Green's Function of the Wave Equation

The Fourier transform technique allows one to obtain Green's functions for a spatially homogeneous infinite-space linear PDE's on a quite general basis—even if the Green's function is actually a *generalized* function. Here we apply this approach to the wave equation.

The wave equation reads (the sound velocity is absorbed in the re-scaled t)

$$u_{tt} = \Delta u \; . \tag{1}$$

Equation (1) is the second-order differential equation with respect to the time derivative. Correspondingly, now we have two initial conditions:

$$u(\mathbf{r}, t=0) = u_0(\mathbf{r}) \; , \tag{2}$$

$$u_t(\mathbf{r}, t=0) = v_0(\mathbf{r}) \; , \tag{3}$$

and have to deal with *two* Green's functions:

$$u(\mathbf{r}, t) = \int G^{(0)}(\mathbf{r}-\mathbf{r}', t)\, u_0(\mathbf{r}')\, d\mathbf{r}' + \int G^{(1)}(\mathbf{r}-\mathbf{r}', t)\, v_0(\mathbf{r}')\, d\mathbf{r}' \; . \tag{4}$$

Both functions satisfy the equation

$$G_{tt} = \Delta G \; , \tag{5}$$

but with *different initial conditions*:

$$G^{(0)}(\mathbf{r}, 0) = \delta(\mathbf{r}) \; , \qquad G^{(0)}_t(\mathbf{r}, 0) = 0 \; , \tag{6}$$

$$G^{(1)}(\mathbf{r}, 0) = 0 \; , \qquad G^{(1)}_t(\mathbf{r}, 0) = \delta(\mathbf{r}) \; , \tag{7}$$

Looking for the solution of (5) in the form

$$G(\mathbf{r}, t) \;=\; \int g(\mathbf{k}, t)\, \mathrm{e}^{i\mathbf{k}\mathbf{r}}\, d\mathbf{k}/(2\pi)^d \; , \tag{8}$$

we get

$$\ddot{g} = -k^2 g \; . \tag{9}$$

That is

$$g(\mathbf{k}, t) = A(\mathbf{k}) \cos(kt) + B(\mathbf{k}) \sin(kt) \; , \tag{10}$$

where the functions $A(\mathbf{k})$ and $B(\mathbf{k})$ are defined by the initial conditions (6)-(7). Plugging (11) into (6)-(7) and taking into account that the Fourier transform of the δ-function is unity:

$$\delta(\mathbf{r}) = \int \mathrm{e}^{i\mathbf{kr}}\, d\mathbf{k}/(2\pi)^d \;, \tag{11}$$

we get

$$g^{(0)}(\mathbf{k},0) = 1 \;, \qquad g_t^{(0)}(\mathbf{k},0) = 0 \;, \tag{12}$$

$$g^{(1)}(\mathbf{k},0) = 0 \;, \qquad g_t^{(1)}(\mathbf{k},0) = 1 \;, \tag{13}$$

and readily find

$$g^{(0)}(\mathbf{k},t) = \cos(kt) \;, \tag{14}$$

$$g^{(1)}(\mathbf{k},t) = (1/k)\sin(kt) \;. \tag{15}$$

Comparing Eqs. (14)-(15) one notices that $g^{(0)}(\mathbf{k},t) = g_t^{(1)}(\mathbf{k},t)$, and thus

$$G^{(0)}(\mathbf{r},t) = G_t^{(1)}(\mathbf{r},t) \;. \tag{16}$$

Hence, it is sufficient to evaluate only $G^{(1)}$, and then find $G^{(0)}$ by differentiating $G^{(1)}$ with respect to t.

From (15) we obtain

$$G^{(1)}(\mathbf{r},t) = \int \frac{\cos(\mathbf{kr})\sin(kt)}{k}\,\frac{d\mathbf{k}}{(2\pi)^d} \;, \tag{17}$$

where we took into account the $\mathbf{k} \to -\mathbf{k}$ symmetry.

Performing the integral (17) essentially depends on the dimension, and we need to consider separately three different cases: $d = 1, 2, 3$.

1D case.

$$G^{(1)}(x,t) = \int_{-\infty}^{\infty} \frac{\cos(kx)\sin(kt)}{k}\,\frac{dk}{2\pi} \;. \tag{18}$$

[Note that despite the fact that in (17) the symbol k stands for the absolute value of vector $\mathbf{k}$, there is no contradiction between (17) and (18) because the integrand of (18) remains the same when $k \to -k$.]

Using

$$\sin\alpha \cdot \cos\beta = [\sin(\alpha+\beta) + \sin(\alpha-\beta)]/2 \;, \tag{19}$$

we rewrite our integral as

$$G^{(1)}(x,t) = \int_{-\infty}^{\infty} \frac{\sin k(t+x) + \sin k(t-x)}{k}\,\frac{dk}{4\pi} \;, \tag{20}$$

and recall that

$$\int_{-\infty}^{\infty} \frac{\sin ky}{k}\, dk \;=\; \pi\, \mathrm{sgn}(y)\;, \tag{21}$$

where

$$\mathrm{sgn}(y) = \begin{cases} \;\;1\,, & y > 0\,, \\ -1\,, & y < 0\,, \\ \;\;0\,, & y = 0\,. \end{cases} \tag{22}$$

This yields the final answer:

$$G^{(1)}(x,t) = [\mathrm{sgn}(t+x) + \mathrm{sgn}(t-x)]/4 = \begin{cases} 1/2\,, & x \in [-t,t]\,, \\ \;\;0\,, & x \notin [-t,t]\,. \end{cases} \tag{23}$$

2D case. In polar coordinates:

$$\mathbf{k} = (k\cos\varphi, k\sin\varphi)\;, \qquad d\mathbf{k} = k\, dk\, d\varphi\;, \tag{24}$$

with φ being the angle between $\mathbf{k}$ and $\mathbf{r}$, we have

$$G^{(1)}(\mathbf{r},t) = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\varphi \int_0^{\infty} \cos[kr\cos\varphi]\cdot \sin(kt)\, dk\;. \tag{25}$$

First, we integrate over k. Once again we use (19) and see that we need to perform

$$I(y) = \int_0^{\infty} \sin(ky)\, dk\;, \tag{26}$$

in terms of which we then would have

$$G^{(1)}(\mathbf{r},t) = \frac{1}{8\pi^2}\int_0^{2\pi} [I(t - r\cos\varphi) + I(t + r\cos\varphi)]\, d\varphi, \tag{27}$$

or simply

$$G^{(1)}(\mathbf{r},t) = \frac{1}{4\pi^2}\int_0^{2\pi} I(t + r\cos\varphi)\, d\varphi\;, \tag{28}$$

because of the symmetry of the cosine function: $\cos(\varphi + \pi) = -\cos\varphi$.

However, the integral (26) is divergent and we should introduce a regularization. With an infinitesimally small positive ε we can write

$$I(y) = \mathrm{Im}\int_0^{\infty} \mathrm{e}^{(iy-\varepsilon)k}\, dk \;=\; \mathrm{Re}\,\frac{1}{y + i\varepsilon}\;. \tag{29}$$

It is too early here to take the limit of $\varepsilon \to 0$: The integral over φ also needs a regularization which is easily done by just keeping the term $i\varepsilon$ in $I(y)$ while doing the integral (28). We thus have

$$G^{(1)}(\mathbf{r}, t) = \frac{1}{4\pi^2} \,\mathrm{Re} \int_0^{2\pi} \frac{d\varphi}{t + r\cos\varphi + i\varepsilon} \;. \tag{30}$$

By a standard trick,

$$z = \mathrm{e}^{i\varphi} \qquad \Rightarrow \qquad d\varphi = -i dz/z \;, \qquad \cos\varphi = (z + 1/z)/2 \;, \tag{31}$$

this integral is reduced to a contour integral along a unity-radius origin-centered circle in a complex plane:

$$I_2 = \int_0^{2\pi} \frac{d\varphi}{t + r\cos\varphi + i\varepsilon} = -2i \oint \frac{dz}{rz^2 + 2(t + i\varepsilon)z + r} \;. \tag{32}$$

Doing the complex integral by residues, we get

$$I_2 = \frac{2\pi}{\sqrt{(t + i\varepsilon)^2 - r^2}} \;. \tag{33}$$

Finally, taking the real part of this integral in the limit of $\epsilon \to 0$, we obtain

$$G^{(1)}(\mathbf{r}, t) \;=\; \frac{1}{2\pi} \frac{\theta(t - r)}{\sqrt{t^2 - r^2}} \;, \tag{34}$$

where

$$\theta(x) = \begin{cases} 1 \;, & x \geq 0 \;, \\ 0 \;, & x < 0 \;. \end{cases} \tag{35}$$

3D case. In spherical coordinates,

$$\mathbf{k} = (k\,\sin\theta\,\cos\varphi,\; k\,\sin\theta\,\sin\varphi,\; k\,\cos\theta) \;, \quad d\mathbf{k} = -k^2 dk\, d\varphi\, d(\cos\theta) \;, \tag{36}$$

with the z-axis along the $\mathbf{r}$ vector, the integrals over φ and θ are readily done, since the integrand is φ-independent, and the only place where the θ-dependence comes from is $\mathbf{kr} = kr\cos\theta$. The result is

$$G^{(1)}(\mathbf{r}, t) = \frac{1}{2\pi^2 r} \int_0^{\infty} \sin(kr)\sin(kt)\, dk \;. \tag{37}$$

Recalling that

$$\sin\alpha \cdot \sin\beta = [\cos(\alpha-\beta) - \cos(\alpha+\beta)]/2 \ , \tag{38}$$

we write it as

$$G^{(1)}(\mathbf{r},t) = \frac{1}{4\pi^2 r}\int_0^\infty [\cos k(r-t) - \cos k(r+t)]\, dk \ , \tag{39}$$

and see that we need to do the integral

$$I_3(y) = \int_0^\infty \cos(ky)\, dk \ . \tag{40}$$

This integral is similar to $I(y)$. It is also divergent and is regularized and calculated the same way:

$$I_3(y) = \mathrm{Re}\int_0^\infty \mathrm{e}^{(iy-\varepsilon)k}\, dk \ = \ \mathrm{Re}\, \frac{i}{y+i\varepsilon} \ = \ \frac{\varepsilon}{y^2+\varepsilon^2} \ = \ \pi\delta(y) \ . \tag{41}$$

We thus have

$$G^{(1)}(\mathbf{r},t) = \frac{1}{4\pi r}\,\delta(t-r) \ . \tag{42}$$

Constructing the solution

The function $G^{(0)} = G^{(1)}_t$ turns out to be a generalized function in any dimensions (note that in 2D the integral with $G^{(0)}$ is divergent). And in 3D even the function $G^{(1)}$ is a generalized function. So we have to establish the final form of the solution free of the generalized functions. In principle, it is sufficient to take care of the function $G^{(1)}$ only, since in view of the relation $G^{(0)} = G^{(1)}_t$ we can always write

$$|u(t)\rangle \ = \ \hat{G}^{(1)}(t)\, |v_0\rangle \ + \ \frac{\partial}{\partial t}\, \hat{G}^{(1)}(t)\, |u_0\rangle \ . \tag{43}$$

That is we act on the function u_0 with the same operator $\hat{G}^{(1)}(t)$ producing thus some smooth—by the nature of the operator $\hat{G}^{(1)}(t)$—time-dependent function, and *then* differentiate this function with respect to t. As we will see, it is also possible to express the operator $\hat{G}^{(0)}(t)$ without resorting to the time-differentiation. However, in 2D and 3D this will lead to a spatial derivative of the function u_0.

1D case. Writing Eq. (43) with $G^{(1)}$ of Eq. (23), we have

$$u(x,t) \;=\; (1/2)\int_{x-t}^{x+t} v_0(x_0)\,dx_0 \;+\; (1/2)\frac{\partial}{\partial t}\int_{x-t}^{x+t} u_0(x_0)\,dx_0\;. \tag{44}$$

The differentiating in the second term can be done explicitly, so that finally we get

$$u(x,t) \;=\; (1/2)\,[u_0(x+t)+u_0(x-t)] \;+\; (1/2)\int_{x-t}^{x+t} v_0(x_0)\,dx_0\;. \tag{45}$$

3D case. Here it is convenient to introduce a shifted variable for integration, $\mathbf{r}_1 = \mathbf{r}_0 - \mathbf{r}$, and to take into account that $G^{(1)}(-\mathbf{r}_1) \equiv G^{(1)}(r_1)$:

$$\int G^{(1)}(\mathbf{r}-\mathbf{r}_0)\,v_0(\mathbf{r}_0)\,d\mathbf{r}_0 \;=\; \int G^{(1)}(r_1)\,v_0(\mathbf{r}_1+\mathbf{r})\,d\mathbf{r}_1\;. \tag{46}$$

We see that without loss of generality we may set $\mathbf{r} = 0$, since the solution at any finite $\mathbf{r}$ is obtained by just translating the initial conditions by the vector $\mathbf{r}$. Writing the integrals with the Green's function (42) in the spherical coordinates (and omitting the subscript 1), we get

$$\begin{aligned} u(\mathbf{r}=0,t) \;=&\; \frac{t}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\; v_0(r=t,\varphi,\theta) \;+ \\ &+ \frac{\partial}{\partial t}\frac{t}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\; u_0(r=t,\varphi,\theta)\;. \end{aligned} \tag{47}$$

Differentiating with respect to time in the second term, we get

$$\begin{aligned} u(\mathbf{r}=0,t) \;=&\; \frac{t}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\; v_0(r=t,\varphi,\theta) \;+ \\ +\frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} & d\theta \sin\theta\,[u_0(r=t,\varphi,\theta) + t\frac{\partial u_0}{\partial r}\,(r=t,\varphi,\theta)]\;. \end{aligned} \tag{48}$$

The meaning of the angular integrals is the averaging over the solid angle:

$$\langle \ldots \rangle \;=\; \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\,(\ldots)\;. \tag{49}$$

Correspondingly, our final result can be written as

$$u(\mathbf{r}=0,t) \;=\; t\,\langle v_0\rangle|_{r=t} \;+\; \langle u_0\rangle|_{r=t} \;+\; t\left\langle \frac{\partial u_0}{\partial r}\right\rangle\Bigg|_{r=t}\;. \tag{50}$$

2D case. In two dimensions the function $G^{(1)}$ (34) is a regular function so that we can simply write

$$u(\mathbf{r},t) \;=\; \frac{1}{2\pi}\int_{|\mathbf{r}-\mathbf{r}_0|\le t}\frac{v_0(\mathbf{r}_0)\,d\mathbf{r}_0}{\sqrt{t^2-|\mathbf{r}-\mathbf{r}_0|^2}} \;+\; \frac{1}{2\pi}\frac{\partial}{\partial t}\int_{|\mathbf{r}-\mathbf{r}_0|\le t}\frac{u_0(\mathbf{r}_0)\,d\mathbf{r}_0}{\sqrt{t^2-|\mathbf{r}-\mathbf{r}_0|^2}}\;. \qquad (51)$$

However, if we want to eliminate the time-derivative in the second term by differentiating under the sign of the integral, we face a problem: The integral becomes divergent. This means that if we differentiate under the sign of the integral, we get a generalized function and need to properly process it. The trick is to replace the time-derivative with a spatial derivative. To this end it is convenient to write the Green's function in such a way that its self-similarity is explicitly seen, and then take advantage of the self-similarity in relating the temporal and spatial derivatives. As is seen, for example, from the dimensional analysis of the wave equation, a proper dimensionless variable is $\xi = t/r$. Correspondingly, we rewrite Eq. (34) in the self-similar form as

$$G^{(1)}(r,t) \;=\; \frac{1}{2\pi r}\,Q(\xi)\;, \qquad (52)$$

where

$$Q(\xi) \;=\; \frac{\tilde{\theta}(\xi)}{\sqrt{\xi^2-1}}\;, \qquad (53)$$

and

$$\tilde{\theta}(\xi) = \begin{cases} 1\,, & \xi \ge 1\,, \\ 0\,, & \xi < 1\,. \end{cases} \qquad (54)$$

Now we have

$$\frac{\partial G^{(1)}}{\partial t} \;=\; \frac{1}{2\pi r}\,Q'(\xi)\,\frac{\partial \xi}{\partial t} \;=\; \frac{1}{2\pi r^2}\,Q'(\xi)\;. \qquad (55)$$

On the other hand,

$$\frac{\partial Q}{\partial r} \;=\; Q'(\xi)\,\frac{\partial \xi}{\partial r} \;=\; -\frac{t}{r^2}\,Q'(\xi)\;. \qquad (56)$$

That is

$$Q'(\xi) \;=\; -\frac{r^2}{t}\,\frac{\partial Q}{\partial r}\;, \qquad (57)$$

and we have

$$\frac{\partial G^{(1)}}{\partial t} = -\frac{1}{2\pi t}\frac{\partial Q}{\partial r} = -\frac{1}{2\pi t}\frac{\partial}{\partial r}\frac{\tilde{\theta}(t/r)}{\sqrt{(t/r)^2-1}} \; . \tag{58}$$

This is a generalized function. To arrive at an ordinary function, we just need to do the integral by parts. Using the representation (46) and setting without loss of generality $\mathbf{r}=0$, in polar coordinates we have

$$\begin{aligned} u(\mathbf{r},t) &= \int_0^\infty dr\,\frac{\theta(t-r)r}{\sqrt{t^2-r^2}}\int_0^{2\pi}\frac{d\varphi}{2\pi}\,v_0 - \\ &- \int_0^{2\pi}\frac{d\varphi}{2\pi t}\int_0^\infty drr\;u_0\frac{\partial}{\partial r}\frac{\tilde{\theta}(t/r)}{\sqrt{(t/r)^2-1}} \; . \end{aligned} \tag{59}$$

Doing the integral in the second term by parts,

$$\int_0^\infty drr\;u_0\frac{\partial}{\partial r}\frac{\tilde{\theta}(t/r)}{\sqrt{(t/r)^2-1}} = -\int_0^\infty dr\;\frac{\tilde{\theta}(t/r)}{\sqrt{(t/r)^2-1}}\left(r\frac{\partial u_0}{\partial r}+u_0\right) \; , \tag{60}$$

we arrive at a regular integral. Taking into account that the θ-functions just fix the upper limit of integration over r, we finally get

$$u(\mathbf{r},t) = \int_0^t \frac{dr}{\sqrt{(t/r)^2-1}}\int_0^{2\pi}\frac{d\varphi}{2\pi}\left(v_0+\frac{r}{t}\frac{\partial u_0}{\partial r}+\frac{u_0}{t}\right) \; . \tag{61}$$