

Many Body Physics

So far we've been considering single particles which are subject to external forces - for example, a projectile subject to air resistance, or a block attached to a spring. Of course, we know that more accurately, forces are simply a result of different particles interacting with each other - molecules in the air scatter off of our projectile, and tension in a spring is a result of the interactions between its constituent atoms. In general, we can imagine a collection of N particles, subject to a variety of interactions amongst themselves. Newton's second law then tells us that

$$m_i \ddot{\mathbf{r}}_i = \sum_{j=1}^N \mathbf{F}_{ji}, \quad (1)$$

where m_i and $\ddot{\mathbf{r}}_i$ are the mass and acceleration of the i^{th} particle, and \mathbf{F}_{ji} is the force that particle j exerts on particle i . We now want to understand what type of behaviour such a system of particles can exhibit. Clearly, our above expression is very general, and describes an enormous variety of different physical systems. However, for the systems we are typically concerned with, there are a variety of simplifications we can make, which will actually let us say quite a bit about the motion of the system, before we specify any particular forces.

Conservation of Momentum

First, we will assume that our forces obey Newton's third law,

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij}, \quad (2)$$

which, as we remember, says that particles must exert equal and opposite forces on each other. Notice that this includes the special case

$$\mathbf{F}_{ii} = -\mathbf{F}_{ii} \Rightarrow \mathbf{F}_{ii} = 0, \quad (3)$$

indicating that a particle cannot exert a force on itself. Newton's third law is certainly a very restrictive assumption. What simplifications does it allow us to make?

We know from our freshman mechanics course that we can define the momentum of an object to be

$$\mathbf{p} = m\mathbf{v}, \quad (4)$$

with m being its mass, and \mathbf{v} being its velocity. We also learned that the total momentum of a collection of bodies,

$$\mathbf{p}_T = \sum_{i=1}^N \mathbf{p}_i, \quad (5)$$

is always conserved in Newtonian mechanics (in E & M we need to modify our definition slightly to account for the momentum of the electromagnetic field).

With Newton's third law, it is in fact easy to prove this. Newton's second law for a given body says that

$$\mathbf{F} = m\mathbf{a}, \quad (6)$$

which we can write as

$$\mathbf{F} = m \frac{d\mathbf{v}}{dt}. \quad (7)$$

Assuming the mass is constant, we can pull it into the time derivative, and we find

$$\mathbf{F} = \frac{d}{dt} (m\mathbf{v}) = \frac{d\mathbf{p}}{dt}. \quad (8)$$

Now, the change in total momentum is given by

$$\frac{d\mathbf{p}_T}{dt} = \sum_{i=1}^N \frac{d\mathbf{p}_i}{dt}. \quad (9)$$

From what we saw above, the change in momentum of particle i is given by the force acting on it, which will be

$$\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_{\text{ext},i} + \sum_{j=1}^N \mathbf{F}_{ji}, \quad (10)$$

where the sum on j is over all of the particles in the system. Notice that we have allowed the possibility of an additional external force, coming from whatever particles exist outside of the collection of N particles we are currently interested in. The vector $\mathbf{F}_{\text{ext},i}$ is taken to be the net sum of all of these forces acting on particle i . Our expression for the change in momentum then tells us

$$\frac{d\mathbf{p}_T}{dt} = \sum_{i=1}^N \left[\mathbf{F}_{\text{ext},i} + \sum_{j=1}^N \mathbf{F}_{ji} \right] = \mathbf{F}_{\text{ext}} + \sum_{i=1}^N \sum_{j=1}^N \mathbf{F}_{ji}, \quad (11)$$

where \mathbf{F}_{ext} is defined as the total collection of external forces acting on the collection of N particles from outside of the system.

Now, because of Newton's third law, the forces two bodies exert on each other are equal and opposite. Thus, when we write out the double sum above, the forces will always come in pairs, and they will cancel out. That is, we can use Newton's third law to write the double sum as

$$\sum_{i=1}^N \sum_{j=1}^N \mathbf{F}_{ji} = - \sum_{i=1}^N \sum_{j=1}^N \mathbf{F}_{ij}. \quad (12)$$

If we change the names of the dummy integration indices on the right, we find

$$\sum_{i=1}^N \sum_{j=1}^N \mathbf{F}_{ji} = - \sum_{j=1}^N \sum_{i=1}^N \mathbf{F}_{ji}. \quad (13)$$

Lastly, if we swap the order of summation on the right, we find

$$\sum_{i=1}^N \sum_{j=1}^N \mathbf{F}_{ji} = - \sum_{i=1}^N \sum_{j=1}^N \mathbf{F}_{ji} \Rightarrow \sum_{i=1}^N \sum_{j=1}^N \mathbf{F}_{ji} = 0. \quad (14)$$

Thus, we find that

$$\frac{d\mathbf{p}_T}{dt} = \mathbf{F}_{\text{ext}}. \quad (15)$$

This tells us that the change in total momentum of a collection of particles is equal to the sum of the external forces acting on all of those particles. If the collection of particles is totally isolated, then its total momentum is conserved. Notice that this result was derived solely on the basis of Newton's second and third laws - no other knowledge of the detailed interactions or physics was needed.

It should be pointed out, however, that while the momentum of an isolated system is conserved, its particular value does depend on our choice of reference frame - it is defined in terms of the velocity of the particles, which as we saw last lecture, will depend on the particular inertial frame of reference. However, so long as we are working in a particular reference frame which obeys Newton's laws, its value in that frame will stay a constant.

Center of Mass

This realization that the time derivative of total momentum only depends on the external forces helps us to define some new concepts which will allow us to understand some of the physical assumptions we've previously been making. Let's define a new object,

$$\mathbf{R} = \frac{\sum_{i=1}^N m_i \mathbf{r}_i}{\sum_{i=1}^N m_i} = \frac{1}{M} \sum_{i=1}^N m_i \mathbf{r}_i, \quad (16)$$

and call it the *center of mass*, where M is the total mass. It is an average of the positions of the particles, weighted by their masses.

Let's also define something called the *center of mass velocity*, which is the time derivative of the center of mass

$$\mathbf{v}_{CM} = \frac{d\mathbf{R}}{dt} = \frac{1}{M} \sum_{i=1}^N m_i \frac{d\mathbf{r}_i}{dt} = \frac{1}{M} \sum_{i=1}^N m_i \mathbf{v}_i = \frac{1}{M} \sum_{i=1}^N \mathbf{p}_i. \quad (17)$$

From this result, we see that the center of mass velocity is related to the total mass and total momentum of the system,

$$\mathbf{p}_T = M \mathbf{v}_{CM}. \quad (18)$$

If we take a second time derivative, we can define the acceleration of the center of mass, and we have

$$\frac{d\mathbf{p}_T}{dt} = M \mathbf{a}_{CM}. \quad (19)$$

However, from the previous section, we know that the change in total momentum is given by the total external force, and so we have

$$\mathbf{F}_{\text{ext}} = M\mathbf{a}_{CM}. \quad (20)$$

This result tells us that the total mass times the acceleration of the center of mass is given by the total external force.

This result actually helps us to explain some subtleties of what we've been doing up until now. I've frequently been making reference to "complicated microscopic forces" that occur inside of a body. I've cited this as the reason for why a coffee cup doesn't fall through a table, and also for why friction doesn't preserve kinetic energy. But if such a complicated microscopic world makes up all of these material bodies, how can I reliably apply Newton's laws to any sort of large bodies? I keep talking about applying forces to large objects like blocks and carts, without worrying at all about the shape or physical deformations of the body, and how this affects the way I interact with them. Why have I been able to treat my block like a single point object, and talk about its location and acceleration, without specifying details about the orientation of the body?

The answer to these questions is that really, when I talk about these large physical objects obeying Newton's second law, what I'm really saying is that the *center of mass* of these large composite bodies obey Newton's laws, using the total external force, and total mass. It doesn't matter precisely how I handle the block, so long as I know what the net force is that I'm applying over all of the atoms which make up its surface. Furthermore, once I know the total external force, I know this is equal to the total mass, times the acceleration of the center of mass. So really, what I am talking about here is the mass-weighted average of all of the positions in the body. When I take an eraser and throw it across the room, it may spin and rotate and move in a weird way, but the *center of mass* is the location in the body which will move according to the projectile motion formulas we derived. Even if the projectile does something incredibly violent like explode in the middle of flight, if we were to look at the constituent pieces, their center of mass would still move along the usual projectile path. These ideas are demonstrated in Figure 1.

Newton's first law of course is also true, since it is a special case of the second law - if the total external force on the object is zero, then its center of mass will not accelerate. Of course, we know that in many cases, a physical body will still react to forces on them, even when the total is zero. If I squeeze an object inwards so that the total force is zero, I can still change the physical size and shape of the object. But its center of mass will not accelerate.

When the mass of an object is constant, then we've seen that we can write Newton's second law in two forms,

$$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt}. \quad (21)$$

However, when the mass changes in time, these two expressions are no longer the same. So the question is, which IS the correct expression, in general? Well,

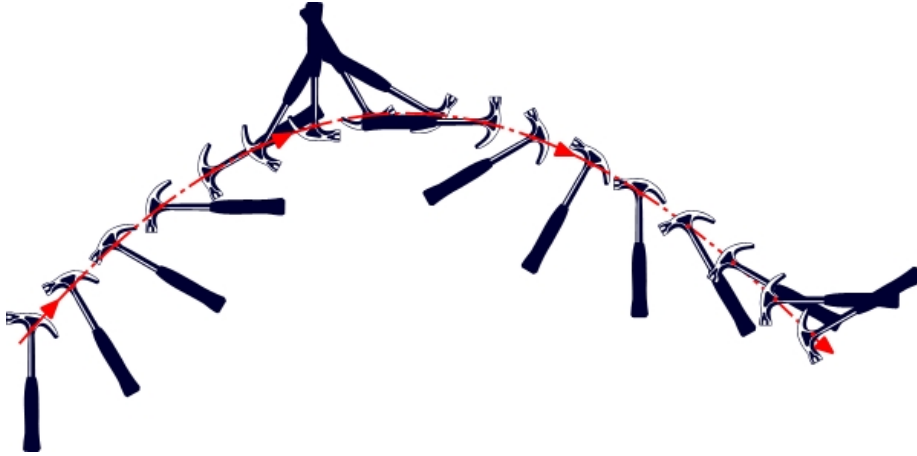


Figure 1: A hammer undergoing projectile motion. While it may rotate and generally move in some complicated way, the center of mass continues to follow the path we previously found for projectile motion.

it turns out I've been lying to you somewhat - the second form is actually the one that is correct in full generality. When the mass of an object changes, I can still use the second form, but I cannot use the first form. In particular, this means that the change in momentum is always equal to the integral of the force over time,

$$\mathbf{p}(t_2) - \mathbf{p}(t_1) = \int_{t_1}^{t_2} \mathbf{F}(t) dt. \quad (22)$$

We define this object to be something called the *impulse*,

$$\mathbf{J} \equiv \int_{t_1}^{t_2} \mathbf{F}(t) dt. \quad (23)$$

This equation can sometimes be useful in situations in which we know the total amount of force applied as a function of time.

Rocket Motion

There are several important situations in which using momentum considerations is crucial, because we are considering a system whose mass is changing. Consider the case of a rocket powering itself by emitting exhaust. In this situation, as the rocket dumps fuel, its mass is changing. This is indicated in Figure 2. With respect to some particular inertial reference frame that we've set up in space, the velocity of the rocket is v_R , and the velocity of the fuel coming out the back, at any particular instant in time, is v_f . Notice that both of these quantities, with respect to the inertial reference frame, will change over time as the rocket accelerates.

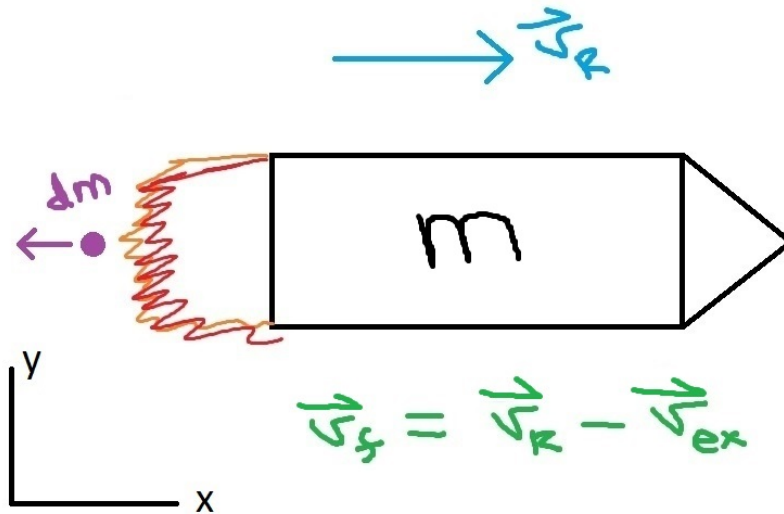


Figure 2: A rocket which is accelerating as a result of burning fuel.

Now we want to know - how does burning fuel affect the motion of the rocket? We'll work in one dimension, for simplicity, so that we will often omit the vector symbols on the relevant velocities - we can always do this if the exhaust is emitted straight out the back (keep in mind, however, that the velocities are still signed quantities). Let's assume that at any given moment, the mass of the rocket is some value m . As the rocket expels fuel, the mass of the rocket will change, but we should still be able to apply momentum conservation. We assume that the infinitesimal change in the rocket's mass is dm . If its infinitesimal change in velocity is dv_R , then the rocket's new momentum, after it has emitted the piece of fuel, will be

$$p_R = (m + dm)(v_R + dv_R). \quad (24)$$

Now, consider the infinitesimal piece of fuel that is emitted during this process. The mass of this piece of fuel will be the negative of the change in the rocket mass, and the momentum it carries away with it is given by

$$p_f = -dm v_f, \quad (25)$$

where v_f is the velocity of the emitted fuel with respect to whatever inertial frame we are using to measure velocities out in space. However, for someone who is on-board the rocket operating its engines, a more natural quantity to consider is the velocity of the fuel with respect to the rocket, since as the rocket operator adjusts the rate of fuel being emitted from the engines, this is the quantity that he can directly adjust. If we use the Galilean velocity transformation formula to write the velocity of the fuel in terms of the velocity of the rocket, emphasizing

with our notation that v_f and v_R are with respect to an inertial frame, we have

$$v_f \equiv v_{fI} = v_{fR} + v_{RI} \equiv v_R + v_{fR}. \quad (26)$$

It is often convention, however, to work instead with the **exhaust velocity**

$$v_{ex} = -v_{fR}. \quad (27)$$

The reason for this is that because fuel is being ejected backwards out of the rocket, v_{fR} will be negative, while v_{ex} will be positive. This is demonstrated in Figure 3. Using this velocity transformation, we can write

$$p_f = -dm(v_R - v_{ex}). \quad (28)$$

The total momentum of the rocket and the emitted piece of fuel is thus

$$p_T = (m + dm)(v_R + dv) - dm(v_R - v_{ex}). \quad (29)$$

If we expand out this expression, we find that

$$p_T = mv_R + mdv + v_{ex}dm + dmdv. \quad (30)$$

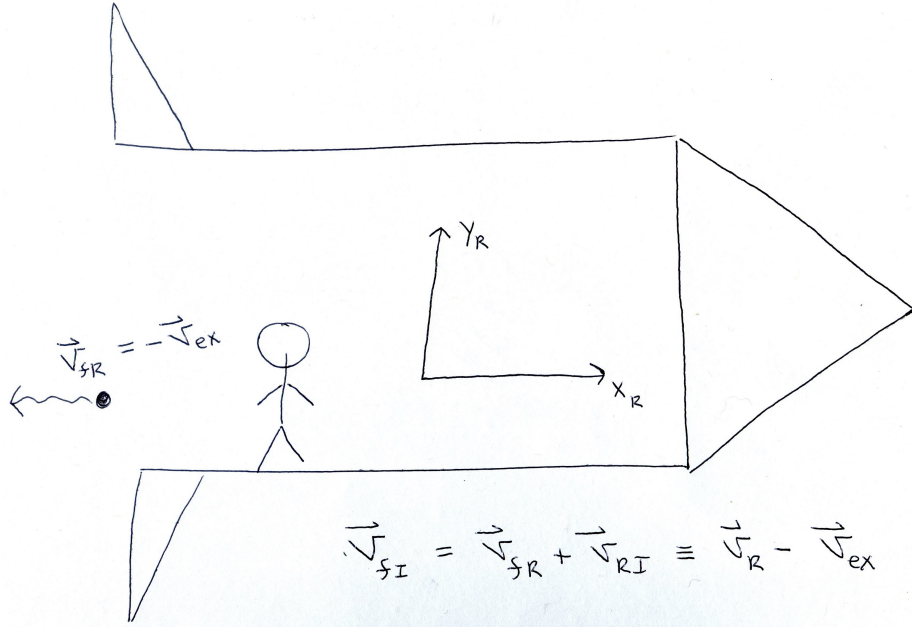


Figure 3: The fuel being exhausted from the perspective of an observer on the rocket.

Now, if we are taking the limit that dm and dv_R both become infinitesimally small, then the last term becomes unimportant compared with the other ones, since it is quadratic in small quantities. Thus, we drop it, and write

$$p_T = mv_R + mdv + v_{ex}dm. \quad (31)$$

Now, because there are no other external forces acting on the system, the net change in momentum must be zero. Before the infinitesimal piece of mass was emitted, it was sitting on the rocket, and the momentum of the two was simply mv_R . Thus, we have

$$mv_R = mv_R + mdv + v_{ex}dm, \quad (32)$$

or

$$mdv = -v_{ex}dm. \quad (33)$$

The above expression can be used to find an equation for the velocity as a function of mass. If we rearrange, we find

$$dv = -v_{ex} \frac{dm}{m}. \quad (34)$$

If we assume the rocket has some initial mass m_0 and some initial velocity v_0 , then we can write

$$\int_{v_0}^{v_R} dv' = -v_{ex} \int_{m_0}^m \frac{dm'}{m'}, \quad (35)$$

which becomes

$$v_R - v_0 = v_{ex} \ln \left(\frac{m_0}{m} \right). \quad (36)$$

So we see that the change in velocity of the rocket depends only on the exhaust speed of the fuel, and the ratio of the original mass to the new mass. Momentum conservation makes this a very easy result to arrive at.

Conservation of Angular Momentum

In addition to momentum (sometimes referred to as linear momentum), there is often an additional conserved quantity in our system of N particles, which we are already familiar with - angular momentum. The angular momentum of a single particle is defined according to

$$\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i, \quad (37)$$

which involves the displacement \mathbf{r}_i and momentum \mathbf{p}_i of the particle. The total angular momentum of the system of N particles is then the vector sum

$$\mathbf{L} = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{p}_i. \quad (38)$$

To see precisely the conditions that guarantee the conservation of angular momentum, we take a time derivative to find

$$\frac{d}{dt} \mathbf{L} = \sum_{i=1}^N \frac{d}{dt} (\mathbf{r}_i \times \mathbf{p}_i) = \sum_{i=1}^N (\mathbf{r}_i \times \dot{\mathbf{p}}_i + \dot{\mathbf{r}}_i \times \mathbf{p}_i), \quad (39)$$

where in the second equality I've made use of the fact that the cross product obeys a derivative product rule similar to that of the dot product. Now,

$$\dot{\mathbf{r}}_i \times \mathbf{p}_i = m_i (\dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i) = 0, \quad (40)$$

since the cross product of any vector with itself is zero. Thus,

$$\frac{d}{dt} \mathbf{L} = \sum_{i=1}^N \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_{i=1}^N \mathbf{r}_i \times \left(\mathbf{F}_{\text{ext},i} + \sum_{j=1}^N \mathbf{F}_{ji} \right). \quad (41)$$

Since the cross product distributes over addition, this becomes

$$\frac{d}{dt} \mathbf{L} = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_{\text{ext},i} + \sum_{i,j=1}^N \mathbf{r}_i \times \mathbf{F}_{ji}. \quad (42)$$

The first term above is the **net external torque** on the system of particles,

$$\tau_{\text{ext}} = \sum_{i=1}^N \tau_{\text{ext},i} = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_{\text{ext},i}. \quad (43)$$

This result is familiar to us from freshman mechanics when we studied the rotational motion of rigid bodies - the time derivative of the angular momentum is equal to the net torque applied to the system.

But what about the second term? Does it immediately cancel? Not necessarily. There is an additional assumption we must make in order for the second term to vanish. To see precisely what it is, let's use Newton's third law to write, for any two particles,

$$\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}. \quad (44)$$

If this term were to vanish, then again, when we expand out the double sum, all of the terms will vanish in pairs. The vanishing of this term will occur, in general, if

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} \parallel (\mathbf{r}_i - \mathbf{r}_j). \quad (45)$$

That is to say, the second sum will vanish if the force between any two particles acts along a line which is *parallel* to the displacement vector between them. In the vast majority of inter-particle forces we will consider, this will indeed be the case. It is the case for Newton's law of gravitation, as we will discuss later, and it is also true for the electrostatic interaction between two point particles. It is not true, however, for general electromagnetic interactions between particles, and in this case we must generalize the definition of angular momentum to include the angular momentum of the electromagnetic field itself. But so long as we restrict ourselves to forces for which this condition holds, then the total angular momentum of an isolated collection of particles will indeed be conserved.

Again, however, we should note that while the angular momentum will be conserved, it will in general depend on the choice of coordinate system. We can

see that if we change our coordinate system so that all displacement vectors shift according to

$$\mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{r}_0, \quad (46)$$

then the angular momentum will change according to

$$\mathbf{L} \rightarrow \mathbf{L} + \mathbf{r}_0 \times \mathbf{p}_T. \quad (47)$$

However, for any particular choice of coordinate system, this quantity will remain a constant, so long as our inter-particle forces obey the appropriate constraint.

Conservation of Energy

In the second week of class, we defined the potential energy for a single particle subject to an external force in one dimension,

$$U(x) = - \int_{x_0}^x F(x') dx'. \quad (48)$$

We found that whenever we could define a potential energy this way, the total energy of the particle was a constant,

$$\frac{dE}{dt} = \frac{d}{dt} (K + U) = 0. \quad (49)$$

We would now like to generalize such a definition to higher dimensions, and then to inter-particle forces. Analogously to the way that I defined potential energy in one dimension to be the (negative) work done on a particle while moving along a path, I might do the same in two or three dimensions. An example of this is sketched in Figure 4, for the case of the gravitational force near the surface of the Earth. Since the gravitational force depends only on my location (trivially, since it is just a constant), then I expect I should be able to do this. I therefore propose that

$$U_g(\vec{r}) = - \oint_C \vec{F} \cdot d\vec{r}, \quad (50)$$

where the notation emphasizes that this is a line integral performed over some curve C . I've defined the line integrals in the figure to start at the origin of some coordinate system I've set up, but this is only one of many possible choices. Now, for gravity, I know that I have

$$\vec{F} = -mg\hat{y}. \quad (51)$$

If I use this in my definition, and I consider traversing a path which points straight upwards, then I have

$$U_g(\vec{r}) = mg \oint_C \hat{y} \cdot d\vec{r} = mg \int_0^h dy = mgh. \quad (52)$$

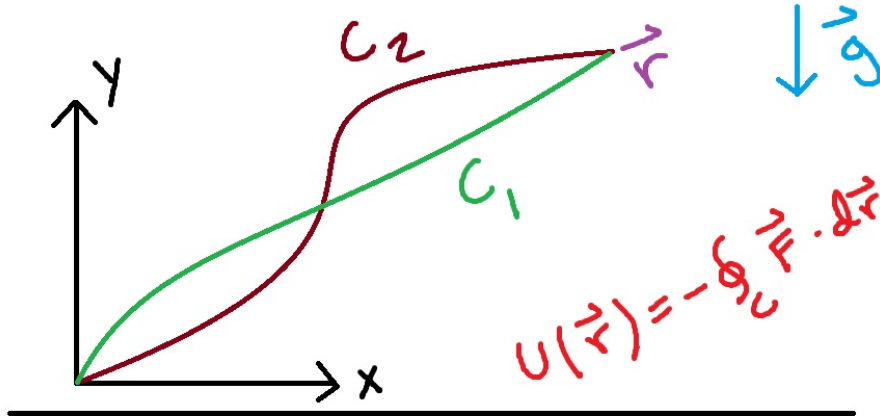


Figure 4: Defining potential energy for an arbitrary vector field.

Therefore, my gravitational potential energy is just a linear function of height (of course, since I can add an arbitrary constant to this, I can take $h = 0$ to be located wherever I want).

However, there is actually a potential issue here (no pun intended). In one dimension, there was only one way to get between any two points - all you can do is move in a straight line. But in more than one dimension, there are actually a lot of ways I can travel to get to a point. This is also demonstrated in Figure 4, where more than one path is drawn. For the case of gravity, my line integral was sufficiently simple to do because the vector field was constant and only pointed in one direction. But for a more general force, I would need to evaluate

$$U(\vec{r}) = -\oint_C \vec{F} \cdot d\vec{r} = -\int \vec{F} \cdot \vec{v} dt. \quad (53)$$

I'm now faced with an important question: does the value of the line integral depend on which path I take? If it does, then I have a problem - because the value of the line integral will depend on which path I take, I can no longer write a simple potential energy function that just depends on my location in space. Notice that while I told you that a line integral of a force (which only depends on position) does not depend on how quickly the path is traversed, I didn't make any promises about how it depends on **which** path you take between the two endpoints.

Of course, we may remember from our vector calculus courses that the line integral will not depend on the path so long as

$$\vec{\nabla} \times \vec{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z} = 0. \quad (54)$$

In other words, so long as the **curl** of the vector field is zero. If this is so, then it is possible to define a potential energy function, which we can find by

computing the line integral in question. The force is then given in terms of the potential energy function by taking a gradient,

$$\vec{F} = -\vec{\nabla}U = -\left(\frac{\partial U}{\partial x}\right)\hat{x} - \left(\frac{\partial U}{\partial y}\right)\hat{y} - \left(\frac{\partial U}{\partial z}\right)\hat{z}. \quad (55)$$

We know that while these considerations may at first seem merely academic, the question of a vector field being curl free is certainly an important one - for example, the gravitational field example given above represents a curl free vector field, and so the result we found for the line integral will be independent of our choice of path. This will also be true for static electric fields, for example, so that they can also be associated with a potential energy function. However, we know that magnetic fields are **never** curl free, and can never be associated with a potential energy function (although they can be associated with a quantity known as a vector potential, which you will discuss in your upper division E & M course). So the question is far from trivial. Proving the above theorem about a vector field admitting a potential if and only if it is curl free is a relatively straight-forward exercise in vector calculus, although we won't be concerned with proving it here.

The conservation of energy for such a one particle system in higher dimensions is now a straight-forward derivation, similar to the one before. We have

$$\frac{dE}{dt} = \frac{d}{dt} \left(\frac{1}{2}m\mathbf{v}^2 + U(\mathbf{r}) \right) = m\mathbf{v} \cdot \mathbf{a} + \frac{d}{dt}U(\mathbf{r}), \quad (56)$$

where the time derivative of the kinetic energy is again found by differentiating the dot product of the velocity with itself. As for the second term, we know that the chain rule in higher dimensions allows us to write

$$\frac{d}{dt} = \frac{dr_x}{dt} \frac{d}{dr_x} + \frac{dr_y}{dt} \frac{d}{dr_y} + \frac{dr_z}{dt} \frac{d}{dr_z} \equiv \mathbf{v} \cdot \nabla. \quad (57)$$

Thus, we find

$$\frac{dE}{dt} = m\mathbf{v} \cdot \mathbf{a} + \mathbf{v} \cdot \nabla U(\mathbf{r}) = \mathbf{v} \cdot [m\mathbf{a} + \nabla U(\mathbf{r})]. \quad (58)$$

However, according to Newton's second law,

$$m\mathbf{a} = \mathbf{F} = -\nabla U(\mathbf{r}), \quad (59)$$

and so again the energy is conserved,

$$\frac{dE}{dt} = 0. \quad (60)$$

This is all well and good for a single particle subject to an external force, but we want to return to our collection of N particles, subject to many inter-particle forces. We can generalize the definition above to a collection of N particles by writing

$$\mathbf{F}_i = -\nabla_i U(\{\mathbf{r}_i\}). \quad (61)$$

The quantity \mathbf{F}_i is the force on particle i , while the function $U(\{\mathbf{r}_i\})$ is the potential energy of the collection of particles. The notation emphasizes that it will, in general, depend in some way on the position vectors of all of the various N particles. The operator acting on it is a gradient operator that acts only with respect to the coordinates of the i^{th} particle,

$$\nabla_i \equiv \left(\frac{\partial}{\partial r_{ix}}, \frac{\partial}{\partial r_{iy}}, \frac{\partial}{\partial r_{iz}} \right). \quad (62)$$

An alternative notation for this object is sometimes written as

$$\nabla_i = \frac{\partial}{\partial \mathbf{r}_i}. \quad (63)$$

The total energy of the system is then defined as

$$E = \sum_{i=1}^N K_i + U(\{\mathbf{r}_i\}) = \frac{1}{2} \sum_{i=1}^N m_i \mathbf{v}_i^2 + U(\{\mathbf{r}_i\}). \quad (64)$$

To see that this total energy is conserved, we take a time derivative to find

$$\frac{dE}{dt} = \sum_{i=1}^N m_i \mathbf{v}_i \cdot \mathbf{a}_i + \frac{d}{dt} U(\{\mathbf{r}_i\}). \quad (65)$$

The time derivative acting on the potential energy can be written according to the chain rule as

$$\frac{d}{dt} = \sum_{i=1}^N \left(\frac{dr_{ix}}{dt} \frac{\partial}{\partial r_{ix}} + \frac{dr_{iy}}{dt} \frac{\partial}{\partial r_{iy}} + \frac{dr_{iz}}{dt} \frac{\partial}{\partial r_{iz}} \right) = \sum_{i=1}^N \mathbf{v}_i \cdot \nabla_i, \quad (66)$$

so that we find

$$\frac{dE}{dt} = \sum_{i=1}^N m_i \mathbf{v}_i \cdot \mathbf{a}_i + \sum_{i=1}^N \mathbf{v}_i \cdot \nabla_i U(\{\mathbf{r}_i\}). \quad (67)$$

Using the relationship between force and potential energy, we find

$$\frac{dE}{dt} = \sum_{i=1}^N m_i \mathbf{v}_i \cdot \mathbf{a}_i - \sum_{i=1}^N \mathbf{v}_i \cdot \mathbf{F}_i, \quad (68)$$

which again vanishes by Newton's second law, applied to each of the i particles.

In some sense, we can imagine our potential energy function living in a large, $3N$ -dimensional space, made up of all of the position vectors of the N particles,

$$\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N). \quad (69)$$

The expression for the force in terms of the potential energy is then something akin to a gradient which acts on this entire space, resulting in one large force “vector” that lives in this $3N$ -dimensional space,

$$\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_N) = -\frac{\partial}{\partial \mathbf{r}} U(\mathbf{r}). \quad (70)$$

We might wonder exactly what general conditions must be placed on the inter-particle forces in order for them to admit a potential energy, something akin to the curl free condition for a single particle. However, it turns out that these conditions are somewhat more mathematically involved than we can discuss here, since it involves generalizing the curl and cross product to higher dimensions (which is not necessarily a simple task). So for our purposes, we will generally be content to *assume* that the inter-particle forces can be derived from a potential energy function. This will be true for any of the inter-particle forces that we will be interested in studying.

Two-Body Central Potentials

By far the most important type of potential energy function we will consider is given according to the form

$$U(\{\mathbf{r}_i\}) = \frac{1}{2} \sum_{i,j=1}^N U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|), \quad (71)$$

which is a sum of **two-body central potentials** U_{ij} . Each two-body central potential depends only on the coordinates of particles i and j , and it is a function only of the distance between them. We will assume that the functions are symmetric,

$$U_{ij} = U_{ji}, \quad (72)$$

so that we can alternatively write

$$U(\mathbf{r}) = \sum_{i < j} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|). \quad (73)$$

For this type of potential, we can define

$$\mathbf{F}_{ji} = -\frac{\partial}{\partial \mathbf{r}_i} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = -\frac{\partial}{\partial \mathbf{r}_i} U_{ji}(|\mathbf{r}_i - \mathbf{r}_j|), \quad (74)$$

so that

$$\mathbf{F}_k = -\frac{\partial}{\partial \mathbf{r}_k} U(\mathbf{r}) = -\frac{1}{2} \sum_{i,j} \frac{\partial}{\partial \mathbf{r}_k} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) = -\frac{1}{2} \left[\sum_{i=1}^N \frac{\partial}{\partial \mathbf{r}_k} U_{ik}(|\mathbf{r}_i - \mathbf{r}_k|) + \sum_{j=1}^N \frac{\partial}{\partial \mathbf{r}_k} U_{kj}(|\mathbf{r}_k - \mathbf{r}_j|) \right]. \quad (75)$$

Notice that we get two sums after taking the derivative, since the derivative will be non-zero whenever $i = k$ **or** $j = k$. Since the two-body potentials are symmetric, we can simplify this as

$$\mathbf{F}_k = - \sum_{i=1}^N \frac{\partial}{\partial \mathbf{r}_k} U_{ki} (|\mathbf{r}_k - \mathbf{r}_i|) = \sum_{i=1}^N \mathbf{F}_{ik}. \quad (76)$$

Thus, when our potential is the sum of two-body potentials, we find that the force on a given particle is simply equal to the vector sum of the forces it feels from each and every other particle. Notice that if we had chosen a more complicated potential energy function, this may not have been the case (there are some branches of physics in which **three**-body potentials and higher must be considered, and the resulting expression for the force on a particle is not so simple). We can also show that our particular choice of potential also correctly reproduces Newton's third law. A short exercise in using the chain rule demonstrates that

$$\mathbf{F}_{ji} = - \frac{\partial}{\partial \mathbf{r}_i} U_{ij} (|\mathbf{r}_i - \mathbf{r}_j|) = f_{ji} (|\mathbf{r}_i - \mathbf{r}_j|) (\mathbf{r}_i - \mathbf{r}_j), \quad (77)$$

where

$$f_{ji} (r) = f_{ij} (r) = - \frac{1}{r} \frac{dU_{ij} (r)}{dr}. \quad (78)$$

From this expression, it is immediately clear that

$$\mathbf{F}_{ji} = f_{ji} (|\mathbf{r}_i - \mathbf{r}_j|) (\mathbf{r}_i - \mathbf{r}_j) = -f_{ij} (|\mathbf{r}_j - \mathbf{r}_i|) (\mathbf{r}_j - \mathbf{r}_i) = -\mathbf{F}_{ij}, \quad (79)$$

which is indeed Newton's third law. In fact, from this expression, we see that we also have

$$(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji} = 0, \quad (80)$$

which was the additional assumption we needed in order for the conservation of angular momentum to hold.

Notice in particular that our choice of potential energy function only depends on the distances between particles, not on their absolute position in space, and not on their angles with respect to some orientation in space. As a result of the particular form of our potential energy function, we found that linear momentum and angular momentum were both conserved. This conclusion is in fact quite general. Next quarter when you study Lagrangian and Hamiltonian mechanics in Physics 104, you will learn why the **translational invariance** of a system always implies momentum conservation, and why the **rotational invariance** of a system always implies angular momentum conservation. These results are part of a broader theorem known as Noether's theorem, which states that for every continuous symmetry of a system, there is a corresponding conserved quantity. The additional continuous symmetry which is responsible for energy conservation is **time translation symmetry** - the fact that none of our expressions for the physics of the system depended explicitly on the time.

Thus, for our system of N particles described by a potential energy function that is a sum of two-body central potentials, we see that the total linear momentum, angular momentum, and energy will be conserved. In the next lecture, we will see how making use of these conservation laws will allow us to dramatically simplify the motion of particles subject to such a set of forces.