Physics 103 Homework 4
Summer 2015

Instructor: Keith Fratus
TA: Michael Rosenthal

Due Date: Friday, July 24$^{\text{th}}$

1. Consider an arbitrary rotation matrix R.

   (a) Use the constraint we derived in class,

   $$R^T R = I,$$

   along with the properties (valid for any two matrices A and B),

   $$\det(AB) = \det(A)\det(B) \;\; ; \;\; \det\left(A^T\right) = \det(A),$$

   in order to derive the condition we found in class,

   $$\det(R) = \pm 1.$$

   (b) The determinant of a 3x3 matrix can be written using the expression

   $$\det(A) = \sum_{i,j,k=1}^{3} A_{1i}A_{2j}A_{3k}\varepsilon_{ijk} = A_{1i}A_{2j}A_{3k}\varepsilon_{ijk},$$

   where the second equality is an example of summation notation. Use the condition we derived in class

   $$R_{im}R_{nj}R_{lk}\varepsilon_{mjk} = \varepsilon_{inl},$$

   along with the above expression for the determinant, in order to show why we must have

   $$\det(R) = +1.$$

2. Consider two sets of coordinate axes, $\mathbf{e}_i$ and $\mathbf{e}'_j$, which are related by the transformation

$$\mathbf{e}'_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2) \;\; ; \;\; \mathbf{e}'_2 = -\mathbf{e}_3 \;\; ; \;\; \mathbf{e}'_3 = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_2)$$

(a) Make a sketch showing how the two coordinate systems are related to each other.

(b) Compute the rotation matrix elements

$$R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j.$$

(c) Imagine that we have a vector whose expression in terms of the old set of coordinates reads

$$\mathbf{r} = \mathbf{e}_1 + 2\mathbf{e}_2 - 5\mathbf{e}_3$$

What is the expression for this vector, written in terms of the new set of coordinates?

(d) Verify explicitly that for this rotation matrix,

$$\det(R) = +1.$$

3. Consider two sets of coordinate axes, $\mathbf{e}_i$ and $\mathbf{e}'_j$, which are related by the transformation

$$\mathbf{e}'_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2) \;\; ; \;\; \mathbf{e}'_2 = 2\mathbf{e}_2 \;\; ; \;\; \mathbf{e}'_3 = (\mathbf{e}_1 + \mathbf{e}_3)$$

(a) Make a sketch showing how the two coordinate systems are related to each other.

(b) Assume that the original coordinate system, $\mathbf{e}_i$, is an orthonormal one. Is the new coordinate system an orthonormal one?

(c) Compute the "rotation" matrix elements

$$R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j.$$

Why did I put the word rotation in quotation marks?

(d) What is the determinant of the above matrix?

(e) Assume that two vectors have an expansions in terms of the old basis,

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 \;\; ; \;\; \mathbf{q} = q_1\mathbf{e}_1 + q_2\mathbf{e}_2 + q_3\mathbf{e}_3,$$

and also an expansion in terms of the new basis

$$\mathbf{r} = r'_1\mathbf{e}'_1 + r'_2\mathbf{e}'_2 + r'_3\mathbf{e}'_3 \;\; ; \;\; \mathbf{q} = q'_1\mathbf{e}'_1 + q'_2\mathbf{e}'_2 + q'_3\mathbf{e}'_3,$$

The dot product between these two vectors is given in terms of the old coordinates by

$$\mathbf{r} \cdot \mathbf{q} = r_1 q_1 + r_2 q_2 + r_3 q_3.$$

Find an expression for the dot product, expressed in terms of the **new** coordinates. What key differences exist between this expression and the one above? **Hint:** Write out the expression

$$\mathbf{r} \cdot \mathbf{q} = (r'_1\mathbf{e}'_1 + r'_2\mathbf{e}'_2 + r'_3\mathbf{e}'_3) \cdot (q'_1\mathbf{e}'_1 + q'_2\mathbf{e}'_2 + q'_3\mathbf{e}'_3)\,,$$

and use the fact that the original basis is orthonormal.

4. Consider the "scalar triple product"

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_i \varepsilon_{ijk} b_j c_k$$

   (a) Using the definition of the Levi-Civita symbol given in class, verify that this symbol possesses the "cyclic permutation" property

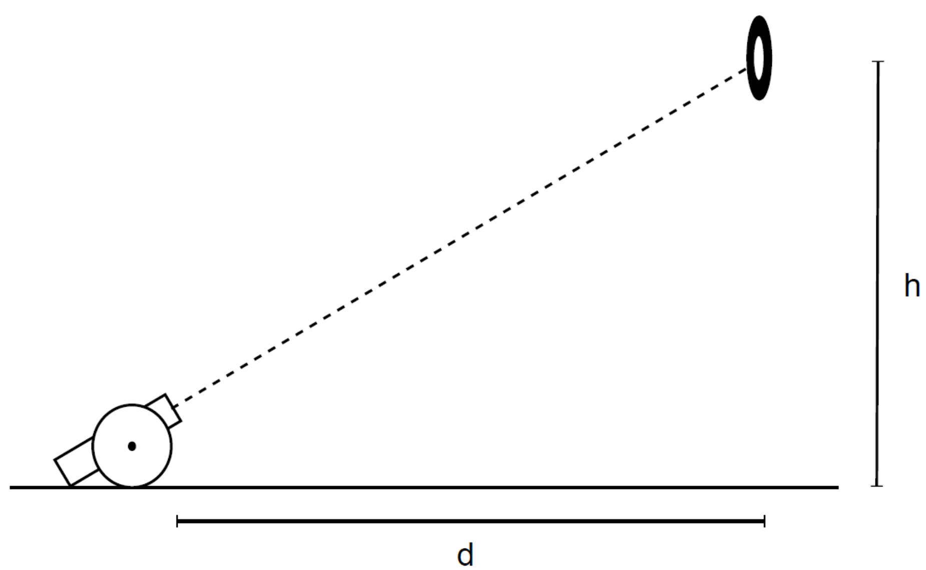
$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki},$$

   for **any** choice of the indices i, j, k.

   (b) Using this cyclic permutation property of the Levi-Civita symbol, prove the identity

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \,.$$

5. A cannon is aimed at a target some height h above the ground, a distance d away, as shown. The cannon is fired at the **same instant** (which for convenience we'll take to be $t = 0$) that the target is released and allowed to fall freely under the influence of gravity. The bullet leaves the cannon with speed v.



   (a) Find the trajectory of the target and the bullet explicitly, and use these expressions to show that the bullet will **always** collide with the target.

   (b) Now, without using **any** math, derive the same conclusion using the gravitational equivalence principle (that is, by assuming that the experiment is being conducted in an accelerating rocket somewhere out in space, with no gravity).

6. A boat's **speed** in water is v_{BW}. That is, when the boat travels through the water, its speed *with respect to the water around it* is v_{BW}. This boat starts at point A on one shore of a river and makes its way in a straight line to a point directly across it on the opposite shore (indicated by the dotted line). The **speed** of the river water flowing downstream, with respect to the ground, is v_{WG}. The width of the river is d.

   (a) **In which direction** must the boat be pointing in order to travel to its destination? (Assume $v_{BW} > v_{WG}$). You should specify your answer in terms of the **angle with respect to the dotted line**, as well as the quantities v_{BW} and v_{WG}. **Hint:** How does the boat's velocity with respect to the water relate to its speed in the water, and the angle in question? Use the velocity addition formula to find the velocity of the boat with respect to the ground, in terms of v_{BW} and v_{WG}. What must be true about the velocity with respect to the ground in order for the boat to travel directly across the river?

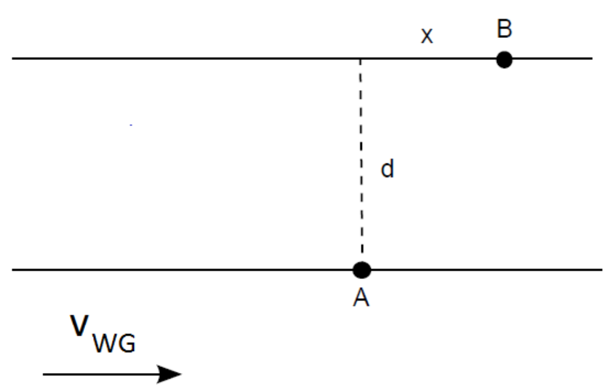
   (b) **How long** will it take the boat to travel directly across the river? Specify your answer in terms of v_{BW}, v_{WG}, and d. Does your answer make sense when $v_{BW} < v_{WG}$? Why or why not? You may want to make use of the trigonometric identity $\cos\theta = \pm\sqrt{1-\sin^2\theta}$ in order to help simplify your formula.

   (c) Now, imagine instead the boat needs to reach a different point B along the opposite shore, a distance x downstream from where it starts, as shown. **In which direction** must the boat face to get there? Specify your answer in terms of v_{BW}, v_{WG}, x, and d. Make sure your answer reduces to your result in part (a) when $x = 0$! You may find handy the formula

$$A\sin\theta + B\cos\theta = \sqrt{A^2+B^2}\,\sin(\theta+\delta),$$

   where $\tan\delta = B/A$. **Hint:** Consider the line that extends from point A to point B - what angle does this line make with respect to the dotted line, and how can you relate that angle to x and d? How is this **same** angle related to the components of the boat's velocity with respect to the ground?

   (d) What is the **minimum value** that v_{BW} can have so that the boat is still able to reach the shore at point B?



7. Consider a particle moving in two dimensions, subject to the potential energy

$$U(x,y) = e^{x^2+y^2-xy}$$

(a) Taylor expand the exponential and keep terms out to second order, so that you have an approximate expression of the form

$$U(x,y) \approx \alpha + \beta x + \gamma y + \epsilon xy + \delta x^2 + \phi y^2$$

The values $\alpha, \beta, \gamma, \epsilon, \delta, \phi$ are numbers which you will find by doing the expansion. This is the two-dimensional version of replacing a potential with its quadratic approximation - the spring approximation.

(b) Find the force that the particle is subject to, by computing

$$F = -\nabla U,$$

using the **approximate** expression for the potential. Write the force in the form

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix},$$

where M is a matrix multiplying the position vector.

(c) Using Newton's second law, we can write

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix},$$

where we've set the particle's mass to one. The above equation is now a **system** of differential equations. Is this system a coupled system of equations, or a decoupled set of equations? In other words, does the expression for $\ddot{x}$ depend on x and y (coupled), or does it only depend on x (decoupled)? Does the expression for $\ddot{y}$ depend on x and y (coupled), or does it only depend on y (decoupled)?

(d) Recall from your linear algebra class that an eigenvector $\mathbf{v}$ of a matrix M with eigenvalue λ satisfies

$$M \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \lambda \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

Find the two eigenvectors and corresponding eigenvalues for this matrix M,

$$M \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} = \lambda_1 \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} \quad ; \quad M \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix} = \lambda_2 \begin{pmatrix} v_{2x} \\ v_{2y} \end{pmatrix}$$

Label the two eigenvalues according to $\lambda_1 < \lambda_2$. That is, assume that λ_1 is the smaller eigenvalue.

(e) Define two new coordinates

$$z = v_{1x}x + v_{1y}y \ ; \ w = v_{2x}x + v_{2y}y,$$

where the coefficients $v_{1x}, v_{1y}, v_{2x}, v_{2y}$ come from the eigenvectors you just found. **Show explicitly** that

$$\ddot{z} = v_{1x}\ddot{x} + v_{1y}\ddot{y} = \lambda_1 z \quad ; \quad \ddot{w} = v_{2x}\ddot{x} + v_{2y}\ddot{y} = \lambda_2 w$$

Why do you think the coordinates w and z are a more useful set of coordinates for this problem, as opposed to the original set of coordinates x and y?

(f) Consider a particle released from rest at the position $x = 1$, $y = 0$. Translate this initial condition into an initial condition in terms of the w and z coordinates, and use that initial condition to find w and z as a function of time. Then, transform this solution back to the x and y coordinates, in order to find x and y as a function of time.

8. Consider the result we found in the first lecture for the motion of a projectile without drag,

$$\boldsymbol{r}(t) = \begin{pmatrix} v_0 \cos\theta t \\ h + v_0 \sin\theta t - \frac{1}{2}gt^2 \end{pmatrix}.$$

Compute the angular momentum of the projectile, along with the torque acting on it. Verify that the time derivative of the angular momentum is indeed equal to the torque.

9. Imagine a rocket which is out in empty space, so that there is negligible gravity. The initial mass of the rocket is m_0. After starting from rest, the rocket begins burning fuel in order to accelerate at a **constant** rate a, until it reaches a final speed of v_f. The rocket has some mass $m(t)$, and some velocity $v(t)$. This is illustrated in the figure below. We are interested in finding the **total work done by the rocket's engines.**

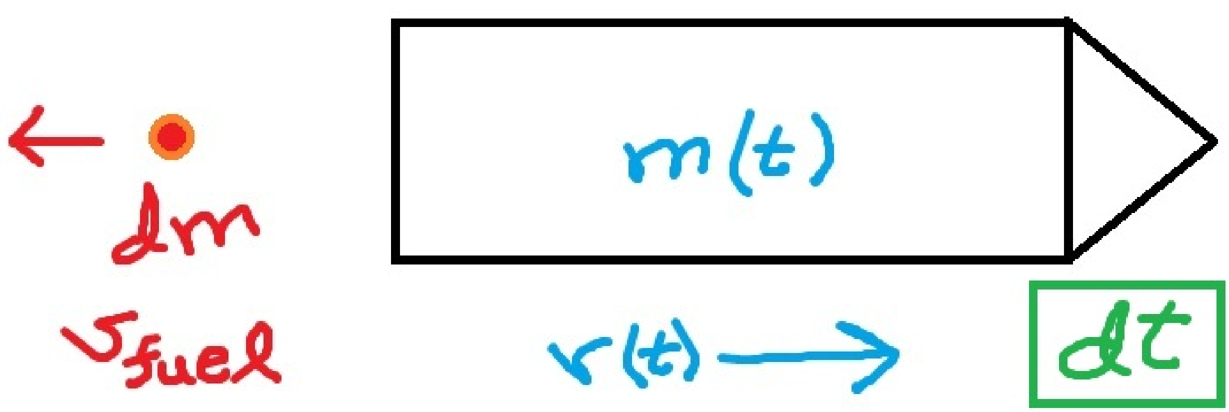


Figure 1: The motion of the rocket as it releases a small amount of fuel in a time dt.

(a) Show that the work done on the **rocket itself** is given by

$$W_r = \int_0^{v_f/a} \left(ma + at\frac{dm}{dt} \right) at\; dt.$$

Hint: Start with the basic definition of work

$$W_r = \int_0^d F(x)\, dx = \int_0^{t(d)} F(t)\, v dt$$

and remember Newton's second law

$$F(t) = \frac{dp}{dt} = \frac{d}{dt}(mv).$$

(b) Show that the total work done on the fuel exhausted out the back is

$$W_{fuel} = -\int_0^{v_f/a} \frac{dm}{dt} \left(v(t) - v_{\mathrm{ex}}\right)^2 dt.$$

Hint: For an infinitesimal piece of exhaust being ejected out the back,

$$dW_{fuel} = F(t)\, v_{fuel}(t)\, dt = \frac{dp_{fuel}}{dt} \left(v(t) - v_{\mathrm{ex}}\right) dt,$$

where $v(t)$ is the velocity of the rocket, and v_{ex} is the speed that the exhaust is emitted from the rocket, with respect to the rocket. Remember that the mass of the infinitesimal amount of ejected fuel is dm.

(c) Add these two expressions to find

$$W_T = \int_0^{v_f/a} ma^2 t + \frac{dm}{dt}\left(2atv_{\mathrm{ex}} - (v_{\mathrm{ex}})^2\right)\; dt$$

(d) Take the rocket equation we found in class

$$v - v_0 = v_{\mathrm{ex}} \ln\left(\frac{m_0}{m}\right),$$

and differentiate both sides with respect to time, in order to find

$$\frac{dv}{dt} = -\frac{v_{\mathrm{ex}}}{m}\frac{dm}{dt}.$$

Use this differential equation to find $m(t)$. **Hint:** Use the chain rule, and remember the acceleration is constant.

(e) Use this expression for $m(t)$ to explicitly compute the work integral, and show that

$$W_T = m_f v_{\mathrm{ex}} v_f.$$

Extra Credit: There is a common mistake people tend to make when computing the total work done by the rocket's engines in the previous problem. They decide to find the kinetic energy of each piece of fuel that leaves the rocket, integrate this to get a total, and then add it to the final kinetic energy of the rocket. But this actually does not give the correct answer, and the reason is because you are missing some work which is done when thinking about it that way. Explain exactly where this missing energy went.