Physics 103 Homework 3
Summer 2015

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Due Date: Friday, July 17$^{\text{th}}$

1. Consider a damped harmonic oscillator subject to a sinusoidal forcing function,

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = \sin(\gamma t).$$

Solve this differential equation in order to find its particular solution (which was quoted in lecture).

2. Consider a particle subject to the potential energy function

$$U(x) = \gamma \cos^2(kx),$$

where $\gamma, k > 0$ are positive constants. Assume the particle has an energy $E = \gamma$. On last week's homework, you computed $t(x)$, the amount of time it took the particle to travel from $x_0 = \pi/2k$ to some other point $\pi/2k < x < \pi/k$.

(a) Invert this expression for $t(x)$ to find the particle's motion, $x(t)$. Make a plot of $kx\left(\widetilde{t}\right)$, where

$$\widetilde{t} \equiv k\sqrt{2\gamma/m}\; t$$

Scaling the coordinates in this way allows you to make a single plot which covers all parameter cases. Do you think this type of motion is realistic for a physical particle? Why or why not?

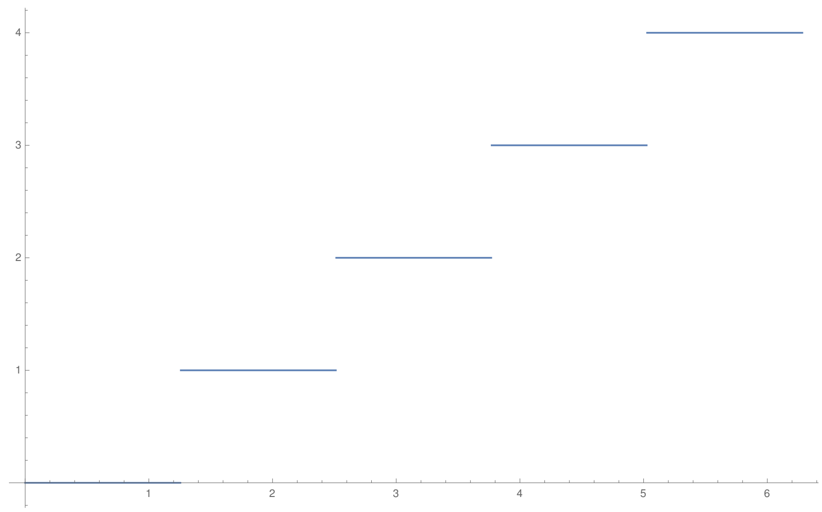
(b) Notice that we were able to specify the motion of the particle without supplying the initial velocity - where did the two initial conditions come from in order to determine this solution?

(c) **Extra Credit:** For $E \neq \gamma$, performing the integral in closed form becomes more difficult. Use Mathematica to perform the integration numerically and make a plot of $\widetilde{t}(kx)$ for $E/\gamma = 0.5, 0.75, 1, 1.01, 1.1, 1.5$ and 5. Notice that the numerical integration command can be nested within the plot command, so it's not necessary to do this as two separate steps. If you flip this plot on its side, you will effectively have a plot of $kx\left(\widetilde{t}\right)$. What is the major qualitative difference between $E/\gamma > 1$ and $E/\gamma < 1$? Why is this the case? What happens to the qualitative shape of the plot as $E/\gamma \gg 1$? Physically, why does this behaviour occur?

3. Consider the driven damped harmonic oscillator, subject to the driving force

$$f(t) = \text{Floor}\left[5t/2\pi\right] \ ; \ 0 < t < 2\pi,$$

where Floor $[x]$ is the "floor function," which is equal to the largest integer less than or equal to x. For example, Floor $[1.5] = 1$ and Floor $[3] = 3$. A plot of this forcing function is shown in the figure below. After the time $t = 2\pi$, this driving force repeats itself periodically.



(a) Work out the Fourier series coefficients, a_n and b_n, for this driving force, through $n = 5$ (you may use a calculator for this). Make a plot of your fifth-order Fourier series approximation, and compare it with the shape of the driving function. **Hint:** Because the function is periodic, you can shift the region of integration when computing the Fourier coefficients, such that

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t')\cos(n\lambda t')\,dt' \rightarrow \frac{2}{T}\int_0^T f(t')\cos(n\lambda t')\,dt',$$

and similarly for the sine term.

(b) Use these Fourier coefficients to write down the particular solution to the driven oscillator under this forcing function. Make a plot of your particular solution for $\omega = \sqrt{2}$ and $\beta = 1$.

(c) **Extra Credit:** Use Mathematica to compute the Fourier expansion of the driving function through $n = 50$. Compare this 50^{th}-order approximation with the shape of the full driving function. Again for $\omega = \sqrt{2}$ and $\beta = 1$, plot the motion of the particle. Does the inclusion of the 45 extra terms make a major qualitative difference to your plot? Now, consider instead the case that $\omega = 10$ and $\beta = 1$. In this case, does it matter if we take $n = 50$ as opposed to $n = 5$? You should make plots for both cases. Why do you think the necessary number of terms depends on ω?

4. Prove the derivative property of the Fourier transform,

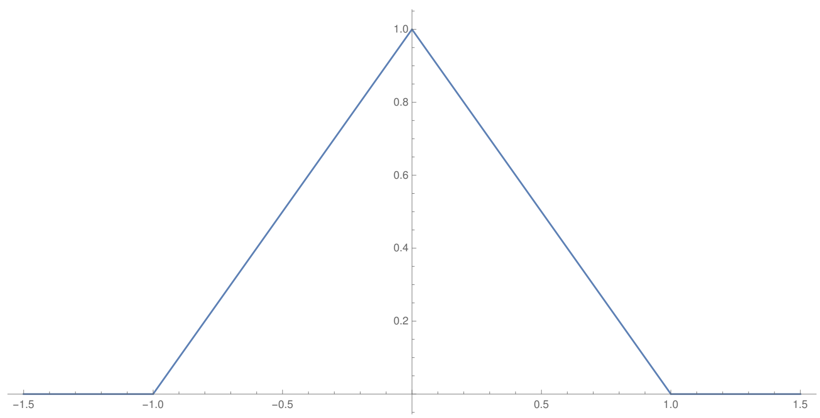
$$\widehat{\left(f^{(n)}\right)}(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^n}{dt^n} \left(f(t)\right) e^{-i\nu t}\, dt = (i\nu)^n \hat{f}(\nu).$$

for an arbitrary integer n. In order to arrive at this relation, do you need to assume that the function $f(t)$ has certain boundary conditions? If so, what are they? (By boundary conditions, we mean the behaviour of $f(t)$ and its derivatives as $t \to \infty$). **Hint:** Use integration by parts, and also induction.

5. Consider the piecewise forcing function

$$f(t) = \begin{cases} t+1, & -1 < x < 0 \\ -t+1, & 0 < x < 1, \\ 0, & \text{otherwise} \end{cases}$$

This forcing function has the triangular shape shown in the figure below.



(a) Solve for the particular solution of a damped harmonic oscillator being driven by this forcing function, using whatever method you deem most appropriate. **Hint:** It may be easiest to specify your answer in the form of a piecewise function. Any integrals which need to be performed can be looked up in a table or computed using a calculator.

(b) Using this particular solution, find the motion of the oscillator as a function of time, assuming it was at rest right before force started acting, at $t = -1$. You should be able to find a closed-form expression in terms of the parameters of the problem. Using this solution, make a plot for $\beta = 1$, and $\omega = 1, 2$ and 5.

6. Check that the Green's function we found in class for the damped harmonic oscillator satisfies the appropriate differential equation. **Hint:** The derivative of the Heaviside step function $\Theta(x)$ is the delta function $\delta(x)$. Also, any term being multiplied by a delta function $\delta(x)$ can be evaluated at the point $x = 0$, since it is zero otherwise.

7. Consider the nonlinear oscillator described by the differential equation

$$\ddot{y} + \omega^2 y + \epsilon y^n = 0,$$

where $0 < \epsilon \ll 1$.

(a) Using the Poincaré-Lindstedt approach to perturbation theory, find the motion of the oscillator as a function of time, accurate to first order in ϵ, for $n = 5$ and $n = 7$. Assume the boundary conditions $y\,(t = 0) = 1$; $\dot{y}\,(t = 0) = 0$. Make sure that your equation does not contain any secular terms! How many higher harmonics are being excited in each case?

(b) Use the short time perturbation technique to find the motion valid through fourth order in time, for the same boundary conditions above, for $n = 5$. Do not make use of the smallness of ϵ in this part.

(c) **Extra Credit:** Using the formula

$$\cos^n \theta = \frac{2}{2^n} \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \cos\left((n-2k)\theta\right),$$

apply the Poincaré-Lindstedt approach to perturbation theory to find the motion of the oscillator as a function of time, accurate to first order in ϵ, for arbitrary **odd** powers n (a different identity holds for even powers). Use the same boundary conditions as above.

8. Derive the resonance condition stated in class,

$$\gamma = \omega_R = \sqrt{\omega^2 - 2\beta^2}.$$

9. Consider the sawtooth forcing function discussed in lecture. If this forcing function is applied to an oscillator with $\beta = 0$ and $\omega = 1$, solve for the motion of the oscillator in the limit of very long times. **Hint:** How many frequency components of the sawtooth wave do you need to consider in this limit? Why?

10. Consider the first-order, linear differential operator

$$\mathcal{L} = \frac{d}{dt} + \alpha$$

for $\alpha > 0$. Consider the Green's function for this differential operator, which satisfies

$$\mathcal{L}G\,(t; t_0) = \delta\,(t - t_0)\,.$$

(a) Derive an integral expression for the Green's Function.

(b) **Extra Credit:** Solve this integral to show explicitly that

$$G\,(t; t_0) = e^{-\alpha(t-t_0)}\Theta\,(t - t_0)$$

(c) **Regular Credit:** Check that this Green's function satisfies the correct differential equation.

11. Consider a weakly damped harmonic oscillator subject to a sudden impulse,

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = \lambda\delta\left(t - t_0\right).$$

The particular solution in this case is then of course given by the Green's function

$$y_p\left(t\right) = \lambda G\left(t; t_0\right) = \frac{\lambda}{\Omega} e^{-\beta(t-t_0)} \sin\left(\Omega\left(t - t_0\right)\right) \Theta\left(t - t_0\right)$$

   (a) Write down the most general solution to the particle's motion, in terms of arbitrary constants to be fixed by initial conditions.

   (b) Assuming that the particle was at rest at equilibrium right before the impulse, find values for these arbitrary constants in order to determine the motion in this case.

   (c) What is the velocity of the particle as a function of time? Does it change continuously?

   (d) How much energy is imparted to the particle by the impulse?

12. Consider a damped harmonic oscillator subject to a cosine forcing function,

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = f_0 \cos\left(\gamma t\right),$$

as we discussed in class. In particular, consider the long-time limit, after any transients have died out.

   (a) What is the **energy** of the oscillator, as a function of time, in this limit? What does this become equal to when $\gamma = \omega$? What is it equal to when $\gamma = \omega_R$?

   (b) How much work does the **driving** force perform on the oscillator over the course of one driving period? How much work does the **drag** force perform on the oscillator over the course of one driving period?

13. **Extra Credit:** Use the method of Fourier transformation to find the most general solution to

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = f_0 e^{-\lambda|t|},$$

where $\lambda > 0$ is a constant. You may need to use Cauchy's residue theorem in order to perform some of the integrals, which I (or the TAs in the PSR) would be happy to explain.

Please choose **five** problems from Set **A**, and also choose **three** problems from Set **B**.

Set A: Problems 1,2,4,8,9,11,12

Set B: Problems 3,5,6,7,10