# Physics 103 Homework 5
# Summer 2015

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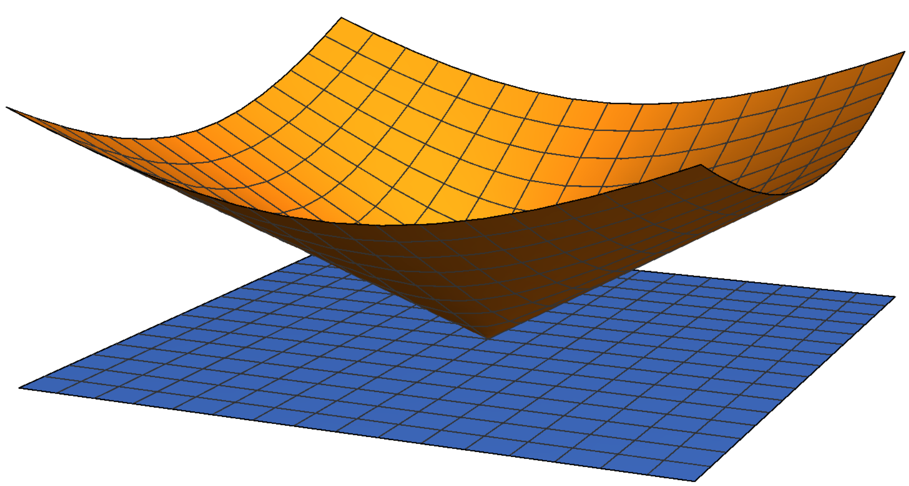
Due Date: Friday, July 31st, IN LECTURE

1. A particle moving in three dimensions is constrained to move frictionlessly on a surface defined by the equation
$$z = r = \sqrt{x^2 + y^2},$$
which is shown in the figure below. Notice that $r$ is the polar coordinate in the x-y plane. Additionally, the particle is subject to a gravitational field described by the potential energy
$$U(z) = mgz.$$



(a) Explain why the z-component of the angular momentum is constant. **Hint:** The forces which keep the particle constrained on the surface always act perpendicularly to the surface. What other forces act on the particle? Visualizing a cross-section of the surface in the x-z plane, for example, may help you understand the problem better.

(b) Now introduce a set of cylindrical coordinates,

$$r = \sqrt{x^2 + y^2}\ ,\ \ \theta = \arctan\left(y/x\right)\ ;\ z = z$$

Find an expression for $L_z$, the z-component of the angular momentum, in terms of these cylindrical coordinates.

(c) Use the above expression for $L_z$, along with the fact that it is conserved, in order to write the total energy of the particle in the form

$$E = \frac{1}{2} m_* \dot{r}^2 + U_{\text{eff}}\left(r\right).$$

What is $m_*$, and what is the expression for the effective potential?

(d) Assuming $g = 10$ and $m_* = 1$, make a sketch of the effective potential for the values $L_z = 0$, 0.25, 0.75, and 1.5 (the plot is best visualized when the range for $r$ is from zero to about 2.5). Which one of these potentials shows a qualitatively distinct behaviour from the other three, and in what way?

(e) Assume that the particle moves in a "circular orbit," in which the r-coordinate is constant. What is the value of $r = r_*$ for this circular orbit, in terms of the other parameters of the problem? Do **not** assume the specific example values given in part c.

(f) For arbitrary motion (not necessarily circular), find an integral expressions which gives the amount of time elapsed, and also the amount of angle traced out, during one oscillation of the radial coordinate.

(g) For the specific values $E = 10$, $g = 10$, $L_z = 1$, and $m_* = 1$, how much angle is traced out during one radial oscillation? Does this "orbit" close on itself?

2. Consider the two-body problem with bare potential

$$U\left(r\right) = -\frac{\gamma}{r^\alpha}\ ;\ \gamma > 0,\ \alpha > 0.$$

For which values of $\alpha$ does the corresponding effective potential **always** admit a minimum? For which values of $\alpha$ does the corresponding effective potential **never** admit a minimum? Sketch the qualitative shape of the effective potential in both of these cases. Is there a cross-over value of $\alpha$ such that the behaviour of the effective potential becomes more subtle? What is this cross-over value, and what is this behaviour?

3. Consider the two-body problem with Yukawa potential

$$U(r) = -k\frac{e^{-\lambda r}}{r} \ ; \ k > 0, \ \lambda > 0.$$

   (a) Consider the condition for a circular orbit

$$U'_{\text{eff}}(r_*) = 0.$$

   Find this condition explicitly, in terms of the parameters of the problem. Show that this condition can **always** be satisfied for **any** choice of $r_*$, as long as the angular momentum is chosen correctly. That is, show that **any** radius can be the radius of a circular orbit, so long as the particle possesses the appropriate angular momentum. Do **not** solve for $r_*$ explicitly.

   (b) A circular orbit will be stable so long as the effective potential has a minimum at that point, as opposed to a maximum. This implies that a circular orbit will be stable so long as

$$U''_{\text{eff}}(r_*) > 0.$$

   Compute the second derivative of the effective potential, and use the equation you found in part a to eliminate the angular momentum from the above condition. How must $r_*$ be related to $\lambda$ in order for the circular orbit at $r_*$ to be stable?

   (c) Take the condition you found in part a for the location of the circular orbit, and solve for $r_*$ **approximately**, as a function of the other parameters $l$, $m$, $k$, and $\lambda$, accurate to **second** order in $\lambda$. Make sure your answer reduces to the correct result for $\lambda = 0$.

4. In the two-body problem, what is the orbital period of a circular orbit, as a function of the circular radius $r_*$? That is, how long does it take for the angular coordinate to rotate from zero to $2\pi$? Do not make any assumptions about the specific form of the bare potential.

5. Consider two particles $i$ and $j$ subject to a central potential, so that

$$\mathbf{F}_{ji} = -\nabla_i U(|\mathbf{r}_i - \mathbf{r}_j|) = -\left(\frac{\partial U}{\partial r_{ix}}, \frac{\partial U}{\partial r_{iy}}, \frac{\partial U}{\partial r_{iz}}\right)$$

   Verify the assertion I made in lecture, which is that

$$\mathbf{F}_{ji} = f_{ji}(r)(\mathbf{r}_i - \mathbf{r}_j),$$

   where,

$$r \equiv |\mathbf{r}_i - \mathbf{r}_j| \ ; \ f_{ji}(r) \equiv -\frac{1}{r}\frac{dU(r)}{dr}.$$

   **Hint:** The function $U$ can originally be thought of as a function of the six Cartesian coordinates of $\mathbf{r}_i$ and $\mathbf{r}_j$,

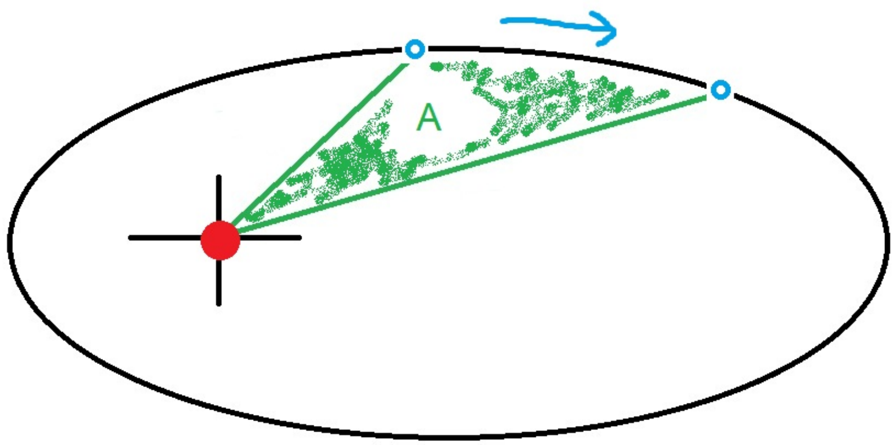
$$\mathbf{r}_i = (r_{ix}, r_{iy}, r_{iz}) \ ; \ \mathbf{r}_j = (r_{jx}, r_{jy}, r_{jz}).$$

However, it can **also** be thought of as a function of the six **spherical** coordinates of the difference vector and center-of-mass position,

$$\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j = (r, \theta_r, \phi_r) \;\; ; \;\; \mathbf{R} = \frac{m_i \mathbf{r}_i + m_j \mathbf{r}_j}{m_i + m_j} = (R, \theta_R, \phi_R) \, .$$

Both of these are a complete set of coordinates for specifying the locations of the particles. Use the multi-dimensional version of chain rule, along with the fact that the potential **only depends on one** of the new coordinates, in order to arrive at the desired result.

6. Prove Kepler's Second law of Planetary motion, which states, "A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time." This idea is sketched in the figure below. **Hint:** Use the conservation of angular momentum in the two-body problem, and find a way to relate $dA/dt$ to $d\theta/dt$.




7. Consider the bounded two-body problem, where the orbital shape of the difference vector is given according to a Kepler ellipse

$$r\left(\theta\right) = \frac{r_*}{1 + e\cos\left(\theta\right)},$$

Remember that the positions of the two particles can be found according to

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r} \;\; ; \;\; \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r}.$$

Assume that $r_* = 1$ and $e = 0.5$. Draw a set of coordinates with the center-of-mass position $\mathbf{R}$ situated at the origin. Sketch the **actual orbital shape of both particles** in the two-body problem, as they orbit their common center of mass, assuming that $m_2 = m_1$. Now, make another plot, except this time with $m_2 = 10m_1$. How do these two plots differ? **Hint:** The position vectors of the two particles **with respect to the center of mass** are given by

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{R} = \frac{m_2}{M}\mathbf{r} \;\; ; \;\; \mathbf{r}_2' = \mathbf{r}_2 - \mathbf{R} = -\frac{m_1}{M}\mathbf{r}$$

8. Consider the projectile shown in Figure 1 of the Scattering lecture notes.

   (a) Show that this incoming projectile, with impact parameter $b$ and initial velocity $v_0$, has an angular momentum whose magnitude is

   $$l = mbv_0.$$

   Make sure to understand that $v_0$ is the velocity when it is very far away to the left.

   (b) Assume now that the projectile experiences the repulsive potential,

   $$U(r) = \frac{\gamma}{r^2}.$$

   For this scattering set up, find the differential scattering cross section,

   $$D(\theta) = \frac{d\sigma}{d\Omega}.$$

   **Hint:** At some point you will need to perform an integral, at which point you should make the substitution $u = 1/r$, and then after that, perform a trigonometric substitution.

9. Consider the two-body problem between the Sun, and one of the other planets in the solar system

   (a) When the other planet is the Earth, what is the distance between the center-of-mass of the two bodies, and the center of the Sun? Write your answer as a multiple of the Sun's radius. **Hint:** See the hint in problem seven.

   (b) When the other planet is Jupiter, what is the distance between the center-of-mass of the two bodies, and the center of the Sun? Write your answer as a multiple of the Sun's radius.

   (c) If we were to exaggerate slightly, why is it often claimed that the solar system is basically a two-body system between Jupiter and the Sun?

10. A muon is a subatomic particle with a half-life of 1.5 $\mu$s. An observer in a laboratory fires a large beam of muons travelling at a speed $v = 0.999c$ with respect to the laboratory. How much time does the laboratory observer measure on his own clock before half of the muons decay?

11. **Extra Credit:** In class, we showed that the closed orbits in the Kepler problem took the form

    $$r(\theta) = \frac{r_*}{1 + e\cos(\theta)},$$

    where $r_*$ was the radius of the effective potential minimum, and $e$ was the eccentricity. Show that this can be written in Cartesian coordinates as

    $$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

    and determine the values of $a$, $b$, and $d$ in terms of $r_*$ and $e$. Make sure your results agree with those given in the lecture notes.

12. **Extra Credit:** For a particle moving according to a Kepler ellipse,

$$r(\theta) = \frac{r_*}{1 + e\cos(\theta)},$$

show that Kepler's third law is true,

$$\pi^2 a^3 r_* = \frac{l^2 T^2}{4 m_*^2},$$

where $T$ is the orbital period, and $a$ is the semi-major axis discussed in the lecture notes. **Hint:** The area of an ellipse if $A = \pi ab$, where $b$ is the semi-minor axis. How is $T$ related to $A$, and how is $a$ related to $b$?