Functions of a Complex Variable

Complex Algebra

Formally, the set of complex numbers can be defined as the set of two-dimensional real vectors, $\{(x, y)\}$, with one extra operation, *complex multiplication*:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1\, x_2 - y_1\, y_2,\ x_1\, y_2 + x_2\, y_1) \ . \tag{1}$$

Together with generic vector addition

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2) \ , \tag{2}$$

the two operations define *complex algebra*.

$\diamond$ With the rules (1)-(2), complex numbers include the real numbers as a subset $\{(x, 0)\}$ with usual real number algebra. This suggests short-hand notation $(x, 0) \equiv x$; in particular: $(1, 0) \equiv 1$.

$\diamond$ Complex algebra features commutativity, distributivity and associativity.

The two numbers, $1 = (1, 0)$ and $i = (0, 1)$ play a special role. They form a basis in the vector space, so that each complex number can be represented in a unique way as [we start using the notation $(x, 0) \equiv x$]

$$(x, y) = x + iy \ . \tag{3}$$

$\diamond$ Terminology: The number i is called imaginary unity. For the complex number $z = (x, y)$, the real umbers x and y are called real and imaginary parts, respectively; corresponding notation is: $x = \mathrm{Re}\, z$ and $y = \mathrm{Im}\, z$.

The following remarkable property of the number i,

$$i^2 \equiv i \cdot i = -1 \ , \tag{4}$$

renders the representation (3) most convenient for practical algebraic manipulations with complex numbers.—One treats x, y, and i the same way as the real numbers.

Another useful parametrization of complex numbers follows from the geometrical interpretation of the complex number $z = (x, y)$ as a point in a 2D plane, referred to in this context as *complex plane.* Introducing polar coordinates, the radius $r = \sqrt{x^2 + y^2}$ and the angle $\theta = \tan^{-1}(y/x)$, one gets

$$x + iy = r(\cos\theta + i\sin\theta) \ . \tag{5}$$

◇ Terminology and notation: Radius r is called *modulus* (and also *magnitude*) of the complex number, $r = |z|$. The angle θ is called *phase* (and also *argument*) of the complex number, $\theta = \arg(z)$. Note an ambiguity in the definition of the phase of a complex number. It is defined up to an additive multiple of 2π.

Since the modulus of a complex number is nothing else than the magnitude of corresponding vector, the standard vector inequalities are applicable:

$$|\,|z_1| - |z_2|\,| \ \leq \ |z_1 + z_2| \ \leq \ |z_1| + |z_2| \ . \tag{6}$$

Parametrization in terms of modulus and phase is convenient for multiplication, because if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \ . \tag{7}$$

Polar parametrization is also convenient for the division which is considered below.

Subtraction and division of complex numbers are defined as the operations opposite to addition and multiplication, respectively. Subtraction thus simply corresponds to vector subtraction. Division of complex numbers can be actually reduced to multiplication. But first we need to introduce one more important operation, *complex conjugation.* For each complex number $z = x + iy$ we define its complex conjugate as

$$z^* = x - iy \tag{8}$$

and note that

$$zz^* = |z|^2 \tag{9}$$

is a real number. Then for any two complex numbers z_1 and z_2 the operation of division can be written as

$$\frac{z_1}{z_2} = |z_2|^{-2} z_1 z_2^* \ . \tag{10}$$

The validity of this relation is checked by multiplying the right-hand side by z_2. In modulus-phase parametrization, Eq. (10) reads

$$z_1/z_2 \; = \; (r_1/r_2) \left[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right] . \tag{11}$$

Problem 1. Establish relations between complex multiplication and inner/outer vector product: Treating the two pairs, (x_1, y_1), (x_2, y_2), as both two vectors in the xy plane, $(x_1, y_1) = \vec{a}$, $(x_2, y_2) = \vec{b}$, and two complex numbers, $(x_1, y_1) = a$, $(x_2, y_2) = b$, make sure that

$$\vec{a} \cdot \vec{b} = (1/2)(a\, b^* + b\, a^*) , \tag{12}$$

$$\vec{a} \times \vec{b} = (i/2)(a\, b^* - b\, a^*)\, \hat{\mathbf{z}} , \tag{13}$$

where $\hat{\mathbf{z}}$ is the unit vector along the z-direction.

Functions of a Complex Variable

A complex function $w = u + iv$ of a complex variable $z = x + iy$ is introduced as a complex-valued function of two real variables, x and y:

$$w(z) = u(x, y) + iv(x, y) . \tag{14}$$

Hence, to specify a complex function it is enough to specify two real functions: $u(x, y)$ and $v(x, y)$.

Partial derivatives are defined as

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\, \frac{\partial v}{\partial x} , \qquad \frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\, \frac{\partial v}{\partial y} . \tag{15}$$

We now formally define partials $\partial/\partial z$ and $\partial/\partial z^*$ as

$$\frac{\partial w}{\partial z} = \frac{1}{2} \left(\frac{\partial w}{\partial x} - i\, \frac{\partial w}{\partial y} \right) , \qquad \frac{\partial w}{\partial z^*} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + i\, \frac{\partial w}{\partial y} \right) . \tag{16}$$

The idea behind definitions (16) is in the following observation. Suppose $w(z)$ is specified not in terms of x and y, but in the form of some finite algebraic expression or infinite series, $w(z, z^*)$, involving z and z^*. In this expression one can formally replace complex numbers z and z^* with two *independent* real variables: $z \to a$, $z^* \to b$ and arrive at the function $w(a, b)$. It is easy to see then from (15)-(16) that $\partial w(z, z^*)/\partial z$ is equal to $\partial w(a,b)/\partial a, \quad a \to z, \; b \to z^*$, and $\partial w(z,z^*)/\partial z^*$ is equal to $\partial w(a,b)/\partial b, \quad a \to z, \; b \to z^*$. That is with respect to the operations $\partial/\partial z$ and $\partial/\partial z^*$ the variables z and z^* behave as independent real variables. This essentially simplifies calculation of partials. [For example, if $w(z,z^*) = zz^*$, then $\partial w/\partial z = z^*$ and $\partial w/\partial z^* = z$.]

Problem 2. Prove the above-mentioned general property of the operations $\partial/\partial z$ and $\partial/\partial z^*$. *Hint.* Since the formal rules of complex and real algebras are the same, the standard differentiating rules are applicable to complex-valued functions when differentiated with respect to x and y.

Consider the variation, δw, of the function $w(z,z^*)$ corresponding to $z \to z+\delta z$, where $\delta z = \delta x + i\delta y$ (and implying $z^* \to z^* + \delta z^*$, $\delta z^* = \delta x - i\delta y$). As can be readily checked with the definitions (16),

$$\delta w = \delta z \, \frac{\partial w}{\partial z} + \delta z^* \frac{\partial w}{\partial z^*} \; . \tag{17}$$

Problem 3. Check Eq. (17).

Note that while δz and δz^* essentially depend on each other, the expression (17) formally looks like they were independent variables.

Relations for partials in modulus-phase parametrization read:

$$r\,\partial/\partial r = x\,\partial/\partial x + y\,\partial/\partial y = z\,\partial/\partial z + z^*\,\partial/\partial z^* \; , \tag{18}$$

$$\partial/\partial\theta = x\,\partial/\partial y - y\,\partial/\partial x = i(z\,\partial/\partial z - z^*\,\partial/\partial z^*) \; , \tag{19}$$

$$z\,\partial/\partial z = (1/2)(r\,\partial/\partial r - i\partial/\partial\theta) \; , \quad z^*\,\partial/\partial z^* = (1/2)(r\,\partial/\partial r + i\partial/\partial\theta) \; . \tag{20}$$

Problem 4. Prove these relations.

How do we construct complex functions? The simplest way is to take a real expression involving four arithmetic operations with one (or two) real numbers a (and b) and replace in it a with a complex variable z (and b with z^*). A more powerful way is to use a power series.

A very important sub-set of complex functions is formed by functions that depend only on z, but not on z^*—in the sense that corresponding real

arithmetic expression (or power series) involves only one variable, a, which is then replaced with z. Clearly, for all such functions the operation $\partial/\partial z^*$ yields zero.
Examples:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \; , \tag{21}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \; , \qquad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \; , \tag{22}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \; , \qquad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \; , \tag{23}$$

All the series in (21)-(23) are convergent and the functions are well defined for any z, coinciding with corresponding real functions at real z. *Very important:* The complex functions defined this way feature *all* the functional and differential relations characteristic of corresponding real functions, because (i) these relations are captured algebraically by the power series and (ii) the real and complex algebras coincide. For example, $e^{z_1+z_2} = e^{z_1}e^{z_2}$, $(e^z)^n = e^{nz}$, $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$, $\partial e^z/\partial z = e^z$, $\partial \sin z/\partial z = \cos z$, etc.

Quite amazingly, *new* functional relations arise.
Examples:

$$e^{iz} = \cos z + i \sin z \; , \tag{24}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \cosh(iz) \; , \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} = -i \sinh(iz) \; , \tag{25}$$

$$\cosh z = \cos(iz) \; , \qquad \sinh z = -i \sin(iz) \; . \tag{26}$$

Problem 5. Choose any two of these relations and prove them by direct comparison of power series.

With the relation (24), the polar representation of a complex number, Eq. (5), can be written as

$$z = re^{i\theta} \; , \tag{27}$$

after which relations (7) and (11) become most transparent.

Analytic Functions

We have considered partial complex derivatives. Now we introduce the notion of a total derivative by the formula

$$\frac{dw}{dz} = \lim_{|\Delta z|\to 0} \frac{\Delta w}{\Delta z} \, , \tag{28}$$

and immediately realize that in a *general* case of a complex-valued function of z, our definition is quite pathological. Indeed, from Eq. (17)

$$\lim_{|\Delta z|\to 0} \frac{\Delta w}{\Delta z} = \frac{\partial w}{\partial z} + \frac{\partial w}{\partial z^*} \lim_{|\Delta z|\to 0} \frac{\Delta z^*}{\Delta z} \, . \tag{29}$$

Now if $\Delta z = |\Delta z| e^{i\theta}$, then $\Delta z^*/\Delta z = e^{-i2\theta}$, and the limit $|\Delta z| \to 0$ does not fix the phase θ. The only meaningful situation arises when

$$\frac{\partial w}{\partial z^*} = 0 \, , \tag{30}$$

in which case the derivative is well defined and is equal to

$$\frac{dw}{dz} = \frac{\partial w}{\partial z} \, . \tag{31}$$

The differentiable in the sense of Eq. (28) functions play an extremely important role. From now on we will be dealing with such functions only. The requirement (30) necessary (and sufficient, if $\partial w/\partial z$ exists) for the function w to be differentiable is the famous Cauchy-Riemann conditions, which in the component notation ($w = u + iv$) read

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \, , \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \, . \tag{32}$$

$\diamond$ Terminology and notation: If function $f(z)$ is differentiable in the sense of Eq. (28) at $z = z_0$ and in some region around z_0, it is called *analytic* at $z = z_0$. If df/dz does not exist at $z = z_0$, then z_0 is called singular point. If $f(z)$ is analytic everywhere in the complex plane, it is called *entire* function. To distinguish analytic functions from generic complex-valued functions of complex variable, we use the notation $f(z)$ for the former and $w(z, z^*)$ for the latter.

Some consequences of Cauchy-Riemann conditions:

(i) The curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal to each other (here c_1 and c_2 are some constants).
(ii) Both u and v are *harmonic functions*, that is they satisfy Laplace's equation:

$$\nabla^2 u = 0 \;, \qquad \nabla^2 v = 0 \;. \tag{33}$$

(iii) For the two real vector fields,

$$\mathbf{A} \;=\; (u,\, -v) \;, \quad \mathbf{B} \;=\; (v,\, u) \;, \tag{34}$$

one has

$$\nabla \cdot \mathbf{A} = 0 \;, \quad \nabla \times \mathbf{A} = 0 \;, \quad \nabla \cdot \mathbf{B} = 0 \;, \quad \nabla \times \mathbf{B} = 0 \;. \tag{35}$$

Problem 6. Show (i)-(iii).

Eqs. (35) mean that each analytic function gives us “for free” two solutions of E&M static problems (in the case when the problem is invariant with respect to translations in the z-direction, so that all the fields depend only on x and y).

Problem 7. Which two E&M problems are solved by the function $f(z) = 1/(z - z_0)$? (Explain why.)

Problem 8. Which of the following functions of the complex variable $z = x + iy = re^{i\theta}$ are analytic (almost everywhere except for some special points or lines), and which ones are not analytic?
(i) $f(z) = x^2 - y^2$
(ii) $f(z) = x^2 + 2ixy - y^2$
(iii) $f(z) = x^2 + iy^2$
(iv) $f(z) = i\theta + \ln r$
(v) $f(z) = r^2(\cos\theta + i\sin\theta)$
(vi) $f(z) = r^\alpha \mathrm{e}^{i\alpha\theta}$, where α is some real number.

Problem 9. Calculate df/dz for the analytic functions of the Problem 8, and $\partial f/\partial z$ for the rest of them.

Contour Integrals

Consider a contour C in the complex plane, going from the point $z_a = x_a + iy_a$ to the point $z_b = x_b + iy_b$. [At this point by *contour* we mean just a smooth arc. It will be clear, however, that the notion of contour is easily generalized to a piecewise smooth arc—a continuous curve consisting of a finite number of smooth arcs.] Without loss of generality, we assume that the contour is specified in a parametric form with a real parameter t: $z = x(t) + iy(t), \;\; t \in [t_a, \, t_b], \;\; z(t_a) = z_a, \;\; z(t_b) = z_b$. We introduce the integral along the contour C the same way one introduces the line integrals: Divide the interval $[t_a, \, t_b]$ into n equal (to be specific) intervals by picking $(n-1)$ intermediate points $t_1, t_2, \ldots$, and define

$$\int_C f(z)\, dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(z(\tilde{t}_j))\, (z_j - z_{j-1}) \;, \tag{36}$$

where $z_j = z(t_j)$, $t_0 \equiv t_a$, $t_n \equiv t_b$, $\tilde{t}_j$ is any point from the interval $[t_j, t_{j+1}]$. If the limit exists, then the integral is defined and is equal to

$$\int_C f(z)\, dz = \int_{t_a}^{t_b} f(z(t))\, z'(t)\, dt \;. \tag{37}$$

Eq. (37) reduces the contour integral to two ordinary real integrals and can be used for practical performing the integration.

Does the integral (36) depend on the particular choice of the parametrization $z(t)$? Actually it is independent not only of the parametrization, but—to a large extent—on the form of the contour, provided the end points are fixed. Substituting in Eq. (37) $u + iv$ for f and $x'(t) + iy'(t)$ for $z'(t)$, and taking advantage of definitions (34), we reduce the complex integral to two line integrals of the real vector fields $\mathbf{A} = (u, -v)$ and $\mathbf{B} = (v, u)$:

$$\int_C f(z)\, dz = \int_C \mathbf{A} \cdot d\mathbf{l} + i \int_C \mathbf{B} \cdot d\mathbf{l} \;. \tag{38}$$

Independence of the parametrization is thus automatically proven. [When deriving (38) we used the standard relation $d\mathbf{l} \equiv (x'(t), y'(t))dt$, which is transparent if one interprets $d\mathbf{l}$ as the infinitesimal displacement of a point along the line, t as the time, and, correspondingly, $(x'(t), y'(t))$ as the velocity vector.]

Cauchy's integral theorem. If a function $f(z)$ is analytic throughout some simply connected region R, then for every closed path C in R the integral of $f(z)$ around C is zero. The proof readily follows from (38). Applying

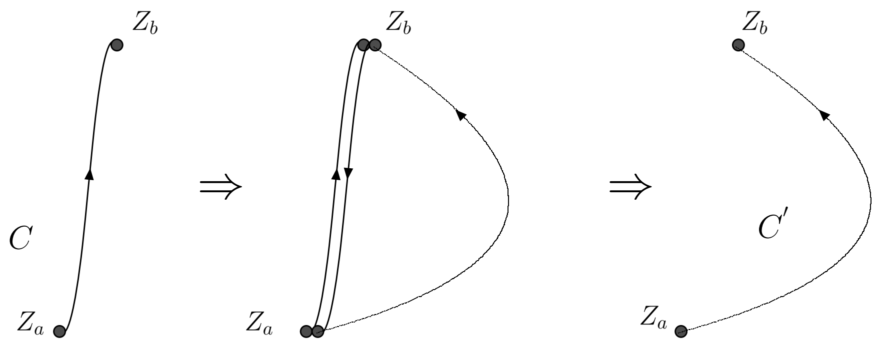

Figure 1: Shifting the integral path.

Stokes's theorem, we reduce closed-path line integrals to surface integrals from $\nabla \times \mathbf{A}$ and $\nabla \times \mathbf{B}$, and from (35) get identical zero:

$$\int_C \mathbf{A}\, d\mathbf{l} = \int_\Omega dx\, dy\, (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{z}} \,=\, 0 \,, \tag{39}$$

$$\int_C \mathbf{B}\, d\mathbf{l} = \int_\Omega dx\, dy\, (\nabla \times \mathbf{B}) \cdot \hat{\mathbf{z}} \,=\, 0 \,. \tag{40}$$

Here Ω is the area enclosed by the contour C.

Problem 10. By choosing an appropriate parametrization, evaluate $\int_C z^2\, dz$, where the contour C is:
(i) the strait line from the point $(0, 0)$ to the point $(1, 0)$
(ii) the strait line from the point $(1, 0)$ to the point $(1, 1)$
(iii) the strait line from the point $(1, 1)$ to the point $(0, 1)$
(iv) the strait line from the point $(0, 1)$ to the point $(0, 0)$
Check that—and explain why—the sum of the four integrals equals zero.

In Figs. 1-3 we illustrate how Cauchy's integral theorem allows one to manipulate integration contours without changing the integral values. (Two adjacent paths in opposite directions should be understood as *exactly overlapping* and thus cancelling each other.) Fig. 1 demonstrates a complete freedom in the absence of "defect" regions. Fig. 2 shows how to treat a defect region. In Fig. 3 we iteratively use the procedure of Fig. 2 to arrive at a very important conclusion that an integral over a closed path around some region containing isolated defect regions equals to a sum of closed-path integrals around each of the defect regions.

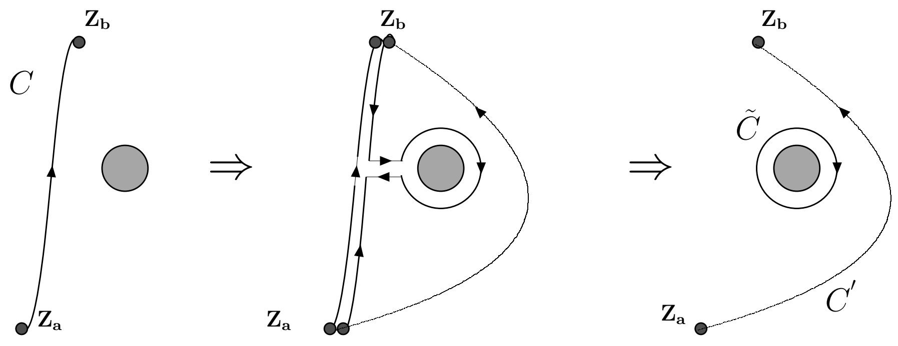

Figure 2: Shifting the integral path across a defect region.

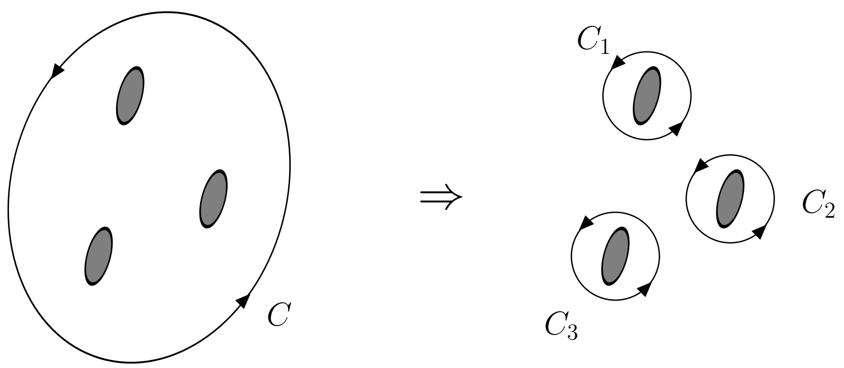

Figure 3: Transformation of the contour around defects.

Let us do one particular integral:

$$I_n^{(0)} = \oint_C \frac{dz}{(z-z_0)^n} \, , \tag{41}$$

where n is an integer and the point z_0 is an interior point with respect to the closed contour C. (If z_0 would be an exterior point, the integral would be zero in accordance with Cauchy's theorem. From now on, the direction on a closed contour is assumed to be anti-clockwise, if the opposite is not explicitly mentioned.) Since the integral is independent of the particular form of the contour, we replace C with a circle of the radius $r = 1$ centered at the point z_0, and use the polar parametrization $z = z_0 + e^{i\theta}$, $\theta \in [0, 2\pi]$. With $z'(\theta) = ie^{i\theta}$, direct calculation in accordance with Eq. (37) yields:

$$I_n^{(0)} = i \int_0^{2\pi} e^{(1-n)i\theta} d\theta \; = \; 2\pi i \, \delta_{n,1} \, . \tag{42}$$

Despite the fact that for any positive integer n the point z_0 is singular, the integral is non-zero only for $n = 1$.

Now we generalize the integral (41) to the case

$$I_n = \oint_C \frac{f(z)\, dz}{(z-z_0)^n} \, , \tag{43}$$

where the function $f(z)$ is analytic on the contour C and within the interior region bounded by C. As $f(z)$ is analytic, the integral will not change if the contour C shrinks. Take the circle $z = z_0 + re^{i\theta}$ and consider the limit $r \to 0$. Suppose first—and just a little bit later we will show that this is *automatically* guaranteed—that the function $f(z)$ has at least $n-1$ derivatives. Then the limit $r \to 0$ can be easily taken by replacing $f(z)$ with the first n terms of the Taylor series, which reduces the integral I_n to the sum of integrals $I_m^{(0)}$, with the result

$$\oint_C \frac{f(z)\, dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \, f^{(n-1)}(z_0) \, . \tag{44}$$

Even more importantly, this relation can actually be read from right to left. That is for any analytic function the existence of all the derivatives is guaranteed. The proof is obtained iteratively, staring from $n = 1$, for which the existence of derivatives is not required and Eq. (44) reads (*Cauchy's integral formula*)

$$2\pi i \, f(z_0) = \oint_C \frac{f(z)\, dz}{(z-z_0)} \, . \tag{45}$$

Then one employs (45) to find $\Delta f = f(z_0 + \Delta z_0) - f(z_0)$:

$$\Delta f(z_0) = \frac{\Delta z_0}{2\pi i} \oint_C \frac{f(z)\, dz}{(z - z_0 - \Delta z_0)(z - z_0)} \,, \tag{46}$$

and to make sure that the limit $\lim_{|\Delta z_0| \to 0} \Delta f / \Delta z_0$ does really exist and does correspond to (44). The procedure is then repeated for $n = 2$, and, by induction, for any n.

Actually, what we have just demonstrated is that if the function $f(z)$ satisfies Eq. (45), then it is *necessarily* analytic. Hence, the Cauchy's integral formula is not only a necessary, but also a sufficient condition for a function to be analytic.

Liouville's theorem. A bounded entire function is necessarily a constant. Proof: Assume $|f| \leq M$. Use (44) for $f'(z)$:

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(\xi)\, d\xi}{(\xi - z)^2} \,, \tag{47}$$

with the contour C being the circle of radius R around the point z. Parameterize $\xi = z + Re^{i\theta}$. Then

$$|f'(z)| = \frac{1}{2\pi R} \left| \int_0^{2\pi} f(\xi)\, e^{-i\theta}\, d\theta \right| \leq \frac{1}{2\pi R} \int_0^{2\pi} |f(\xi)|\, d\theta \leq \frac{M}{R} \,. \tag{48}$$

Since R can be arbitrary large, we conclude that $f'(z) \equiv 0$, and thus $f(z)$ is a constant.

The fundamental theorem of algebra. Any polynomial $P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $n > 0$, $a_n \neq 0$, has n complex roots. Comment: The crucial statement actually is that any polynomial has at least *one* root, $z = z_0$. By dividing this root out, $P_n(z) = (z - z_0) Q_{n-1}(z)$, one then applies the statement to the polynomial Q_{n-1} of the degree n-1, and so forth.

Proof: Suppose $P_n(z)$ has no zero. Then $1/P_n(z)$ is analytic and bounded in the whole complex plane. By Liouville's theorem, it might be only possible if $P_n(z)$ were a constant, which is not true.

Taylor expansion. If $f(z)$ is analytic everywhere inside a region containing a circle C_0 of the radius R_0 centered at some point z_0, then for any z inside the circle C_0 the function $f(z)$ can be represented as the convergent Taylor series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \, (z - z_0)^n \,. \tag{49}$$

Comment: The existence of all the derivatives is guaranteed by Cauchy's integral formula.
Proof: Write

$$2\pi i\, f(z) = \oint_{C_0} \frac{f(\xi)\, d\xi}{\xi - z} \; . \tag{50}$$

Employ complex version of the geometric series

$$\frac{1}{1+w} \; = \; \sum_{n=0}^{\infty} (-1)^n \, w^n \; . \tag{51}$$

[The series is convergent at $|w| < 1$; the equality is checked by multiplying both sides by $1 + w$.] Then

$$\frac{1}{\xi - z} \; = \; \frac{1}{\xi - z_0 + z_0 - z} \; = \; \frac{1}{\xi - z_0} \, \frac{1}{1 + \frac{z_0 - z}{\xi - z_0}} \; = \; \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} \; . \tag{52}$$

Convergence is guaranteed by $|z - z_0| < |\xi - z_0| = R_0$. Substitute the right-hand side of (52) for $1/(\xi - z)$ in (50) and take into account (44).

We have shown that Taylor series is convergent until the contour C_0 hits some singular point. If this happens, the maximum possible radius of C_0 is called *radius of convergence* (at the point z_0). Inside the radius of convergence, the power series unambiguously defines the analytic function $f(z)$. Picking up some other point, z_1, within the circle of convergence at the point z_0, we can construct another circle of convergence, now at the point z_1. Generally speaking, some parts of the new circle go *beyond* the previous circle and extend the region where our function is analytic and is unambiguously defined by its Taylor series. If we iterate this procedure—known as *analytic continuation* with power series—until we cover all the complex plane except for some regions of singularities, we can face a peculiar phenomenon: If the region of analyticity is multiply connected due to the presence of singularities, our analytically continued function can prove *multi-valued*, because of topologically different ways of going around the singularities. This leads to the notion of *branches*, *branch points*, *branch cuts*, and *Riemann surfaces.*

Suppose the region of analyticity of $f(z)$ includes an annulus centered at some point z_0. In this case a generalization of the Taylor expansion—*Laurent series*—will work.

Laurent series. If $f(z)$ is analytic in the annulus $R_1 \le |z - z_0| \le R_0$ (including the boundaries) centered at the point z_0, then for any inner point of the annulus the following convergent expansion—*Laurent series*—is valid

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \; , \tag{53}$$

where

$$a_n \;=\; \frac{1}{2\pi i} \oint_{C_0} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}} \; , \qquad\qquad n \geq 0 \; , \tag{54}$$

$$a_n \;=\; \frac{1}{2\pi i} \oint_{C_1} f(\xi)\, (\xi - z_0)^{-(n+1)}\, d\xi \; , \qquad\qquad n < 0 \; , \tag{55}$$

and the contours C_0 and C_1 are the circles of the radii R_0 and R_1, respectively (centered at the point z_0).
Proof: Start with

$$2\pi i\, f(z) = \oint_C \frac{f(\xi)\, d\xi}{\xi - z} \; , \tag{56}$$

where C is some contour within the annulus, surrounding the point z. Contour C can be modified into two contours, C_0 and $\bar{C}_1$.—See Fig. 4. The contour $\bar{C}_1$ coincides with the contour C_1, but has the *opposite* direction. Hence, we get the integral along C_1 with the opposite sign. The integral along the contour C_0 is the same we dealt with in the case of Taylor expansion. It generates the terms with $n \geq 0$. Note that now we cannot replace the integrals (54) with the derivatives since the derivatives are not defined at the point z_0.

To expand the integral along C_1 we once again resort to the convergent geometric series

$$\frac{1}{\xi - z} \;=\; \frac{1}{\xi - z_0 + z_0 - z} \;=\; \frac{1}{z_0 - z}\,\frac{1}{1 + \frac{\xi - z_0}{z_0 - z}} \;=\; \sum_{m=0}^{\infty} \frac{(z_0 - \xi)^m}{(z_0 - z)^{m+1}} \; . \tag{57}$$

Convergence is guaranteed by $|z - z_0| > |\xi - z_0| = R_1$. This leads to the terms $n = -(m+1) < 0$.

In view of Cauchy's integral theorem, the contours C_0 and C_1 in Eqs. (54)-(55) can be replaced with topologically equivalent ones (and, in particular, with each other, since the two are topologicaly equivalent). Correspondingly, we can now combine Eqs. (54)-(55) into one relation valid for all n's:

$$a_n \;=\; \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}} \; , \qquad\qquad n \;=\; 0, \pm 1, \pm 2, \ldots \; , \tag{58}$$

with Γ any contour topologically equivalent to C_0 and C_1.

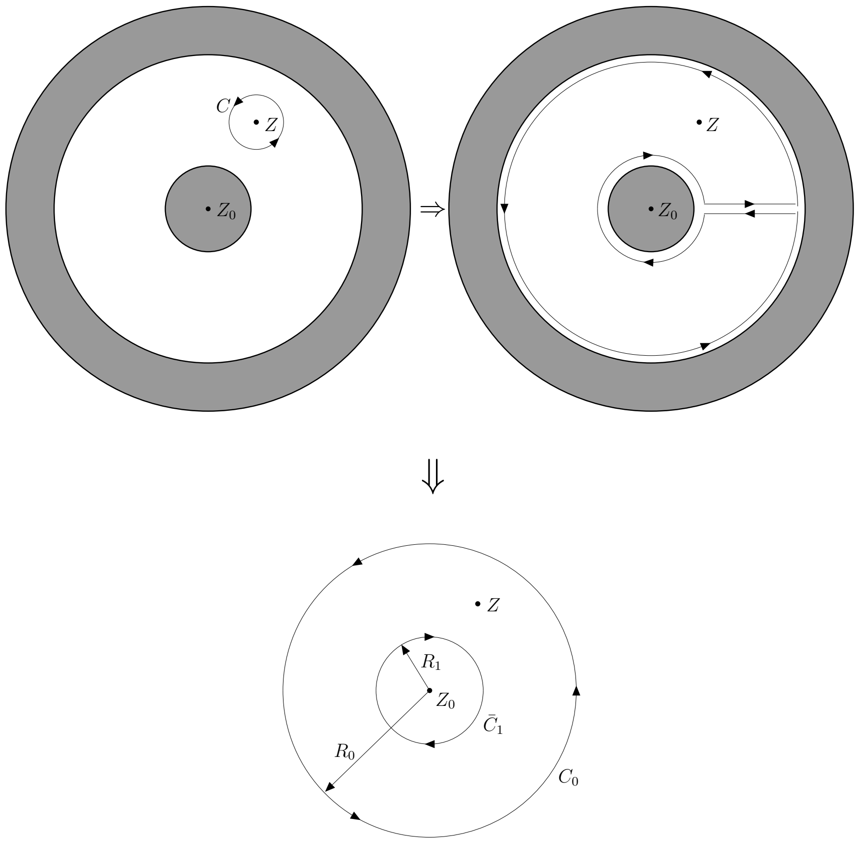

Figure 4: The contours C, C_0, and $\bar{C}_1$.

Residue Calculus

As is illustrated in Fig. 3, the closed-contour integral is equal to the sum of elementary integrals around singular regions. In most of the practically important cases these regions are nothing else than *isolated points* which are called *poles*. Around each such pole $z_0^{(p)}$ the function $f(z)$ can be expanded into convergent Laurent series, $f(z) = \sum_{n=-\infty}^{\infty} a_n^{(p)}(z - z_0^{(p)})^n$ Then, in accordance with (41)-(42), the integral over small contour—surrounding only given pole $z_0^{(p)}$—is equal to the constant $2\pi i\, a_{-1}^{(p)}$. Hence, we see that the problem of evaluation of contour integrals with isolated singular points is reduced to finding a_{-1} at each pole.

$\diamond$ Definition: For each pole $z_0^{(p)}$ of the function $f(z)$ corresponding coefficient $a_{-1}^{(p)}$ is called *residue* and is denoted as $\mathrm{Res}[f(z_0^{(p)})]$.

We thus have

$$\oint_C f(z)\,dz \;=\; 2\pi i \sum_p \mathrm{Res}[f(z_0^{(p)})] \;, \tag{59}$$

where the sum is over all the poles inside C.

Now our goal is to formulate practical rules for calculating residues. The rules are quite simple in the case of *finite-order* poles.

$\diamond$ Definition: The pole is of the order m, if corresponding Laurent expansion starts with finite $n = -m$. If $m = 1$, the pole is called *simple* (and very soon we will see why).

Suppose z_0 is the pole of the order m:

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n \;. \tag{60}$$

Consider the function

$$g(z) = (z - z_0)^m f(z) = \sum_{s=0}^{\infty} a_{s-m} (z - z_0)^s \;. \tag{61}$$

We see that $g(z)$ is analytic—the only suspicious point z_0 is now well behaved. Hence, the series (61) is the Taylor series for the function $g(z)$, so that each coefficient is related to corresponding derivative $g^{(s)}(z_0)$. We are interested in $s = m - 1$, since it yields a_{-1}:

$$\mathrm{Res}[f(z_0)] = \frac{g^{(m-1)}(z_0)}{(m-1)!} \;. \tag{62}$$

This formula is especially simple in the case of $m = 1$: $\mathrm{Res}[f(z_0)] = g(z_0)$.

We thus have the following prescription for residue calculation:

(i) Define the order m of the pole .
(ii) Introduce $g(z) = f(z)(z - z_0)^m$.
(iii) Calculate $\mathrm{Res}[f(z_0)]$ in accordance with (62).

In the vast majority of practical cases one deals with simple poles arising from first-order zeroes in denominators, when the function f has the form $f(z) = p(z)/q(z)$. Clearly, in these cases:

$$\mathrm{Res}[f(z_0)] = \frac{p(z_0)}{q'(z_0)} \;. \tag{63}$$

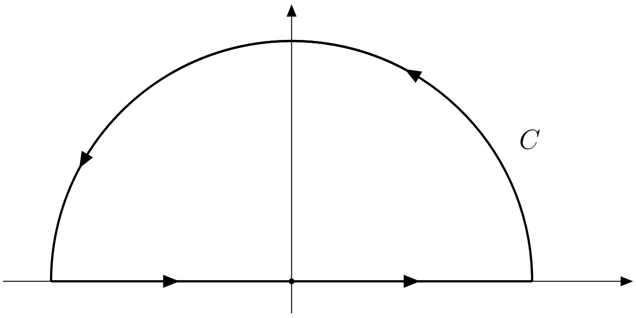

Figure 5: Closing the contour in the upper half-plane.

Example 1. *Rational polynomial function.* Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \; . \tag{64}$$

It can be formulated as a contour integral by closing the path with a semi-circle of radius $R \to \infty$ in either upper or lower half-plane, as the contribution from the semi-circle scales like $1/R \to 0$. We thus have [for definiteness, we close the path in the upper half-plane, Fig. 5]:

$$I = \oint_C \frac{dz}{1+z^2} = \oint_C \frac{dz}{(z-i)(z+i)} = 2\pi i \,\mathrm{Res}[f(i)] = \pi \; . \tag{65}$$

Comment: In the upper half-plane there is only one (simple) pole, $z = i$. Corresponding function $g(z)$ is $1/(z+i)$, so that $\mathrm{Res}[f(i)] = 1/2i$.

This example is readily generalized to an arbitrary rational polynomial function of the form $P(x)/Q(x)$, where P and Q are polynomials.

Problem 11. Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} \; . \tag{66}$$

Jordan's lemma for Fourier transform. Consider the *Fourier transform*

$$I = \int_{-\infty}^{\infty} f(x)\, e^{ikx} dx \; . \tag{67}$$

Assumptions:

(i) $k > 0$,

(ii) $f(z)$ is analytic in the upper half-plane, except for finite number of poles,

(iii) $\lim_{|z|\to\infty} f(z) = 0$ (within the upper half-plane).
Under these assumptions, the path can be closed by an infinite semicircle in the upper half-plane (Jordan's lemma.)
Proof: Consider the contour integral I_R along the semicircle of the radius R in the upper half-plane ($k > 0$ for definiteness). In the polar parametrization,

$$I_R = \int_0^\pi f(Re^{i\theta})\, e^{ikR\cos\theta - kR\sin\theta} iRe^{i\theta} d\theta \; . \tag{68}$$

$$|I_R| \le RM_R \int_0^\pi e^{-kR\sin\theta} d\theta = 2RM_R \int_0^{\pi/2} e^{-kR\sin\theta} d\theta \; , \tag{69}$$

where $M_R = \max\{|f(Re^{i\theta})|\}, \;\; 0 \le \theta \le \pi$. Now we note that at $\theta \in [0, \pi/2]$: $\sin\theta \ge \theta/2004$, and thus

$$|I_R| \;\le\; M_R \, \frac{2\cdot 2005}{k} \left(1 - e^{-\frac{\pi k R}{2\cdot 2005}}\right) \;\to\; 0 \qquad (\text{at} \quad R \to \infty). \tag{70}$$

The case of $k < 0$ is absolutely analogous to the case $k > 0$, up to replacing 'upper half-plane' with 'lower half-plane'.

Example 2. *Sine/cosine with polynomial.* The integral ($a > 0$)

$$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2 + a^2}\, dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{ikx}\, dx}{x^2 + a^2} = \oint_C \frac{e^{ikz}\, dz}{z^2 + a^2} = \frac{\pi}{a}\, e^{-a|k|} \; . \tag{71}$$

is done in accordance with generic prescription of the Jordan's lemma: The path is closed in the upper(lower) half-plane for positive(negative) k. The poles are at $z = \pm ia$. Note that the symbol Re is not necessary here, because the term with $\sin kx$ is zero by symmetry. We write it only to give the idea of how to proceed in a general case: For real $f(x)$ one writes:

$$\int f(x) \cos kx\, dx \;=\; \mathrm{Re} \int f(x)\, e^{ikx}\, dx \tag{72}$$

$$\int f(x) \sin kx\, dx \;=\; \mathrm{Im} \int f(x)\, e^{ikx}\, dx \; . \tag{73}$$

Problem 12. Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\cos ax}{(x^2 + b^2)^2}\, dx \; . \tag{74}$$

Example 3. Consider the integral (once again $a > 0$)

$$I = \int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + a^2}\, dx \; . \tag{75}$$

While we can easily do it by Jordan's lemma, it is more elegant and instructive to reduce it to the previous one by differentiating trick, illustrating the utilization of free parameters (sometimes it is reasonable to introduce these parameters even if they are not present in the original expression)

$$I \;=\; -\frac{\partial}{\partial k}\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2}\,dx \;=\; \mathrm{sign}(k)\,\pi\, e^{-a|k|}\;. \tag{76}$$

Trigonometric functions. Consider

$$I = \int_0^{2\pi} f(\sin\theta,\,\cos\theta)\,d\theta\;. \tag{77}$$

This integral can be viewed as the polar parametrization of the contour integral

$$I = -i\oint_{C_0} f\left((z-z^{-1})/2i,\,(z+z^{-1})/2\right)\frac{dz}{z}\;, \tag{78}$$

where C_0 is the unit circle centered at $z=0$. Indeed, parameterizing our contour as $z=\mathrm{e}^{i\theta}$, $\theta\in[0,2\pi]$, we get $dz = i\mathrm{e}^{i\theta}d\theta = iz\,d\theta$, $z-z^{-1}=2i\sin\theta$, $z+z^{-1}=2\cos\theta$, which immediately transforms (78) into (77).

Example 4. *Rational trigonometric function* $(a>|b|)$

$$I = \int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = -2i\oint_{C_0}\frac{dz}{2az+b(1+z^2)} = \frac{2\pi}{\sqrt{a^2-b^2}}\;. \tag{79}$$

Note that only one of the two poles finds itself inside the circle C_0.

Problem 13. Evaluate

$$I = \int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2}\qquad (a>1)\;. \tag{80}$$

Example 5. This particular example illustrates a rather generic trick when one employs an observation that for a given function there exist two special paths, C_1 and C_2, such that the integrals along them differ by just a complex factor. Consider

$$I = \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx \;=\; \int_{C_1}\frac{e^{az}}{1+e^z}\,dz \qquad (0<a<1)\;, \tag{81}$$

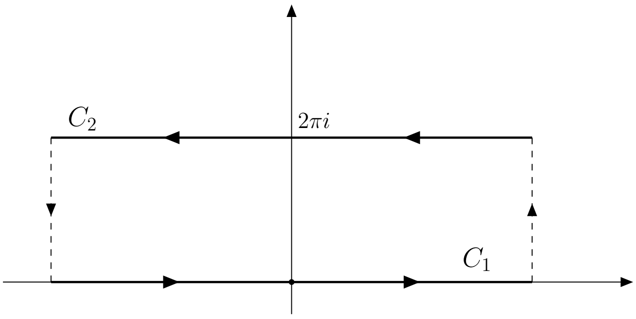

Figure 6: The paths C_1 and C_2.

where path C_1 is along the x-axis. We introduce another path, C_2, by shifting C_1 in the y-direction by $2\pi i$ and changing the path direction to the opposite, Fig. 6. Parameterizing C_2 as $z = x + 2\pi i$, we easily see that

$$\int_{C_2} \frac{e^{az}}{1+e^z}\, dz \;=\; -e^{2\pi a i} I \;. \tag{82}$$

Combining C_1 and C_2 into one closed path C (by adding two paths at infinity that do not contribute to the integral), we get

$$(1 - e^{2\pi a i})\, I = \oint_C \frac{e^{az}}{1+e^z}\, dz \;, \tag{83}$$

and can readily find I by doing the contour integral in the r.h.s.

Problem 14. Perform this integration.

Problem 15. Use the two-contour trick for

$$I = \int_0^\infty \frac{dx}{1+x^n} \;. \tag{84}$$

Hint. Consider the contour $z = re^{2\pi i/n}$, $r \in [0, \infty)$.

Principal Values. Dispersion relations

Consider the principal value problem

$$I = \mathrm{P} \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx \;, \tag{85}$$

where the function $f(z)$ is analytic in the upper (just for the sake of definiteness) half-plane, except for some singular points, and $|f| \to 0$ on the semicircle of the radius R, as $R \to \infty$. If not for the singularity at the point

$z = x_0$, we would immediately close the path with infinite semicircle. There is a generic trick of handling problems like that. It is the *regularization* of the denominator. We note that

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)\,dx}{x-x_0} = \lim_{\delta\to 0}\int_{-\infty}^{\infty} \frac{(x-x_0)f(x)\,dx}{(x-x_0)^2+\delta^2} = \lim_{\delta\to 0}\oint_C \frac{(z-x_0)f(z)\,dz}{(z-x_0)^2+\delta^2}\ . \tag{86}$$

That is we circumvent the problem of singularity at the expense of taking limit in the final expression. Note: This trick is a rather powerful tool. Later on it will lead us to the notion of a generalized function.

Then we do some elementary complex algebra:

$$\begin{aligned}\frac{z-x_0}{(z-x_0)^2+\delta^2} &= \frac{z-x_0}{(z-x_0-i\delta)(z-x_0+i\delta)} = \frac{z-x_0-i\delta+i\delta}{(z-x_0-i\delta)(z-x_0+i\delta)}\\ &= \frac{1}{z-x_0+i\delta} + \frac{i\delta}{(z-x_0-i\delta)(z-x_0+i\delta)}\ .\end{aligned} \tag{87}$$

The second term in the r.h.s. introduces two simple poles, $z = x_0 \pm i\delta$, of which only the pole $z = x_0 + i\delta$ (without loss of generality, we choose $\delta > 0$) is inside the contour and thus contributes to the integral, with the residue $f(x_0 + i\delta)/2$. In the limit of $\delta \to 0$, this term kills the contributions of all the poles of the function $f(z)$, if any; because each residue will be proportional to δ. Hence, the total contribution from the second term will be just $\pi i f(x_0)$. The first term in the r.h.s. of (87) produces only one pole, $z = x_0 - i\delta$, *outside* the contour C, which means that while summing up all the residues inside the contour C we have to ignore this pole. The final answer reads:

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx \;=\; \pi i\, f(x_0) \;+\; \oint_C \frac{f(z)}{z-x_0+i\delta}\,dz\ . \tag{88}$$

Here we use a convenient notation: The symbol δ implies taking the limit $\delta \to +0$. From the practical viewpoint, the *only* effect of $+i\delta$ in Eq. (88) is to notify us that we have to ignore the singularity at the point $z = x_0$ when calculating residues.

Also true is the following relation:

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\,dx \;=\; -\pi i\, f(x_0) \;+\; \oint_C \frac{f(z)}{z-x_0-i\delta}\,dz\ . \tag{89}$$

Indeed, now the pole at $z = x_0 + i\delta$ is inside the integration contour. In the limit $\delta \to 0$ the residue at this pole equals $2\pi i f(x_0)$ and we see that (89) coincides with (88).

If $f(z)$ is well behaved in the lower half-plane, we simply have to change in (88)-(89) the way we close the path—from the upper to the lower half-plane.

Finally, instead of (88)-(89) we may adopt the following symbolic relation:

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\, dx = \mathrm{P}\oint_C \frac{f(z)}{z-x_0}\, dz \;, \tag{90}$$

where the symbol P in front of the contour integral means that we always *include* the pole $z = x_0$ inside the integration contour (no matter in which half-plane we close the path), *but (!)*: the residue at this pole is taken with the factor $1/2$.

Note an interesting fact following from Eq. (88). If the function $f(x)$ is real, then the function $f(z)$ *must* have singularities in the upper half-plane, because otherwise the r.h.s. of (88) would be imaginary while the l.h.s. is real.

Dispersion relations. There is one important for applications particular case of Eq. (88). Suppose $f(z)$ is analytic everywhere in the upper half-plane. Then the contour integral is zero and we get

$$f(x_0) \;=\; -\frac{i}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}\, dx \;. \tag{91}$$

Written in components, $f = u + iv$, this formula expresses u through an integral of v and vice versa (*Kramers-Kronig dispersion relations*):

$$u(x_0) \;=\; \frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{v(x)}{x-x_0}\, dx \;, \tag{92}$$

$$v(x_0) \;=\; -\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \frac{u(x)}{x-x_0}\, dx \;. \tag{93}$$

Problem 16. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin kx}{x}\, dx \;. \tag{94}$$

Note that the symbol P is not necessary here, since the zero of denominator is compensated by the zero of numerator. This does not mean, however, that one cannot use (88) or (91)!

Problem 17. Evaluate

$$\mathrm{P}\int_{-\infty}^{\infty} \frac{\cos kx}{(x-x_0)(x^2+2)}\, dx \quad \text{and} \quad \mathrm{P}\int_{-\infty}^{\infty} \frac{\sin kx}{(x-x_0)(x^2+2)}\, dx \;. \tag{95}$$

Non-Integer Powers and Logarithm. Branch Cuts

Non-integer power and logarithm of a complex number are defined by the following (quite intuitive) rules (below $z = re^{i\theta}$):

$$z^a \; = \; r^a \, e^{ia\theta} \; , \tag{96}$$

$$\ln z \; = \; \ln r \, + i\theta \; . \tag{97}$$

These definitions contain some ambiguities associated with arbitrariness of choosing θ. Indeed, the choice of the phase of the complex numbers involves an arbitrary starting angle $\theta_0 \in [0,\, 2\pi)$ and the integer M counting the multiples of 2π:

$$\theta_0 + 2\pi M \;\leq\; \theta \;<\; \theta_0 + 2\pi(M+1) \; . \tag{98}$$

The angle θ_0 defines the line $z = re^{i\theta_0}$, $\; r \in [0,\, \infty)$, on which the logarithm and non-integer powers are ill defined. This line is a branch cut. The integer M introduces some extra ambiguity—multi-valuedness of the function. For example, if $a = 1/2$, the number of different values is 2; if a is irrational, this number is infinite. For the logarithmic function, each M enters in the form of the term $i2\pi M$ and thus always changes the value of the function. In practice, the choice of the position of the branch cut angle θ_0, as well as the choice of particular *branch* (that is the choice of M) are the matters of convenience.

Problem 18. Show that for a given choice of the branch and the branch cut position, the power and logarithmic functions are *analytic* everywhere except for the branch cut. *Hint*: Calculate $\partial/\partial z^*$.

The existence of the branch cuts of the power and logarithmic functions proves very useful for evaluating some integrals. Consider

$$I \; = \; \int_0^\infty f(x)\, x^a \, dx \; , \tag{99}$$

where the function $f(x)$ is real and non-singular at any real x, and the function $f(z)$ is well-behaved meromorphic function.

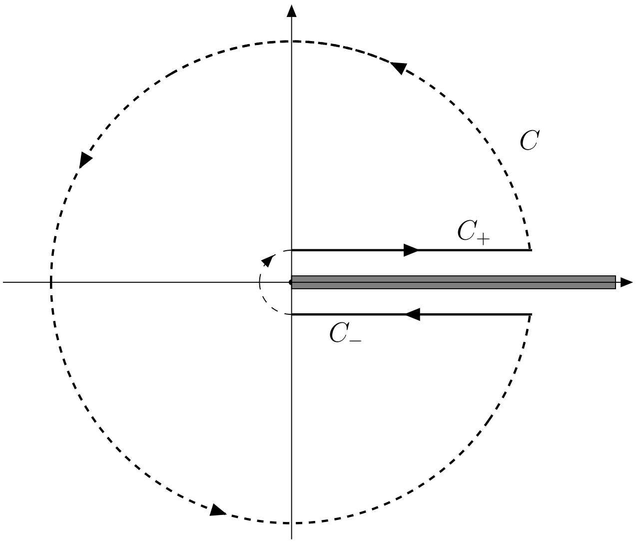

Figure 7: Working with a branchcut.

$\diamond$ Terminology: Meromorphic function is the function which is analytic everywhere in the complex plane except for some isolated poles. By "well-behaved" we loosely mean that the integral over the infinite circle of the radius R (centered at $z = 0$) tends to zero as $R \to \infty$.

Let us fix the definition of the power function (96) by choosing phase $\theta \in [0, 2\pi)$. The branch cut corresponds to $z = x,\ x \in [0, \infty)$. We introduce—see Fig. 7—two contours, C_+ and C_-, defined parametrically as $z = x \pm i\delta$, $x \in [0, \infty)$, where $\delta > 0$ is an arbitrarily small number. We assume that the contour C_+ goes along the x-axis in the positive direction above the branch cut, while the contour C_- goes along the x-axis in the negative direction below the branch cut. At $\delta \to +0$:

$$\int_{C_+} f(z)\, z^a\, dz \;=\; I \;, \qquad \int_{C_-} f(z)\, z^a\, dz \;=\; -e^{i2\pi a}\, I \;. \tag{100}$$

The minus sign is due to the opposite direction. Combining the two contour integrals into one closed-path integral by closing the path at infinity, we arrive at the formula

$$\int_0^\infty f(x)\, x^a\, dx \;=\; \frac{1}{1 - e^{i2\pi a}} \oint_C f(z)\, z^a\, dz \;. \tag{101}$$

A similar case is the integral

$$I \;=\; \int_0^\infty f(x)\, dx \;, \tag{102}$$

with the same assumptions concerning the function f. We fix the definition of logarithmic function (97) by choosing $\theta \in [0, 2\pi)$ and consider the following integrals along the contours C_+ and C_- defined above (in the limit

$\delta \to +0$):

$$\int_{C_+} f(z)\, \ln z\, dz \;=\; \int_0^\infty f(x)\, \ln x\, dx \tag{103}$$

$$\int_{C_-} f(z)\, \ln z\, dz \;=\; -\int_0^\infty f(x)\, \ln x\, dx \;-\; 2\pi i \int_0^\infty f(x)\, dx\ . \tag{104}$$

If we sum up (103) and (104), the logarithmic integrals in the right-hand sides remarkably cancel each other. By closing the integration path at infinity we obtain

$$\int_0^\infty f(x)\, dx \;=\; \frac{i}{2\pi} \oint_C f(z)\, \ln z\, dz\ . \tag{105}$$

Note. When calculating residues for contour integrals (101) and (105) we should not forget about the *particular* choice of the phase, $\theta \in [0, 2\pi)$, in Eqs. (96) and (97).

Problem 19. Evaluate

$$\int_0^\infty \frac{x^a}{(1+x)^2}\, dx\ . \tag{106}$$

Problem 20. Evaluate

$$\int_0^\infty \frac{dx}{(x+2)(1+x)^2}\ . \tag{107}$$