# Physics 103 Homework 1
# Summer 2015

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Due Date: Thursday, July 2nd

1. Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, show that their sum $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ satisfies the law of cosines:

$$c^2 = a^2 + b^2 + 2ab\cos\theta$$

where $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ (Hint: remember that by definition, $c^2 \equiv \boldsymbol{c} \cdot \boldsymbol{c}$). Using this result, explain why the triangle inequality

$$|\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}|$$

is true.

2. The position vector of a particle that moves on a two-dimensional surface, as a function of time, is given by

$$\boldsymbol{r}(t) = \lambda R \sin(\omega t)\,\hat{x} + R\cos(\omega t)\,\hat{y},$$

where $\lambda > 0$, $R > 0$, and $\omega > 0$ are all constants, the first of which is dimensionless, the second of which has units of length, and the last of which has units of inverse time.

   (a) Find the velocity, acceleration, and speed of the particle as a function of time.

   (b) Find the angle between the velocity vector and the acceleration vector as a function of time. In particular, what is this angle at a time $t = \pi/2\omega$? What does your expression reduce to when $\lambda = 1$?

   (c) What type of shape does the particle trace out as it moves through space? Can you explicitly show that the particle's trajectory satisfies the defining equation for this shape?

3. Consider the projectile motion problem from lecture, with linear (n=1) air resistance included.

   (a) How long does it take for the projectile to reach its maximum height? Call this time duration $t_M$. You should be able to solve this exactly, without any approximations. You can make use of anything derived in the lecture notes, including the expression for the velocity as a function of time.

(b) What is the maximum height itself? Call this maximum height $h_M$. You should also be able to solve this exactly.

(c) Take your expressions for $t_M$ and $h_M$ and Taylor expand them in the drag coefficient, $k$, out to third order. Do your results make sense for the case that $k = 0$?

(d) Compared with the result for motion in a vacuum, does air resistance increase or decrease the time it takes for the projectile to reach its maximum height?

(e) What is the acceleration of the projectile when it is at its maximum height?

4. Consider the equation

$$\lambda + \cos(\lambda x) = x,$$

where $\lambda$ is some small parameter.

(a) Assume that there is a solution to this equation, $x_S(\lambda)$, which depends on $\lambda$ and has a Taylor series expansion

$$x_S(\lambda) = x_{S0} + x_{S1}\lambda + x_{S2}\lambda^2 + x_{S3}\lambda^3 + \ldots$$

Find the first three terms in this expansion, using the methods we discussed in lecture. Does your answer make sense in the limit that $\lambda = 0$? You may want to look up the Taylor series expansion for the cosine function.

(b) Make a plot of the functions $f(x) = x$ and $g(x) = \lambda + \cos(\lambda x)$ for the values $\lambda = 0.5, 1.5, 2.0, 2.25, 2.5, 7.5$. Examine how the two functions intersect each other, for each value of $\lambda$. Based on how the different intersections behave for different values of $\lambda$, do you think the perturbative result you derived in part a might have some limited range of validity? Why or why not?

5. Consider the projectile motion problem from lecture, with linear (n=1) air resistance included. We found that the time the projectile hit the ground was defined by the equation

$$hk^2 + \left[gm^2 + \phi mk\right]\left[1 - e^{-kt_R/m}\right] = gmkt_R$$

Let's make the simplifying assumption that the projectile is fired from the ground, so that $h = 0$. In this case, our equation can be written

$$t_R = \left[\frac{m}{k} + \frac{\phi}{g}\right]\left[1 - e^{-kt_R/m}\right]$$

When there is no air resistance, we know that the time the projectile hits the ground is given by

$$t_R^{(0)} = \frac{2\phi}{g}$$

(a) Assume that $m = 1$ kilogram, $\phi = v_0 \sin\theta = 100$ meter/second, $k = 0.01$ kilogram/second, and $g = 10$ meter/second$^2$. What is $t_R^{(0)}$ in this case?

(b) If the drag coefficient $k$ is suitably small, then we know that

$$t_R \approx t_R^{(0)},$$

with the difference being some small correction. As a result, we should expect that

$$\left[\frac{m}{k} + \frac{\phi}{g}\right]\left[1 - e^{-kt_R/m}\right] \approx \left[\frac{m}{k} + \frac{\phi}{g}\right]\left[1 - e^{-kt_R^{(0)}/m}\right]$$

With this in mind, let's define the quantity

$$t_R^{(1)} \equiv \left[\frac{m}{k} + \frac{\phi}{g}\right]\left[1 - e^{-kt_R^{(0)}/m}\right].$$

Using the numerical values in the problem, along with your value for $t_R^{(0)}$, compute the value of $t_R^{(1)}$. How do $t_R^{(1)}$ and $t_R^{(0)}$ compare?

(c) Continuing this idea, for a general $n$, define the quantity

$$t_R^{(n+1)} \equiv \left[\frac{m}{k} + \frac{\phi}{g}\right]\left[1 - e^{-kt_R^{(n)}/m}\right].$$

Compute each $t_R^{(n)}$, up through $n = 6$. What is happening to the value of each successive $t_R^{(n)}$?

(d) The result of a sophisticated computer calculation tells us that, for these parameter values,

$$t_R \approx 19.3748,$$

accurate to this many decimal places. How do the values you found compare with this quantity?

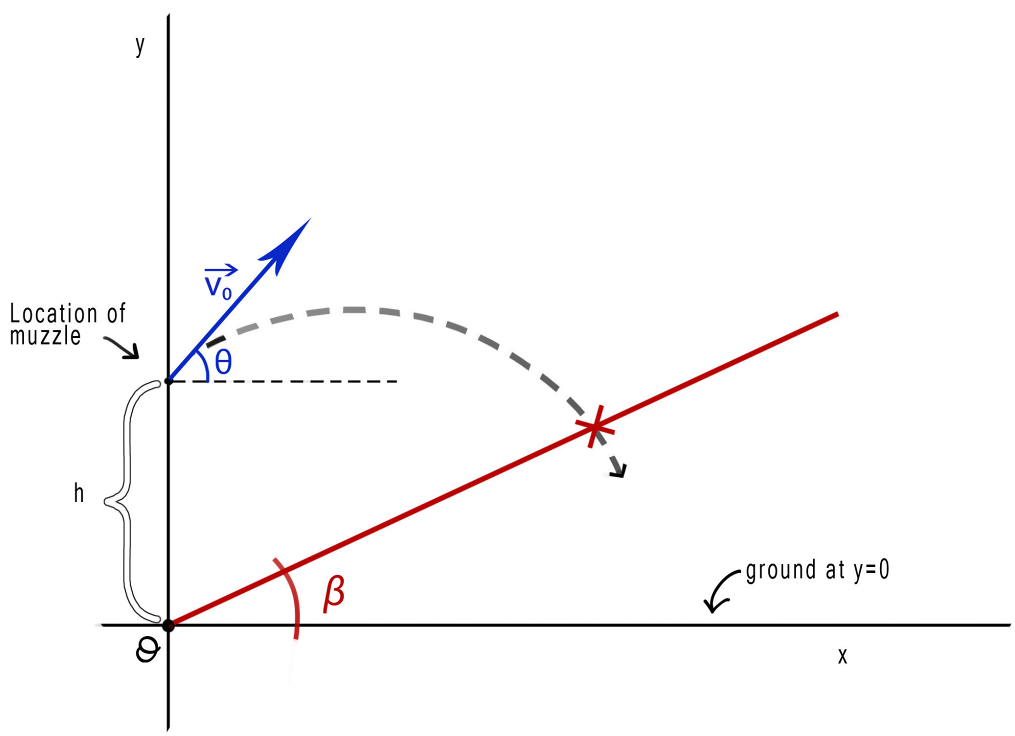
(e) The method we've developed here is an example of a *root-finding algorithm* - it is a numerical approach for trying to find better and better approximations to the solution of an equation. Many different types of root-finding algorithms exist, and deciding on which algorithm best suits a particular equation can be a very involved subject. The method we have developed here is somewhat primitive, and most computer software makes use of more advanced techniques. However, it still gives us a sense as to how a computer, which is capable of doing many numerical calculations very quickly, might be able to find the solution to an equation. **Comparing this method with the perturbative technique we developed in class, what do you think are some pros and cons of each method?**

(f) **Extra Credit:** Use Mathematica on the computers in the Physics Study Room to write a **For Loop** that finds the first 100 terms in the sequence of $t_R^{(n)}$ values. How quickly do the values approach the "exact" result given above? Print out your Mathematica notebook and attach it to your homework.

6. Consider again the projectile motion problem from lecture, with linear (n=1) air resistance included. This time, however, imagine that we are standing at the base of a hill, firing the projectile towards the incline. This is shown in the figure below. The red line represents the inclined hill, which we will assume is a straight line passing through the origin, making an angle $\beta$ with the horizontal. All of the other details of the problem are the same as they were in class.

   (a) Find an equation that defines the time, $t_H$, when the projectile runs into the hill. You can use all of the results already derived in the lecture notes. **Hint:** What is the equation that defines the shape of the hill?

   (b) Accurate to first order in the drag coefficient $k$, solve for the time $t_H$. **Hint:** With some clever thought, this problem involves very little work!




7. In class, we used the Taylor series expansion

$$\frac{1}{gm + \phi k} = \frac{1}{gm}\left(\frac{1}{1 + \frac{\phi}{gm}k}\right) = \frac{1}{gm} - \frac{\phi}{g^2m^2}k + \frac{\phi^2}{g^3m^3}k^2 + ...$$

when we were deriving the range of a projectile subject to air resistance. However, the Taylor series expansion for this particular function only converges when

$$\frac{\phi}{gm}k < 1 \;\Rightarrow\; k < \frac{mg}{v_0 \sin\theta}.$$

This means that values of $k$ which violate this inequality can no longer be considered "small" enough for our perturbative result to be accurate. Considering the physics of the problem, explain qualitatively why $mg$ appears in the numerator of this equation, and why $v_0 \sin\theta$ shows up in the denominator.

8. Consider a projectile fired straight upwards into the air, starting from the ground. Since there is no motion in the horizontal direction, this is effectively a one-dimensional problem. For simplicity, we'll refer to the height of the projectile simply as $y(t)$, and the velocity simply as $v(t)$. Additionally, assume that the projectile is subject to the drag force

$$\boldsymbol{F}_R = -kv^4\hat{v},$$

in addition to the usual force of gravity. That is, the air resistance acting on the particle is proportional to the **fourth** power of velocity.

   (a) First consider the motion of the projectile right after it is fired, when it is travelling upwards. What is the total force on the projectile? **Hint:** Be careful when considering the sign of the drag force!

   (b) Find a differential equation for $y(t)$ when the particle is travelling upwards. Then, convert this to a differential equation for $v(t)$.

   (c) Using the chain rule for derivatives, we have

$$\dot{v} = \frac{dv}{dt} = \frac{dy}{dt}\frac{dv}{dy} = v\frac{dv}{dy},$$

   Use this fact to convert your differential equation into an expression of the form

$$\frac{dv}{dy} = f(v),$$

   where $f(v)$ is some function of $v$. This change of variables for the differential equation lets us consider the velocity of the projectile as being a function of its height, $v(y)$, rather than a function of time. In some applications this form of the differential equation is much more useful, as we'll see shortly.

   (d) As long as the projectile is travelling upwards, its velocity should steadily decrease as its height increases - the larger the height, the smaller the velocity. Because of this, we expect that $v(y)$ should be an *invertible* function of $y$, so that the inverse function $y(v)$ also exists. In this case, the inverse function theorem tells us that we can write

$$\frac{dv}{dy} = f(v) \;\Rightarrow\; \frac{dy}{dv} = \frac{1}{f(v)}$$

Writing the differential equation in this way, solve the differential equation for $y(v)$, assuming that the initial velocity is $v_0$.

(e) Setting $v = 0$ in the equation you just found, what is the maximum height, $y_M$, of the projectile? **Notice that we never had to explicitly find $y(t)$ in order to answer this question.** This is where the magic of the change of variables pays off.

(f) Repeat these steps for the **downward** motion of the projectile, finding the function $y(v)$ as it is falling from its maximum height. Be careful that you have the correct sign for the drag force! Also, remember that in this case, the initial velocity is zero, since the projectile comes to rest at its maximum height.

(g) By setting $y(v_g) = 0$ in the above equation you found, find the velocity $v_g$ of the projectile right before it crashes back into the ground. You should be able to solve this exactly, without any approximations.