Physics 103 - Discussion Notes #3

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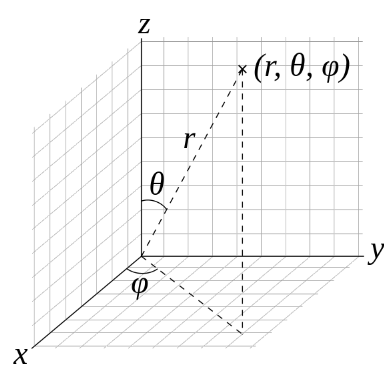
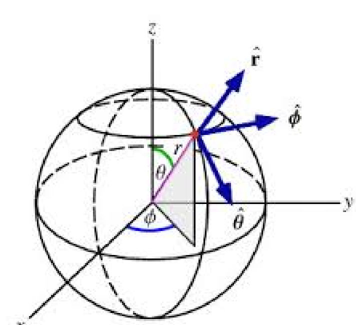
Spherical Coordinates

- All of you are used to Cartesian coordinates, which serve as a convenient coordinate system for many physical problems.
- However, there are a large number of physical situations where the use of cartesian coordinates is extremely cumbersome, and is it is far more natural to describe our system in what are known as *spherical coordinates*.
- In Cartesian coordinates, we describe an arbitrary vector $\mathbf{a}$ by specifying its *components* along the three basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.
- It's important to remember, however, that this vector exists *independently* of the coordinate system we use to describe it, and that we can describe the same vector by specifying its components with respect to any three linearly independent vectors.
- In spherical coordinates, we specify a point vector by giving the radial coordinate r, the distance from the origin to the point, the polar angle θ, the angle the radial vector makes with respect to the z axis, and the azimuthal angle ϕ, which is the normal polar coordinate in the $x-y$ plane.



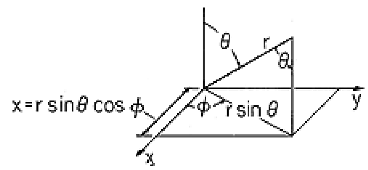
- We can specify a *vector* in spherical coordinates as well. Before we do this we need to discuss how we define our basis vectors in a general coordinate system. In Cartesian coordinates our basis vectors are simple and unchanging, but in spherical things aren't quite so simple.
- The basis vector with respect to a certain coordinate direction, $\hat{\mathbf{a}}$, is *the direction the position vector will move in if we increase the coordinate a while leaving all other coordinates fixed* (and normalized to 1 of course).
- If we look at the basis vectors $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\phi}$ below we see that they agree with this idea.



- Notice now however, *that our basis vectors are now dependent on position.* If we move to a different point on the sphere, our unit vectors will point in a different direction. This is emphatically NOT true in Cartesian coordinates, where the unit vectors are the same no matter what point we're describing. This makes some operations in spherical much more complex than their cartesian counterparts, which we'll come back to soon.

Spherical Unit Vectors in Terms of Cartesian

- We can explicitly show that the spherical unit vectors depend on position by calculating their components in Cartesian coordinates.
- To begin, we first must determine how to convert between Spherical and Cartesian coordinates.



- From the above picture, we can see that, in terms of Cartesian coordinates, the spherical coordinates are given by

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

- On the other hand, from the Pythagorean theorem we have

$$r = \sqrt{x^2 + y^2 + z^2}$$

while from simple trig its easy to see that

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

- We can use these expressions to convert spherical coordinates into cartesian and vice-versa.
- To determine the spherical unit vectors in terms of cartesian coordinates, we go back to how we defined the unit vectors. From our definition, we see that we can write

$$\hat{\mathbf{r}} = \frac{\frac{d\mathbf{r}}{dr}}{\left|\frac{d\mathbf{r}}{dr}\right|}$$

Now in Cartesian the position vector is simply given by

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$
$$\mathbf{r} = r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}$$

so

$$\frac{d\mathbf{r}}{dr} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

and

$$\left|\frac{d\mathbf{r}}{dr}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos\theta^2}$$
$$\left|\frac{d\mathbf{r}}{dr}\right| = \sqrt{\sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos\theta^2}$$
$$\left|\frac{d\mathbf{r}}{dr}\right| = \sqrt{\sin^2\theta + \cos\theta^2} = 1$$

Thus

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

Notice that $\hat{\mathbf{r}}$ depends on the angles θ and ϕ and thus depends on position, as expected. We can do the exact same thing to determine $\hat{\phi}$ and $\hat{\theta}$

$$\hat{\theta} = \frac{\frac{d\mathbf{r}}{d\theta}}{\left|\frac{d\mathbf{r}}{d\theta}\right|}$$

$$\hat{\phi} = \frac{\frac{d\mathbf{r}}{d\phi}}{\left|\frac{d\mathbf{r}}{d\phi}\right|}$$

I won't go through the details of the calculation here - it's straightforward, though a little tedious. The result is

$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$

$$\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

Non-Constancy of Unit Vectors

- We now discuss the issues raised by the fact that the spherical unit vectors are dependent on position. Consider a particle with position vector $\mathbf{r}$, with Cartesian components (r_x, r_y, r_z) . Suppose now we wish to calculate the velocity of the particle, as we did in the first homework. The answer of course, is simply

$$\mathbf{v} = \frac{dr_x}{dt}\hat{\mathbf{x}} + \frac{dr_y}{dt}\hat{\mathbf{y}} + \frac{dr_z}{dt}\hat{\mathbf{z}}$$

This may seem straightforward, but there's an extremely important subtlety that many of you are probably missing. Consider the first term - we need to take the time derivative of $r_x\hat{\mathbf{x}}$. *Really*, this is

$$\frac{d}{dt}\left(r_x\hat{\mathbf{x}}\right) = \frac{dr_x}{dt}\hat{\mathbf{x}} + r_x\frac{d\hat{\mathbf{x}}}{dt}$$

Now we come to what makes Cartesian coordinates so useful, apart from their conceptual simplicity - $\hat{\mathbf{x}}$ *is not a function of time, even if the particle is changing position.* None of the Carteisan unit vectors are dependent on position, and therefore aren't dependent on time. This is not true of e.g. $\hat{\mathbf{r}}$, as we showed earlier. Since $\hat{\mathbf{r}}$ is dependent on position if the particle is moving then $\hat{\mathbf{r}}$ is time dependent. With this in mind, let's calculate the velocity of a particle in spherical coordinates. The position vector is now given simply by

$$\mathbf{r} = r\hat{\mathbf{r}}$$

so

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}}$$

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\left(\frac{d\hat{\mathbf{r}}}{dr}\frac{d\mathbf{r}}{dt} + \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} + \frac{d\hat{\mathbf{r}}}{d\phi}\frac{d\phi}{dt}\right)$$

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\left(0 + \left(\cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}\right)\dot{\theta} + \left(-\sin\theta\sin\phi\hat{\mathbf{x}} + \sin\theta\cos\phi\hat{\mathbf{y}}\right)\dot{\phi}\right)$$

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\left(\hat{\theta}\dot{\theta} + \dot{\phi}\sin\theta\hat{\phi}\right)$$

We can do a similar calculation for the acceleration - it proceeds exactly as with the velocity, so it's straightforward but again rather tedious. When the dust settles we get

$$\mathbf{a} = \hat{\mathbf{r}}\left(\ddot{r} - r\dot{\phi}^2\sin^2\theta\right) + \hat{\theta}\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right) + \hat{\phi}\left(r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + 2\dot{r}\dot{\phi}\sin\theta\right)$$

- The fact that the unit vectors are not constant means there are other subtleties when working in spherical coordinates as well. For instance when integrating vector function in Cartesian coordinates we can take the unit vectors outside the integral, since they are constant. *This is no longer the case in spherical!* Often it's better to convert your unit vectors back to Cartesian before attempting to do any integration.

Line and Volume Integrals in Spherical Coordintes

- Let us now determine the line elements in spherical coordinates - that is how much a particle moves when we infinitesimally displace it in one of the coordinate directions. To begin with, if we infinitesimally displace an object in the $\hat{\mathbf{r}}$ the change in position is just dr

$$dl_r = dr$$

Now clearly this does not work in the $\hat{\theta}$ direction, since $d\theta$ doesn't have units of length - instead if we increment the θ coordinate of the position vector, the particle moves by a distance

$$dl_\theta = r d\theta$$

Incrementing the ϕ coordinate does something similar, but we now have to take into account that the relevant radius is no longer just r, but the projection of r into the $x - y$ plane, $r\sin\theta$. This gives

$$dl_\phi = r\sin\theta d\phi$$

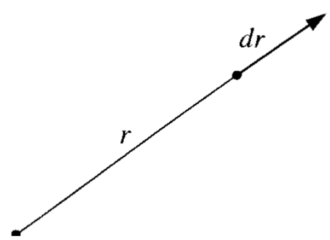
And thus a general line element is given by

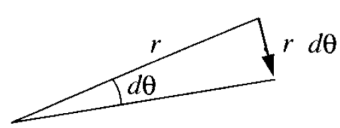
$$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$$

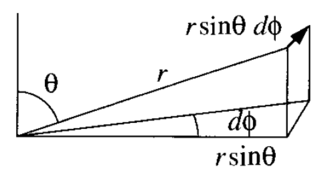
and the volume element is given by

$$dV = dl_r dl_\theta dl_\phi = r^2 \sin\theta dr d\theta d\phi$$

DO NOT forget this when doing integrals in spherical coordinates! Doing so will give you nonsensical results when performing triple integrals in spherical coordinates.







Grad, Curl, Divergence and Laplacian in Spherical Coordinates

- In principle, converting the gradient operator into spherical coordinates is straightforward. Recall that in Cartesian coordinates, the gradient operator is given by

$$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

where T is a generic scalar function. Thus, to calculate e.g. the $\hat{\mathbf{x}}$ component of the gradient, we would simply employ the chain rule

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial T}{\partial \phi}\frac{\partial \phi}{\partial x}$$

and so on for the y and z components. Unfortunately, this method is extremely tedious, and it would take an extremely long time to calculate the gradient this way, not to mention the Laplacian, which is even more complex. Fortunately, there exist more sophisticated methods to treat general system of coordinates, from which we can obtain the gradient much more quickly. I won't go into these methods in detail here, but Griffiths' *Introduction to Electrodynamics* has a nice treatment in Appendix A. Instead I will simply quote the result for the gradient, as well as the curl, divergence, and Laplacian in spherical

$$
\begin{aligned}
\nabla T &= \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi} \\
\nabla \cdot \mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \\
\nabla \times \mathbf{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta})\left(\sin\theta v_\phi\right) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r v_\phi\right)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi} \\
\nabla^2 T &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}
\end{aligned}
$$

where v_r is the $\hat{\mathbf{r}}$ component, v_θ is the $\hat{\theta}$ component, and v_ϕ is the $\hat{\phi}$ component.