force (the Coulomb force from the other particle). However, the combined two-particle system does *not* feel any external electrical force, and therefore we expect that the total momentum conservation law should hold:

$$\frac{d}{dt}\left(m\boldsymbol{v}_1 + q\boldsymbol{A}_{12}\right) + \frac{d}{dt}\left(m\boldsymbol{v}_2 + q\boldsymbol{A}_{21}\right) = 0$$

Let's begin by evaluating

$$\begin{aligned}\frac{d}{dt}\left(m\boldsymbol{v}_1 + q\boldsymbol{A}_{12}\right) &= m\boldsymbol{a}_1 + q\frac{d\boldsymbol{A}_{12}}{dt} \\ &= \boldsymbol{F}_1 + q\frac{dr_1^i}{dt}\left.\frac{d\boldsymbol{A}_{12}}{dr_1^i}\right|_{\boldsymbol{r}_1=\boldsymbol{0},\boldsymbol{r}_2=-d\hat{y}} \\ &= \boldsymbol{F}_1 + qv_1^i\left.\frac{d\boldsymbol{A}_{12}}{dr_1^i}\right|_{\boldsymbol{r}_1=\boldsymbol{0},\boldsymbol{r}_2=-d\hat{y}}\end{aligned}$$

where I used Newton's second law to write $\boldsymbol{F} = m\boldsymbol{a}$, and used the chain rule to write $d\boldsymbol{A}/dt = (d\boldsymbol{A}/dr^i)(dr^i/dt)$ (remember that there's an implied sum over the index $i = x, y, z$); the derivative is to be evaluated at the positions $\boldsymbol{r}_1 = \boldsymbol{0}, \boldsymbol{r}_2 = -d\hat{y}$. Since $\boldsymbol{v}_1 = v_1\hat{x}$, we get

$$\frac{d}{dt}\left(m\boldsymbol{v}_1 + q\boldsymbol{A}_{12}\right) = \boldsymbol{F}_1 + qv_1\left.\frac{d\boldsymbol{A}_{12}}{dx_1}\right|_{\boldsymbol{r}_1=\boldsymbol{0},\boldsymbol{r}_2=-d\hat{y}}$$

The problem is now to calculate the vector potential felt by particle 1 due to particle 2, $\boldsymbol{A}_{12}$. There is no unique choice for the vector potential (because of gauge freedom), but a handy choice for the (nonrelativistic) vector potential due to a point charge with velocity $\boldsymbol{v}$ at position $\boldsymbol{r}'$ is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

Thus

$$\begin{aligned}\boldsymbol{A}_{12} &= \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \\ &= \frac{\mu_0}{4\pi}\frac{qv_2}{r}\hat{y}\end{aligned}$$

where I've now left the distance between particles 1 and 2 a variable r (instead of fixed d) because I'll be differentiating with respect to it; in particular,

$$r \equiv \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

We now have

$$\begin{aligned}\frac{d}{dt}\left(m\boldsymbol{v}_1 + q\boldsymbol{A}_{12}\right) &= \boldsymbol{F}_1 + qv_1 \left.\frac{d\boldsymbol{A}_{12}}{dx_1}\right|_{\boldsymbol{r}_1=\boldsymbol{0},\boldsymbol{r}_2=-d\hat{y}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{y} + \frac{\mu_0}{4\pi}q^2 v_1 v_2 \left.\frac{d}{dx_1}\frac{1}{r}\right|_{\boldsymbol{r}_1=\boldsymbol{0},\boldsymbol{r}_2=-d\hat{y}} \hat{y} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{y} + \frac{\mu_0}{4\pi}q^2 v_1 v_2 \left[\frac{x_2 - x_1}{r^3}\right]_{\boldsymbol{r}_1=\boldsymbol{0},\boldsymbol{r}_2=-d\hat{y}} \hat{y}\end{aligned}$$

where I used the fact that

$$\frac{d}{dx}\frac{1}{r} = \frac{x_2 - x_1}{r^3}$$

Now, note that at $\boldsymbol{r}_1 = \boldsymbol{0}$ and $\boldsymbol{r}_2 = -d\hat{y}$, we have $x_2 - x_1 = 0$, and therefore the second term just evaluates to zero. Thus

$$\frac{d}{dt}\left(m\boldsymbol{v}_1 + q\boldsymbol{A}_{12}\right) = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{y}$$

As promised, momentum is not conserved for this single particle, because of the electric force from the second particle (had we ignored electric forces and only dealt with magnetic forces, then this expression would indeed have been zero). Now, on to the second particle:

$$\begin{aligned}\frac{d}{dt}\left(m\boldsymbol{v}_2 + q\boldsymbol{A}_{21}\right) &= m\boldsymbol{a}_2 + q\frac{d\boldsymbol{A}_{21}}{dt} \\ &= \boldsymbol{F}_2 + q\frac{dr_2^i}{dt}\left.\frac{d\boldsymbol{A}_{21}}{dr_2^i}\right|_{\boldsymbol{r}_1=\boldsymbol{0},\boldsymbol{r}_2=-d\hat{y}} \\ &= \boldsymbol{F}_2 + qv_2^i \left.\frac{d\boldsymbol{A}_{21}}{dr_2^i}\right|_{\boldsymbol{r}_1=\boldsymbol{0},\boldsymbol{r}_2=-d\hat{y}}\end{aligned}$$

Using $\boldsymbol{v}_2 = v_2\hat{y}$, we get

$$\frac{d}{dt}\left(m\boldsymbol{v}_2 + q\boldsymbol{A}_{21}\right) = \boldsymbol{F}_2 + qv_2 \left.\frac{d\boldsymbol{A}_{21}}{dy_2}\right|_{\boldsymbol{r}_1=\boldsymbol{0},\boldsymbol{r}_2=-d\hat{y}}$$

This time, we have

$$\begin{aligned}
\boldsymbol{A}_{21} &= \frac{\mu_0}{4\pi} \frac{q\boldsymbol{v}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} \\
&= \frac{\mu_0}{4\pi} \frac{qv_1}{r} \hat{x}
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dt}\left(m\boldsymbol{v}_2 + q\boldsymbol{A}_{21}\right) &= \boldsymbol{F}_2 + qv_2 \left.\frac{d\boldsymbol{A}_{21}}{dy_2}\right|_{\boldsymbol{r}_1=\boldsymbol{0}, \boldsymbol{r}_2=-d\hat{x}} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{y} - \frac{\mu_0}{4\pi}\frac{q^2 v_1 v_2}{d^2}\hat{x} + \frac{\mu_0}{4\pi} q^2 v_1 v_2 \left.\frac{d}{dy_2}\frac{1}{r}\right|_{\boldsymbol{r}_1=\boldsymbol{0}, \boldsymbol{r}_2=-d\hat{y}} \hat{x} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{y} - \frac{\mu_0}{4\pi}\frac{q^2 v_1 v_2}{d^2}\hat{x} + \frac{\mu_0}{4\pi} q^2 v_1 v_2 \left[\frac{y_1 - y_2}{r^3}\right]_{\boldsymbol{r}_1=\boldsymbol{0}, \boldsymbol{r}_2=-d\hat{y}} \hat{x}
\end{aligned}$$

But at $\boldsymbol{r}_1 = \boldsymbol{0}$ and $\boldsymbol{r}_2 = -d\hat{y}$, $y_1 - y_2 = d$ and $r = d$, so

$$\begin{aligned}
\frac{d}{dt}\left(m\boldsymbol{v}_2 + q\boldsymbol{A}_{21}\right) &= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{y} - \frac{\mu_0}{4\pi}\frac{q^2 v_1 v_2}{d^2}\hat{x} + \frac{\mu_0}{4\pi} q^2 v_1 v_2 \frac{d}{d^3}\hat{x} \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\hat{y}
\end{aligned}$$

Note that the contributions from the magnetic fields do indeed cancel out, leaving just the electric field stuff; again, if we were to ignore the Coulomb force and just focus on the magnetic fields, we would have found that the "complete" momentum was conserved for just this one particle.

Combining these results, we find that

$$\boxed{\frac{d}{dt}\left(m\boldsymbol{v}_1 + q\boldsymbol{A}_{12}\right) + \frac{d}{dt}\left(m\boldsymbol{v}_2 + q\boldsymbol{A}_{21}\right) = 0}$$

So the "complete" momentum is indeed conserved when considering the entire two-particle system, because there are no external electrical forces acting on it.

Problem 3

Let's first think about this system physically. Before we change the magnetic field, the system consists of a stationary ring and static magnetic and electric fields. In general, electromagnetic fields can carry momentum (linear and angular), but because of the rotational symmetry of the system, we expect the electromagnetic fields to carry no total momentum.

When we increase the magnetic field, there is a corresponding increase of magnetic flux through the ring. The conductor will oppose this increase in flux by generating a current to oppose it; if we take the $+z$ direction parallel to the magnetic field, then the ring will generate a current rotating clockwise when viewed from above (depending on whether the ring is conducting or insulating, it may remain stationary or begin to rotate, but either way the charge carriers will carry some mechanical angular momentum). This will be accompanied by a net angular momentum, which we expect will be cancelled out by a corresponding change in the angular momentum of the electromagnetic field.

Looking back at our equations, our goal is to show that

$$\frac{d}{dt}\left(\boldsymbol{r} \times \boldsymbol{p}_{\text{canon}}\right) = 0$$

Using our expression for the canonical momentum, we want to show that

$$\frac{d}{dt}\left(m\boldsymbol{r} \times \boldsymbol{v} + q\boldsymbol{r} \times \boldsymbol{A}\right) = 0$$

or

$$\frac{d}{dt}\left(m\boldsymbol{r} \times \boldsymbol{v}\right) = -\frac{d}{dt}\left(q\boldsymbol{r} \times \boldsymbol{A}\right)$$

The left-hand side of the above equation represents the change in angular momentum of the ring, while the right-hand side represents the change in the angular momentum stored in the electromagnetic field, as discussed.

To show that this equality is true, let's begin by working on the left-hand side, i.e. by working out the change in the angular momentum of the ring. From the angular version of Newton's second law, $d\boldsymbol{L}/dt = \boldsymbol{\tau}$, we have that

$$\frac{d}{dt}\left(m\boldsymbol{r} \times \boldsymbol{v}\right) = \boldsymbol{\tau}$$

So the left-hand side is nothing more than the torque exerted on the ring during the change in the magnetic field. Next, if we integrate Maxwell's equation

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

over the area inside the ring and apply Stokes' theorem, we get Faraday's law:

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt}$$

where Φ is the magnetic flux through the loop, and the integral is taken around the loop. The area of the loop is πR^2, so the change in the flux is

$$\frac{d\Phi}{dt} = \pi R^2 \frac{\delta B}{\delta t}$$

(I'll denote the change in the magnetic field by δB in time δt, rather than use dB and dt, just to make clear what's a derivative and what's just an ordinary fraction). Thus from Faraday's law, we have

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\pi R^2 \frac{\delta B}{\delta t}$$

Now, the force on a segment $d\ell$ of the ring is $dF = Edq$, where $dq = qd\ell/2\pi R$ is the charge of the segment (the charge per unit length is $q/2\pi R$). The torque on this small segment is thus

$$\begin{aligned} d\tau &= |\boldsymbol{r} \times \boldsymbol{dF}| \\ &= RdF \\ &= \frac{REqd\ell}{2\pi R} \\ &= \frac{q}{2\pi} Ed\ell \end{aligned}$$

The total torque is thus

$$\tau = \frac{q}{2\pi} \oint Ed\ell$$

But Faraday's law tells us what the line integral of the electric field around the loop is; using our previous result (and momentarily ignoring overall minus signs, which we'll restore later), we find that the total torque is

$$\begin{aligned} \tau &= \frac{q}{2\pi} \oint Ed\ell \\ &= \frac{qR^2 \delta B}{2\delta t} \end{aligned}$$

One final issue: we need the *vector* torque, not just its magnitude. To figure out which direction the torque points, recall that we had set up our coordinate system so that the z-axis is parallel to $\boldsymbol{B}$. Now, an increase in the magnetic field will induce a current in the ring to try to oppose it: thus positive charges in the ring will travel clockwise as viewed from above. By the right-hand rule, the torque necessary to induce this motion of the charges must point in the negative z-direction; or, alternatively, in the $-\boldsymbol{\delta B}$ direction, so the vector torque is

$$\boldsymbol{\tau} = -\frac{1}{2} q R^2 \frac{\boldsymbol{\delta B}}{\delta t}$$

(Note that if the sign of the charges is changes or the field is decreased instead of increased, the direction of $\boldsymbol{\tau}$ changes appropriately). In conclusion, we have found that the change in the mechanical angular momentum of the ring is

$$\frac{d}{dt}\left(m\boldsymbol{r}\times\boldsymbol{v}\right) = -\frac{1}{2}qR^2\frac{\boldsymbol{\delta B}}{\delta t}$$

Alrighty. Next, we move on to calculating the change in the angular momentum stored in the fields:

$$\frac{d}{dt}\left(q\boldsymbol{r}\times\boldsymbol{A}\right)$$

To calculate this quantity, we'll need to pick a gauge for $\boldsymbol{A}$. For our constant magnetic field, a gauge choice with a nice rotational symmetry well-adapted to this problem is the Coulomb gauge $\nabla\cdot\boldsymbol{A}=0$. In this gauge, we can write the vector potential for a uniform and constant magnetic field as

$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r}$$

Then we obtain

$$\begin{aligned}\frac{d}{dt}\left(q\boldsymbol{r}\times\boldsymbol{A}\right) &= q\boldsymbol{r}\times\frac{d\boldsymbol{A}}{dt}\\ &= \frac{1}{2}q\boldsymbol{r}\times\left(\frac{d\boldsymbol{B}}{dt}\times\boldsymbol{r}\right)\\ &= \frac{1}{2\delta t}q\boldsymbol{r}\times\left(\boldsymbol{\delta B}\times\boldsymbol{r}\right)\end{aligned}$$

(Since the radius of the ring doesn't change during the process, I freely carried time deriviatives through any $\boldsymbol{r}$s). Now, $\boldsymbol{\delta B}$ points along the axis of the ring, while $\boldsymbol{r}$ points perpendicular to it; by the right-hand rule, $\boldsymbol{\delta B}\times\boldsymbol{r}$ will have magnitude $R\delta B$ and point along a counterclockwise tangent to the circle (again, as viewed from above). This, too, is perpendicular to $\boldsymbol{r}$, so $\boldsymbol{r}\times(\boldsymbol{\delta B}\times\boldsymbol{r})$ will have magnitude $R^2\delta B$ and point along the axis of the circle, in the $\boldsymbol{\delta B}$ direction. Thus

$$\boldsymbol{r}\times\left(\boldsymbol{\delta B}\times\boldsymbol{r}\right) = R^2\boldsymbol{\delta B}$$

(this can also be computed by evaluating the cross products by brute force, but with orthogonal vectors like we have here, it's easier to just figure out via the right-hand rules). Then

$$\frac{d}{dt}\left(q\boldsymbol{r}\times\boldsymbol{A}\right) = \frac{1}{2}qR^2\frac{\boldsymbol{\delta B}}{\delta t}$$

and thus, lo and behold, we have shown explicitly that

$$\begin{aligned}
\frac{d\boldsymbol{L}_{\text{canon}}}{dt} &= \frac{d}{dt}\left(m\boldsymbol{r} \times \boldsymbol{v} + q\boldsymbol{r} \times \boldsymbol{A}\right) \\
&= \frac{d}{dt}\left(m\boldsymbol{r} \times \boldsymbol{v}\right) + \frac{d}{dt}\left(q\boldsymbol{r} \times \boldsymbol{A}\right) \\
&= -\frac{1}{2}qR^2\frac{\boldsymbol{\delta B}}{\delta t} + \frac{1}{2}qR^2\frac{\boldsymbol{\delta B}}{\delta t} \\
&= 0
\end{aligned}$$

Remarkably, even though the mechanical angular momentum of the charges in the ring and the angular momentum stored in the EM fields were not conserved separately, the grand total canonical angular momentum was indeed conserved. Cool!

Problem 4

To start, let's calculate the resonant frequency for a transition from the ground state to the first excited state of this infinite square well. This is given by

$$\begin{aligned}\omega_{12} &\equiv \frac{E_2 - E_1}{\hbar} \\ &= \frac{4E_1 - E_1}{\hbar} \\ &= \frac{3E_1}{\hbar} \\ &= \frac{3\pi^2\hbar}{2mL^2}\end{aligned}$$

Plugging in our numbers (with m the mass of an electron), we get

$$\omega_{12} \approx 1.714 \times 10^{15} \text{ rad/s}$$

This is very close to the three driving frequencies we're asked to consider, which means this perturbation should excite a resonance between these two energy levels. Now, let's get calculating: from first-order perturbation theory, the probability of transitioning from the ground state to the first excited state is $|c_{12}|^2$, where

$$c_{12} = \delta_{12} - \frac{i}{\hbar}\int_0^T e^{-i(E_2-E_1)t/\hbar} \langle 2|\, H' \,|1\rangle \; dt$$

where in our case $T = 1 \times 10^{-15}$ s, and the perturbing Hamiltonian is

$$H' = V_0 x \cos\omega t$$

Let's begin by calculating the matrix element:

$$\begin{aligned}\langle 2|\, H' \,|1\rangle &= V_0 \cos\omega t \,\langle 2|\, x \,|1\rangle \\ &= V_0 \cos\omega t \frac{2}{L}\int_0^L x \sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{\pi}{L}x\right)\; dx\end{aligned}$$

The integral should be familiar to you from 115A (I did it on Homework 4 this past fall, if you took it this year). It comes out to be

$$\int_0^L x \sin\left(\frac{2\pi}{L}x\right)\sin\left(\frac{\pi}{L}x\right)\; dx = -\frac{8L^2}{9\pi^2}$$

and thus

$$\begin{aligned}\langle 2|\, H' \,|1\rangle &= -V_0 \cos\omega t \frac{2}{L}\frac{8L^2}{9\pi^2} \\ &= -\frac{16V_0 L}{9\pi^2}\cos\omega t\end{aligned}$$

This will give

$$
\begin{aligned}
c_{12} &= \frac{i}{\hbar}\frac{16V_0L}{9\pi^2}\int_0^T e^{-i(E_2-E_1)t/\hbar}\cos\omega t\,dt \\
&= \frac{16iV_0L}{9\pi^2\hbar}\int_0^T e^{-i\omega_{12}t}\cos\omega t\,dt \\
&= \frac{16iV_0L}{9\pi^2\hbar}\int_0^T e^{-i\omega_{12}t}\frac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)\,dt \\
&= \frac{8iV_0L}{9\pi^2\hbar}\int_0^T \left(e^{-i(\omega_{12}-\omega)t}+e^{-i(\omega_{12}+\omega)t}\right)\,dt \\
&= \frac{8iV_0L}{9\pi^2\hbar}\left[\frac{1}{-i(\omega_{12}-\omega)}e^{-i(\omega_{12}-\omega)t}+\frac{1}{-i(\omega_{12}+\omega)}e^{-i(\omega_{12}+\omega)t}\right]_0^T \\
&= \frac{8iV_0L}{9\pi^2\hbar}\left(\frac{e^{-i(\omega_{12}-\omega)T}-1}{-i(\omega_{12}-\omega)}+\frac{e^{-i(\omega_{12}+\omega)T}-1}{-i(\omega_{12}+\omega)}\right)
\end{aligned}
$$

At this point, we could go ahead and square this expression exactly, but we can make our lives easier by remembering that we're close to resonance, i.e. that our driving frequency ω is close to the resonant frequency ω_{12}. In particular, $\omega_{12}-\omega \ll \omega_{12}+\omega$, so the first term in the above expression dominates, and we can go ahead and drop the second term. Thus near resonance,

$$
c_{12} \approx -\frac{8V_0L}{9\pi^2\hbar}\frac{e^{-i(\omega_{12}-\omega)T}-1}{\omega_{12}-\omega}
$$

The transition probability is then

$$
\begin{aligned}
P_{12} &= |c_{12}|^2 \\
&\approx \left|\frac{8V_0L}{9\pi^2\hbar}\frac{e^{-i(\omega_{12}-\omega)T}-1}{\omega_{12}-\omega}\right|^2 \\
&= \left(\frac{8V_0L}{9\pi^2\hbar}\right)^2\frac{\left(e^{-i(\omega_{12}-\omega)T}-1\right)\left(e^{i(\omega_{12}-\omega)T}-1\right)}{(\omega_{12}-\omega)^2} \\
&= \left(\frac{8V_0L}{9\pi^2\hbar}\right)^2\frac{2-2\cos((\omega_{12}-\omega)T)}{(\omega_{12}-\omega)^2} \\
&= \left(\frac{16V_0L}{9\pi^2\hbar}\right)^2\frac{\sin^2((\omega_{12}-\omega)T/2)}{(\omega_{12}-\omega)^2}
\end{aligned}
$$

Now, we had $V_0 = 1\times10^{-2}$ eV/m, and $L = 1\times10^{-9}$ m, so

$$
\left(\frac{16V_0L}{9\pi^2\hbar}\right) \approx 2.74\times10^3 \text{ rad/s}
$$

so

$$
P_{12} = \left(\frac{2.74\times10^3 \text{ rad/s}}{(\omega_{12}-\omega)^2}\right)^2\sin^2((\omega_{12}-\omega)T/2)
$$

Now, let's start plugging in our numbers. First, we have $\omega = 1.537 \times 10^{15}$ rad/s, so

$$P_{12} = \left(\frac{2.74 \times 10^3 \text{ rad/s}}{(1.714 - 1.537) \times 10^{15} \text{ rad/s}}\right)^2 \sin^2(((1.714 - 1.537) \times 10^{15} \text{ rad/s})(1.0 \times 10^{-15} \text{ s})/2)$$

$$\boxed{P_{12} \approx 1.872 \times 10^{-24}}$$

Next, for $\omega = 1.691 \times 10^{15}$ rad/s, we have

$$P_{12} = \left(\frac{2.74 \times 10^3 \text{ rad/s}}{(1.714 - 1.691) \times 10^{15} \text{ rad/s}}\right)^2 \sin^2(((1.714 - 1.691) \times 10^{15} \text{ rad/s})(1.0 \times 10^{-15} \text{ s})/2)$$

$$\boxed{P_{12} \approx 1.877 \times 10^{-24}}$$

Finally, for $\omega = 1.706 \times 10^{15}$ rad/s, we have

$$P_{12} = \left(\frac{2.74 \times 10^3 \text{ rad/s}}{(1.714 - 1.706) \times 10^{15} \text{ rad/s}}\right)^2 \sin^2(((1.714 - 1.706) \times 10^{15} \text{ rad/s})(1.0 \times 10^{-15} \text{ s})/2)$$

$$\boxed{P_{12} \approx 1.877 \times 10^{-24}}$$

Note that these probabilities are almost identical. That's because near resonance (and for small enough T), we can expand

$$\sin^2((\omega_{12} - \omega)T/2) \approx \frac{1}{4}(\omega_{12} - \omega)^2 T^2$$

giving

$$P_{12} \approx \left(\frac{8V_0 LT}{9\pi^2 \hbar}\right)^2$$

So near resonance, the transition probability attains its maximum value (for fixed T), and depends only on T (but not not ω).