Physics 103 Homework 2
Summer 2015

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1. Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ which are both functions of time, show that

$$\frac{d}{dt}\left(\boldsymbol{a}\cdot\boldsymbol{b}\right) = \frac{d\boldsymbol{a}}{dt}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\frac{d\boldsymbol{b}}{dt}.$$

2. For the following forces, compute the corresponding potential energy function. For each of the forces, also answer the following two questions: Is it possible for a particle, with a finite amount of energy E, to travel from $x = -\infty$ to $x = +\infty$? If so, what is the minimum amount of energy it needs to do so?

   (a) $F(x) = -kx - \lambda x^3 \ ; \ k, \lambda > 0$

   (b) $F(x) = -kx + \lambda x^3 \ ; \ k, \lambda > 0$

   (c) $F(x) = \sin(kx) \ ; \ k > 0$

   (d) $F(x) = \frac{1}{\gamma x^2 + 1} \ ; \ \gamma > 0$

   (e) $F(x) = \tanh(kx) \ ; \ k > 0$

3. A block with cross-sectional area A, made of some unknown material, is floating in a fluid. At equilibrium, the block displaces a volume of fluid equal to V. If the block is pushed down slightly, when it is let go, it will bob up and down, due to the combined effects of gravity and buoyancy, oscillating around its equilibrium position.

   (a) Find the frequency of these small oscillations, in terms of A, V, and the acceleration due to gravity, g. **Hint:** Use Archimedes' principle.

   (b) A cube of lithium, with side lengths 1 cm by 1 cm by 1 cm, is floating in mineral oil at room temperature, on the surface of the Earth. Find an explicit numerical value for the frequency of small oscillations of this cube.

4. A very strange floor has the property that its coefficient of kinetic friction is given by

$$\mu_k = \alpha \sin^2(\theta),$$

where α is a constant parameter, and θ is the angle around the center of the room I'm in. I now imagine that there is a block which is sitting on the floor, and that the block is constrained to move around the center of the room on a circular track of radius R. The mass of the block is m, and the acceleration due to gravity is g.

(a) I give the block an initial shove, so that its initial speed is v_0, and it starts out at $\theta = 0$. The block will then travel some arc length around the circle before coming to a stop. We will call this arc length s. Find an equation that relates s, R, α, m, v_0, and g. Don't worry if you can't solve this equation for s. **Hint:** Use the work-energy principle, and the fact that the exact velocity function is not important for doing work calculations.

(b) Imagine that the term α is very large, or that v_0 is very small, so that the block does not travel very far, $s \ll R$. In this case, use the approximation formula

$$x - \cos(x)\sin(x) \approx \frac{2}{3}x^3$$

to solve explicitly for s, in terms of the other quantities in the problem.

(c) Imagine that we had attempted to find a solution for s of the form

$$s \equiv s(v_0) = \sum_{n=0}^{\infty} s_n v_0^n$$

Based on your results in part b, do you think this would have been successful?

5. Consider a function $f(x)$ which has a Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

for a real variable x. We can naturally extend the domain of f to all complex numbers z by simply defining $f(z)$ in terms of the original Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

Because taking sums and powers of complex numbers are well-defined mathematical ideas, it seems intuitive that this is a well-defined expression (of course, proving the convergence of such an expression is a more subtle matter, although we won't be concerned with that here). As for real variables, if I have another function g with a Taylor series expansion,

$$g(z) = \sum_{n=0}^{\infty} g_n z^n,$$

then the two functions are equal if and only if all of the coefficients in their Taylor series expansions are equal,

$$g(z) = f(z) \;\;\Leftrightarrow\;\; g_n = f_n \;\;\forall\, n \in \mathbb{Z}.$$

Using this fact, prove Euler's formula for any arbitrary complex number z,

$$e^{iz} = \cos(z) + i\sin(z).$$

That is, show that the Taylor series expansions are equal on both sides, for **every** term. **DO NOT** simply take a handful of terms, and then assume the pattern continues! **Hint:** Take the Taylor series expansion for e^{iz} and split the summation into two terms, one for even powers of z and one for odd powers.

6. For a particle subject to an arbitrary force $F(x)$ in one dimension, we define the potential energy to be.

$$U(x) = -\int_{x_0}^{x} F(x')\, dx',$$

where x_0 is some arbitrary location.

   (a) Show that if the lower limit in the integration is replaced with a new value, x_1, that this only shifts the potential energy by a constant. What is this constant?

   (b) Show explicitly that shifting the potential energy by a constant has no effect on the actual motion of the particle. **Hint:** Consider Newton's Second Law.

7. Consider a particle subject to the potential energy function

$$U(x) = \gamma \cos^2(kx),$$

where $\gamma, k > 0$ are positive constants.

   (a) Assume the particle has an energy $E = \gamma$. How long does it take for the particle to travel in the positive direction from the point $x_0 = \pi/2k$ to some other point $x_f < \pi/k$? You should be able to write this as a relatively simple closed-form expression.

   (b) What happens to the travel time as $x_f \to \pi/k$? Physically, what is happening at this point that causes this behaviour?

8. Consider the weakly damped harmonic oscillator, with position and velocity given by

$$x(t) = x_* + Ce^{-\beta t}\cos(\Omega t - \delta) \;\; ; \;\; v(t) = -Ce^{-\beta t}\left[\beta\cos(\Omega t - \delta) + \Omega\sin(\Omega t - \delta)\right],$$

where C and δ are determined by initial conditions, and all of the other parameters are defined the same way as in lecture.

(a) Using the same trigonometric identity as in lecture, write the velocity as one single cosine term. Notice that the position and velocity both oscillate with the same frequency, but with a different phase. What is the phase difference between the position oscillations and the velocity oscillations? What does this phase difference become when $\beta = 0$ (the case of no damping)? What does it become as $\beta \to \omega$ (critical damping)?

(b) Consider the power delivered to the spring by the force of drag

$$P_d = F_d v = -bv^2.$$

Using the expression you just derived for the velocity in part a, explicitly compute P_d as a function of time. How does this compare to the rate of change of energy,

$$\frac{dE}{dt} = -\beta m\omega^2 C^2 e^{-2\beta t}\left[1 + \cos\left(2\Omega t - 2\delta - \phi\right)\right] \;\; ; \;\; \phi \equiv \arctan\left(\frac{2\beta\Omega}{\beta^2 - \Omega^2}\right).$$

which we derived in lecture? **Hint:** You may want to use the two trigonometric identities

$$\cos^2(\theta) = \frac{1}{2}\left(1 + \cos(2\theta)\right) \;\; ; \;\; \arctan\left(\frac{2x}{1 - x^2}\right) = 2\arctan(x).$$

(c) Based on your conclusions in part b, explain why the spring loses energy most quickly at points in time when the velocity is at a maximum.

9. Consider again the differential equation for the damped harmonic oscillator

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = 0$$

(a) Generalizing the technique that we used when considering critical damping, assume that this equation has a solution of the form

$$y(t) = z(t)\, e^{-\beta t}.$$

Plug this proposed solution into the differential equation for y, and derive the resulting differential equation for z. Write this differential equation in terms of

$$\Omega = \sqrt{\omega^2 - \beta^2},$$

which could be imaginary, depending on the damping regime. Remember that in this more general case, $\beta \neq \omega$.

(b) Is there a sense in which the differential equation for z is simpler than the differential equation for y? Solve this differential equation for z in all three damping regimes.

(c) Explain why the quantity

$$\mathcal{E} = \dot{z}^2 + \Omega^2 z^2$$

should be a constant, for all time.

10. Consider a particle subject to the potential energy function

$$U(x) = \lambda x^4.$$

(a) Assume the particle is given an initial displacement $x_0 = a$, with no initial velocity. That is, it is released from rest at the position a. How long does it take for the particle to travel from $x = a$ to $x = -a$? Write your answer in the form

$$T = \sqrt{\frac{m}{2}} f(\lambda; a) I,$$

where m is the particle's mass, $f(\lambda; a)$ is some function which depends only on a and λ, and I is an integral expression which does not depend on any parameters - it is some universal number. What is the approximate numerical value of I? **Hint:** Make a change of variables in the integral.

(b) Does anything seem unusual about the behaviour of $f(\lambda; a)$ as a is varied? What happens to the time T as a becomes very small? What happens as a becomes very large?

(c) Imagine now that the potential is modified slightly, so that

$$\widetilde{U}(x) = kx^2 + \lambda x^4.$$

Assuming that the initial conditions for the particle are the same as before, now how much time does it take for the particle to travel from $x = a$ to $x = -a$? Write your answer in the form

$$T = \sqrt{\frac{m}{2k}} \int_{-1}^{1} \frac{dy}{\left[g(y)\sqrt{1 + (\lambda/k) a^2 h(y)}\right]}.$$

What are the functions $g(y)$ and $h(y)$?

(d) Argue that, no matter how large λ/k is, there is always a value of a small enough such that

$$(\lambda/k) a^2 h(y) \ll 1$$

for all values of y that are integrated over.

(e) For small enough x, we have the Taylor series expansion

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \binom{-1/2}{n} x^n$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n(n-1)(n-2)\cdots 1}$$

is a binomial coefficient, generalized to include non-integer values of α. Assuming a is small enough, use this expansion to write the expression for the time T as

$$T = \sqrt{\frac{m}{2k}} \sum_{n=0}^{\infty} c_n I_n \left(\frac{\lambda a^2}{k} \right)^n ,$$

where I_n is an integral expression that only depends on the integer n. What is the expression I_n?

(f) Tabulate the first three numerical values for I_n - this is something that Wolfram Alpha or a graphing calculator should be able to do. Use this to write out the first three terms in the expansion of T.

(g) Based on the results in parts e and f, argue that, no matter how small k is, adding the quadratic term to the potential has dramatically changed the behaviour of T for small values of a. Given that any "real" system is most likely not described by a perfect quartic potential, what do you think this result implies for the behaviour of "real" systems?

(h) **Extra Credit:** Using Mathematica on the computers in the PSR, compute $\sqrt{\frac{2k}{m}}T$ for a variety of values of $\lambda a^2/k < 1$. Do this by using the NSum command to numerically sum up all of the infinitely many terms in the sum. Using these values, sketch a plot for how $\sqrt{\frac{2k}{m}}T$ varies with the quantity $\lambda a^2/k$.