Physics 103 Final Exam
Summer 2015

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Friday, July 31st

Name: ______________________ **Perm:** ______________

- This exam is **closed book**, other than a "cheat sheet" - you may use one 8.5"x11" sheet of paper, with whatever notes you want, *hand-written*, on **one side** of the paper.
- You will have a total of **three hours** to complete the exam.
- A basic graphing calculator is allowed on the exam. However, any other electronics, such as phones, laptops, or anything else that can communicate with the outside world, is prohibited. **A smartphone on airplane mode is not a valid calculator.** If you are seen using any of these devices during the exam, they will be confiscated until the end of the exam.
- Please choose **five out of seven** problems to solve. Please mark clearly **which five** problems you want graded, either by circling the problem numbers, or in some other way clearly marking five problems.
- **If you fail to mark five problems clearly, the TAs and I make no promises as to how we will grade your exam.**
- All of the problems are designed to be roughly equal in length and difficulty. Doing all seven problems will **not** get you extra credit, although you may want to think about all seven problems before picking five of them to do.

Good luck!

1. A particle of mass m is **released from rest** in a constant gravitational field, and falls downwards near the surface of the Earth (in other words, the magnitude of the acceleration due to gravity is simply g). Additionally, the particle encounters a drag force which is proportional to the **fourth power** of the velocity,

$$\boldsymbol{F} = -kv^4\hat{v}.$$

As the particle accelerates from some velocity v_0 to some other velocity v_1, find the amount of distance it travels, as a function of the velocities,

$$s\left(v_0 \to v_1\right) = ?$$

Hint: Use the chain rule when solving the differential equation.

2. Consider a particle of mass m, in one dimension, subject to the force

$$F(x) = -\phi x^7,$$

where ϕ is some positive constant (make sure to notice that x is raised to the **seventh** power). Assume that the particle is released, **from rest**, at a position $x = -a$, where a is some positive constant.

(a) **How much time** does it take for the particle to travel from $x = -a$ to $x = +a$? Write your answer in the form

$$T(-a \to +a) = N f(m, \phi, a),$$

where f is a function of m, ϕ, and a, and N is a numerical constant with **no** dependence on these parameters (you do **not** need to determine what N is). Notice that this quantity is **half** of the oscillation period.

(b) Now imagine that a **kinetic friction** force is **added**,

$$F_f = \begin{cases} -b\hat{v} & \text{if the particle is moving} \\ 0 & \text{if the particle is not moving} \end{cases}$$

where b is a positive constant. If the particle is released again from a position $x = -a$, **how much total work** is done on the particle as it travels from $x = -a$ to $x = 0$? Remember to include the force from part (a)! Based on this expression, is it ever possible for the particle to simply not make it to the origin? If so, how must a be related to the other two parameters so that it **can** make it to the origin?

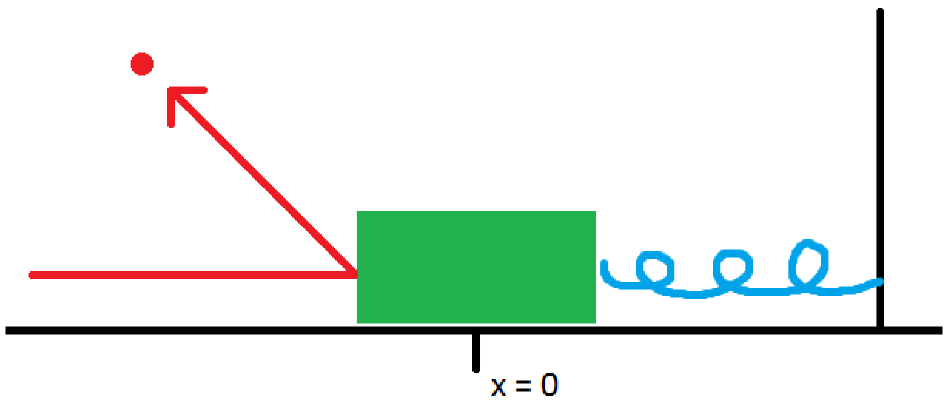
3. Consider the equation
$$x^2 + x + \tanh(\lambda x) = 0.$$

(a) When $\lambda = 0$, how many solution(s) does this equation have? What are these solutions?

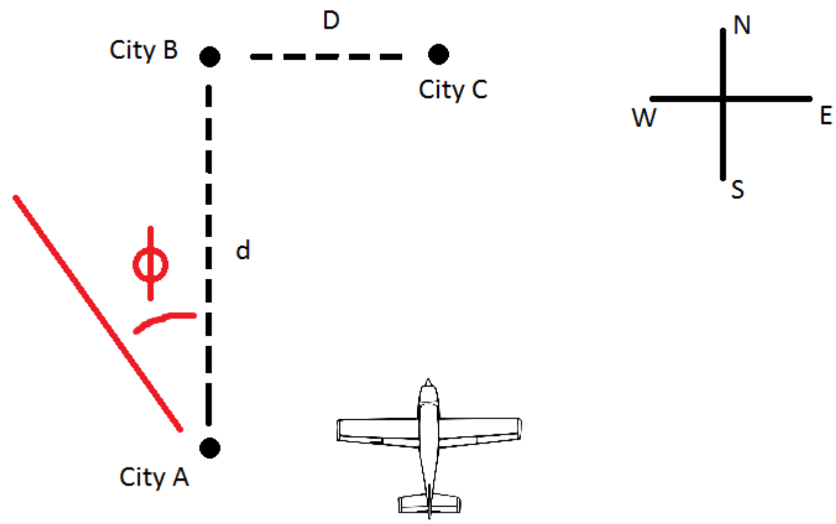
(b) Denote the **largest** solution from part (a) as x_0^L. Show that x_0^L is **always** a solution to the equation, even when $\lambda \neq 0$.

(c) Focusing on the other solution(s) (that is, the solution(s) which are **not** x_0^L), assume that there is some series expansion of x, such that
$$x(\lambda) = x_0 + x_1\lambda + x_2\lambda^2 + x_3\lambda^3 + ...$$
Find the values of these four coefficients. In other words, find the expansion of $x(\lambda)$ **through third order** in λ.

4. Consider a damped harmonic oscillator, with un-damped frequency ω and damping parameter β. This oscillator describes a block with is attached to a spring, as shown in the Figure below. We assume that before $t = 0$, the block is sitting motionless at $x = 0$, which is its equilibrium point. At time $t = 0$, a bullet fired from a gun bounces off of the block. The impact is virtually instantaneous, and during the collision, the block gains an energy equal to E. Find the motion of the block as a function of time, for **both** the over-damped and under-damped cases. Your answer should involve E, ω, and β.



5. A plane sets out on a journey from City A, as shown in the Figure below. The plane flies *through the air around it* at a speed v. As the plane flies through the air, there is a **cross-wind**: the air is moving East with a speed w, *with respect to the ground*.



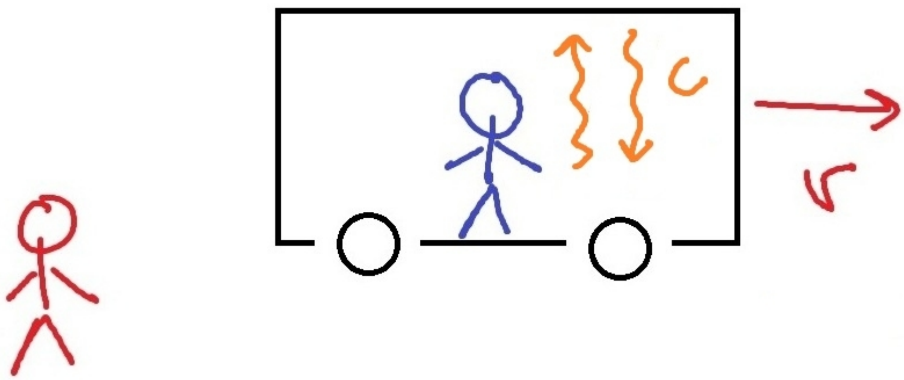
(a) The pilot first considers a trip to City B, which is due north of City A, a distance d away. If the pilot wants to fly straight North to City B, at what angle, ϕ, does he need to orient his plane?

(b) How long does the journey in part (a) take?

(c) The next day, the pilot decides to fly to City C, again departing from City A. City C is due east from City B, a distance D away. If the pilot wants to fly straight to City C from City A, at what angle, ϕ, does he need to orient his plane?

Please specify your answers in terms of the parameters of the problem, v, w, d, and D, and measure the angle as indicated in the figure (increasing counter-clockwise from North).

6. Consider two observers, one on the ground, and one in a passing train, shown below. According to the ground-based observer, the train passes by with velocity v. In this scenario, the ground-based observer will use a coordinate t to measure times, while the train-based observer will use a coordinate t'. Their clocks are synchronized so that $t = t' = 0$ when their origins coincide.



(a) The train-based observer fires a light-pulse from the floor of the train, which then reflects off of the roof, and hits the floor again, in the same spot. If the train-based observer believes that this takes an amount of time dt', how long does it take, dt, according to the ground-based observer? Specify your answer in terms of v. **Hint:** Which time duration should be the longer one?

(b) Assume now that the train travels some distance down the track, turns around, and meets up with the ground-based observer. If the ground is approximately an inertial frame, which of the two observers has **aged less** in this process?

(c) Assume now that the velocity of the train, as measured by the ground-based observer, is not constant, but rather

$$v(t) = c \tanh(t/\tau) \;\; ; \;\; \tau > 0,$$

where c is the speed of light. Assume that the relation you found in part (a) still holds in this case, with $v \to v(t)$, for infinitesimal time-steps dt and dt'. If the ground-based observer's watch has ticked away an amount of time T since the origins coincided, how much time does he say has elapsed on the train's clock, T'? **Hint:** Make sure to notice it is $v(t)$ that you are working with, and **NOT** $v(t')$!

(d) Viewing T' as a function of T, what happens to T' as $T \to \infty$? If the ground-based observer spends all of eternity waiting on the station platform, what is the maximum amount of time that elapses on the train?

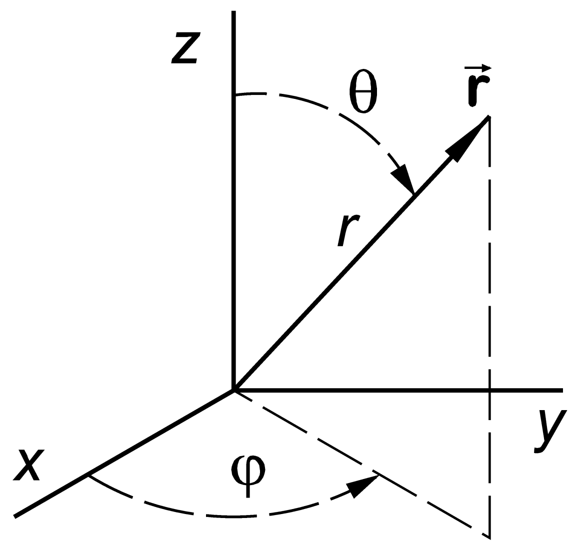
7. A particle of mass m is constrained to move frictionlessly on the surface of a **sphere of radius** R. In addition to the constraint forces which keep the particle on the sphere, there is a gravitational force described by the potential

$$U(z) = mgz$$

To describe the motion of this particle, we will use a spherical coordinate system,

$$x = r \sin\theta \cos\phi \ ; \ y = r \sin\theta \sin\phi \ ; \ z = r\cos\theta,$$

shown below. The origin of the coordinates will coincide with the center of the sphere.



(a) Explain why the z-component of the angular momentum, L_z, is conserved.

(b) Use the expressions for the kinetic energy and angular momentum,

$$K = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) \ ; \ L_z = mr^2\sin^2\theta\dot{\varphi},$$

in order to write the total (conserved) energy as

$$E = \frac{1}{2}m_*\dot{\theta}^2 + U_{\text{eff}}(\theta).$$

What are m_* and U_{eff}? Your answer should involve m, g, R, and L_z.

(c) Assuming $L_z \neq 0$, which point(s) on the sphere can the particle never visit?

(d) As the coordinate θ moves in the effective potential, it oscillates between some smallest value θ_a and some largest value θ_b. Write down an **integral expression** for **how much time**, $T_{a\to b}$, it takes the particle to travel from θ_a to θ_b. Do **not** make any effort to solve this integral, and do **not** attempt to find θ_a or θ_b.

(e) Write down an integral expression for **how much polar angle**, $\varphi_{a\to b}$, is swept out as the particle travels from θ_a to θ_b. Do **not** make any effort to solve this integral.

Extra Work Space

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Some potentially useful facts:

$$e^{ix} = \cos(x) + i\sin(x) \quad \Rightarrow \quad \cos(x) = \frac{e^{ix}+e^{-ix}}{2} \ ; \ \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$$

$$\hat{f}(\nu) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\nu t} dt \ \Leftrightarrow \ f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\nu)\, e^{+i\nu t} d\nu$$

$$\int_a^b \frac{x^{n-1}}{\alpha+\beta x^n} dx = \frac{1}{n\beta}\ln\left[\frac{\alpha+\beta b^n}{\alpha+\beta a^n}\right]$$

$$\int_a^b \frac{x^{n-1}}{\alpha-\beta x^{2n}} dx = \frac{1}{n\sqrt{\alpha\beta}}\left[\operatorname{arctanh}\left(\sqrt{\frac{\beta}{\alpha}}b^n\right) - \operatorname{arctanh}\left(\sqrt{\frac{\beta}{\alpha}}a^n\right)\right]$$

$$\int_{-\infty}^{\infty} e^{+ixt}\ dt = \int_{-\infty}^{\infty} e^{-ixt}\ dt = 2\pi\delta(x)$$

$$\tanh(\phi) \approx \phi - \frac{1}{3}\phi^3 + \frac{2}{15}\phi^5 - \frac{17}{315}\phi^7 + \ldots$$

$$\left(\frac{d}{dt}+\phi\right) G(t;t_0) = \lambda\delta(t-t_0) \quad \Rightarrow \quad G(t;t_0) = \lambda e^{-\phi(t-t_0)}\Theta(t-t_0)$$

$$\begin{cases} \left(\frac{d^2}{dt^2}+2\beta\frac{d}{dt}+\omega^2\right) G(t;t_0) = \lambda\delta(t-t_0) \quad \Rightarrow \quad G(t;t_0) = \frac{\lambda}{2\Omega i}\left[e^{\alpha_+(t-t_0)} - e^{\alpha_-(t-t_0)}\right]\Theta(t-t_0) \\ \Omega = \sqrt{\omega^2-\beta^2} \ ; \ \alpha_\pm = -\beta \pm \sqrt{\beta^2-\omega^2} \end{cases}$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$A\sin\theta + B\cos\theta = \sqrt{A^2+B^2}\,\sin(\theta+\delta) \ ; \ \tan\delta = B/A$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\cosh^2(x) - \sinh^2(x) = 1 \ ; \ 1-\tanh(x)^2 = \operatorname{sech}^2(x) \ ; \ \operatorname{csch}^2(x) = \coth^2(x) - 1$$

$$\int \operatorname{sech}(ax)dx = a^{-1}\arctan(\sinh(ax)) + C$$

$$\arctan(\infty) = \pi/2 \ ; \ \arctan(0) = 0 \ ; \ \sinh(0) = 0 \ ; \ \sinh(\infty) = \infty$$

The fundamental theorem of algebra: “every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n roots.”