Physics 103 - Discussion Notes #2

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- Last week we discussed one infinite series representation of a function - the Taylor series.
- This week we will talk about another such representation, *the Fourier Series.*

Preliminary Ideas

- Before we discuss Fourier series, we need to discuss three other concepts: even and odd functions, and orthogonality of functions.
- A function f is *periodic* with period T, if for all x in the domain of f

$$f(x+T) = f(x)$$

- $\sin(x)$, $\cos(x)$, and $\tan(x)$ are all common examples of periodic functions.
- Functions can have many different periods when defined this way, e.g. for the sinusoidal functions $T = 2\pi, 4\pi$, etc. all obey this definition. The *smallest* possible period is sometimes called the *fundamental period*, though we'll often just refer to it as the period.
- Any linear combination of periodic functions with period T

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x)$$

  is itself periodic with period T
- A function that obeys

$$f(x) = -f(x)$$

  is called *odd*, some examples are x, x^3, and $\sin x$.
- A function that obeys

$$f(-x) = f(x)$$

  is called *even*. Some examples are x^2, x^4, and $\cos x$.
- A sum of odd functions is itself odd, and a sum of even functions is itself even.
- A product of two odd functions or two even functions is even. A product of an odd function and an even function is odd.
- If $f(x)$ is even, then it obeys the equation

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

- If $f(x)$ is odd, then it obeys the equation

$$\int_{-a}^{a} f(x)\,dx = 0$$

- Two functions are orthonormal on the interval $[a, b]$ if

$$\int_a^b f_m(x) f_n(x)\, dx = \delta_{mn}$$

where δ_{mn} is the *Kronecker Delta* function, defined by

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

- If the above functions integrate to 0, we say they are simply *orthogonal.*

- One important result we'll need for Fourier series is that $\frac{1}{\sqrt{L}} \cos\left(\frac{m\pi x}{L}\right)$ is orthonormal to $\frac{1}{\sqrt{L}} \cos\left(\frac{n\pi x}{L}\right)$ as are $\frac{1}{\sqrt{L}} \sin\left(\frac{m\pi x}{L}\right)$ and $\frac{1}{\sqrt{L}} \sin\left(\frac{n\pi x}{L}\right)$, on the interval $-L$ to L. Also, $\frac{1}{\sqrt{L}} \cos\left(\frac{m\pi x}{L}\right)$ and $\frac{1}{\sqrt{L}} \sin\left(\frac{n\pi x}{L}\right)$ are orthogonal. Let's prove this. First consider

$$\int_{-L}^{L} \frac{1}{\sqrt{L}} \sin\left(\frac{m\pi x}{L}\right) \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi x}{L}\right) dx$$

Using trigonometric identities, one can show that

$$\sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2} \sin \frac{(m-n)\pi x}{L} + \frac{1}{2} \sin \frac{(m+n)\pi x}{L}$$

Plugging this in above gives

$$\frac{1}{2L} \int_{-L}^{L} \sin \frac{(m-n)\pi x}{L} + \sin \frac{(m+n)\pi x}{L}$$

But this is just the integral of two sine functions over some whole number of periods, which is 0. Thus

$$\int_{-L}^{L} \frac{1}{\sqrt{L}} \sin\left(\frac{m\pi x}{L}\right) \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi x}{L}\right) dx = 0$$

- Next consider

$$\int_{-L}^{L} \frac{1}{\sqrt{L}} \sin\left(\frac{m\pi x}{L}\right) \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi x}{L}\right) dx$$

Again using trig IDs we can show that

$$\sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2} \cos \frac{(m-n)\pi x}{L} - \frac{1}{2} \cos \frac{(m+n)\pi x}{L}$$

so the above integral becomes

$$\frac{1}{2L} \int_{-L}^{L} \cos \frac{(m-n)\pi x}{L} - \cos \frac{(m+n)\pi x}{L}$$

If $m \neq n$ then we can make the same argument as above, but if $m = n$ then while the second term will still integrate to 0, the first will become simply $\cos(0) = 1$. Because the integral ranges over an interval of length $2L$, the result of the integral is simply $2L$. Thus

$$\int_{-L}^{L} \frac{1}{\sqrt{L}} \sin\left(\frac{m\pi x}{L}\right) \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi x}{L}\right) dx = \delta_{mn}$$

- Finally, consider
$$\int_{-L}^{L} \frac{1}{\sqrt{L}} \cos\left(\frac{m\pi x}{L}\right) \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi x}{L}\right) dx$$
Again, we can use the trig addition formulas to show that
$$\cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2} \cos\frac{(m-n)\,\pi x}{L} + \frac{1}{2} \cos\frac{(m+n)\,\pi x}{L}$$
so with the same arguments as before we have
$$\int_{-L}^{L} \frac{1}{\sqrt{L}} \cos\left(\frac{m\pi x}{L}\right) \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi x}{L}\right) dx = \delta_{mn}$$

- Now we are ready to discuss the ideas of Fourier series.

Fourier Series

- Consider a function $f(x)$ which can be written as a sum of trigonometric functions
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$
where the coefficients a_n and b_n are yet to be determined.

- Because it is written this way $f(x)$ must periodic with period $2L$ since all the functions in the above sum are $2L$ periodic.

- It turns out that *almost any function* can actually be written in this way! The proof of this is not something we're interested in as physicists, nor is the question of when this is or is not possible. Suffice it to say that essentially all functions we deal with in physics can be written like this.

- We now want to determine the coefficients in the above sum. To do so, first consider integrating both sides of the above equation from $-L$ to L
$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) dx$$
$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \frac{a_0}{2} dx$$
Since the integral of a sinusoidal function over a whole number of periods is 0. Thus we have
$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx$$
This gives us a_0. To find a_m, we multiply both sides of the equation by $\cos\frac{m\pi x}{L}$ and integrate from $-L$ to L
$$\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L} dx = \int_{-L}^{L} \frac{a_0}{2} \cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} + b_n \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\right) dx$$
$$\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} a_n L \delta_{mn}$$
$$\int_{-L}^{L} f(x) \cos\frac{m\pi x}{L} dx = a_m L$$

so

$$a_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

Similarly, to find b_n we multiply both sides of the equation by $\sin\frac{m\pi x}{L}$ and integrate from $-L$ to L

$$\int\limits_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = \int\limits_{-L}^{L} \frac{a_0}{2}\sin\frac{m\pi x}{L} + \sum_{n=1}^{\infty}\left(a_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\right)dx$$

$$\int\limits_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} b_n L\delta_{mn}$$

so

$$b_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

- Thus we can now define the Fourier series - a function defined on the interval $-L$ to L has the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

- Notice that by writing the function this way we are *making* it periodic on the whole real line, with period $2L$. This is called the *periodic extension* of the function. Let's see how this works in an example.

Example 1

Calculate the Fourier series for the function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

on the interval $[-\pi, \pi]$.

- Here $L = \pi$. Also note that $f(x)$ is odd, so the integral for the a_n

$$a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0$$

since the integrand is odd.

- For the b_n on the other hand, we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)\,dx$$

$$b_n = \frac{2}{\pi n}\left[-\cos(nx)\right]_0^{\pi}$$

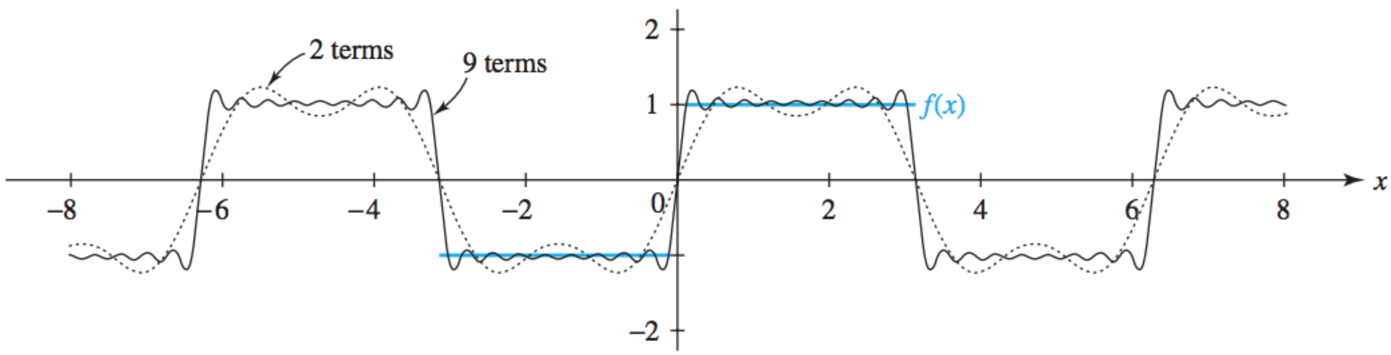
$$b_n = -\frac{2}{\pi n}\left[(-1)^n - 11\right]_0^{\pi}$$

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd} \end{cases}$$

- Thus the Fourier series is given by

$$f(x) = \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty} \frac{\sin(nx)}{n} = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$$

- A plot of the function is shown below. Notice that the function itself is not periodic, but by taking $L = \pi$ we've *extended* it to be periodic with period 2π



Applications of Fourier Series

- Fourier series show up in a *huge* number of areas of physics. Perhaps one of the most basic is the solution for a wave on a string.
- Consider a string of length L, fixed at both ends. By going through the physics of the problem we can show that we need to solve the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial y^2}{\partial t^2}$$

where c is the velocity that waves on the string travel at. By doing some math that I won't go into here, we can show that our solutions take the form

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\omega t + B_n \cos\omega t\right)$$

where $\omega = \frac{n\pi}{L}c$. But how do we determine the A_n and B_n? Suppose the string is initially at rest, but has some initial stretched length given by $y(x,0)$. Then

$$y(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

and

$$v\left(x,0\right)=\dot{y}\left(x,0\right)=\sum_{n=1}^{\infty}\omega B_n\sin\left(\frac{n\pi x}{L}\right)=0$$

so we can immediately say that $B_n=0$ for all n, but what about A_n? The equation for $y\left(x,0\right)$ is just the Fourier series for $y\left(x,0\right)$! Actually it's technically the Fourier *sine* series, but that distinction isn't really important for our purposes. We can get the coefficients in the same way we did above, by multiplying by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating. We now only integrate from 0 to L, because that's the domain we're interested in.

$$\int_0^L y\left(x,0\right)\sin\left(\frac{m\pi x}{L}\right)dx=\sum_{n=1}^{\infty}\int_0^L A_n\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

But the sine function is orthogonal on this interval as well, as we can easily show, and we get

$$\int_0^L y\left(x,0\right)\sin\left(\frac{m\pi x}{L}\right)dx=\sum_{n=1}^{\infty}A_n\frac{L}{2}\delta_{mn}$$

$$A_n=\frac{2}{L}\int_0^L y\left(x,0\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

So using Fourier series we can determine the position of the string at all later times!

Complex Fourier Series and the Fourier Transform

- It turns out there is nicer way to write the Fourier series if we allow the coefficients to be complex.
- I won't go into the details here, suffice it to say that using Euler's Formula

$$e^{i\theta}=\cos\theta+i\sin\theta$$

we can write the Fourier series of a function as

$$f\left(x\right)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi x}{L}}$$

where the c_n are now in general *complex* numbers, and are given by

$$c_n=\frac{1}{2L}\int_{-L}^{L}f\left(x\right)e^{\frac{in\pi x}{L}}dx$$

Note that $n$can now be both positive and negative, rather than just positive.

- This form is useful because it allows us to define the Fourier transform. To proceed, let's rewrite the above expression as

$$f\left(x\right)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{in\pi x}{L}}\Delta n$$

where $\Delta n=1$ is just the spacing between integers. Now define

$$k\equiv\frac{\pi n}{L}$$

$$\Delta k\equiv\frac{\pi\Delta n}{L}$$

and

$$A\left(k\right)=\frac{\sqrt{2\pi}La_n}{\pi}$$

This turns our Fourier series into

$$f\left(x\right)=\sum_k\frac{A\left(k\right)}{\sqrt{2\pi}}e^{ikx}\Delta k$$

- Now consider the case that $L \to \infty$. In this limit Δk becomes infinitesimally small, so the sum over discrete values turns into an integral

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ikx} dk$$

This function defines $f(x)$ in terms of the *Fourier transform* of the function $A(k)$. This is often referred to as the "backwards transform." Again, one can show, though we won't here, that $A(k)$ is given by

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx$$

- Like the Fourier series, the Fourier transform has applications in a huge number of physics problems. One example is a wave traveling on a string. Because the string no longer has fixed length, it turns out we need to consider the Fourier transform to solve the behavior of the wave, and not the Fourier series.

The Dirac Delta Function

- We now shift gears and discuss the Dirac delta function. One way to motivate this is to consider the mass density of a point particle. Suppose we have a particle centered at the origin with mass m - what is the mass density of the particle? It should satisfy the condition that

$$\int_D \rho(x)\, dx = \begin{cases} m & \text{if } D \text{ contains the origin} \\ 0 & \text{else} \end{cases}$$

But what kind of function obeys this condition? The answer is the Dirac delta function, which can be defined via

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

The most useful property of the delta function is that it "picks up" the value of our integrand at some point if we integrate over all values of x, that is

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a) \int_{-\infty}^{\infty} \delta(x-a)\, dx = f(a)$$

since the integrand is zero at all values of x except $x = a$. Returning to our example of mass density, we see that we can write the density of the particle as

$$\rho(x) = m\delta(x)$$

which has the properties we discussed above. Again, the delta function shows up in a large number of physical situations, and is an invaluable tool for a physicist.

Fourier Transform of the Dirac Delta Function

- Finally, consider the Fourier transform of the Dirac delta function

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - x_0)\, e^{-ikx} dx$$

$$A(k) = \frac{1}{\sqrt{2\pi}} e^{-ikx_0}$$

- But Fourier transforms act in pairs, so this implies that the reverse must be true as well, that is the Fourier transform of the complex exponential is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{-ikx_0} e^{ikx} dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k)\, e^{ik(x-x_0)} dk = \frac{\delta(x-x_0)}{\sqrt{2\pi}}$$

- This is another extremely useful result that comes up frequently in physics.