Driven Harmonic Motion

Let's again consider the differential equation for the (damped) harmonic oscillator,

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = \mathcal{L}_\beta y = 0, \tag{1}$$

where

$$\mathcal{L}_\beta \equiv \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}}{\mathrm{d}t} + \omega^2 \tag{2}$$

is a linear differential operator. Our physical interpretation of this differential equation was a vibrating spring with angular frequency

$$\omega = \sqrt{k/m}, \tag{3}$$

where k is the spring constant and m is the particle's mass, subject to an additional drag force described by

$$F_d = -b\dot{x} = -bv \ ; \ \beta = b/2m \tag{4}$$

However, there are plenty of other physical interpretations of this equation - for example, an electronic circuit with a capacitor, resistor, and an inductor will also satisfy this equation, with the coordinate y representing the circuit's current (with the constants b and ω replaced with the appropriate electronic quantities). In the last lecture we spent some time studying the solutions to this differential equation in detail.

However, we can imagine some further generalizations of this equation. In many cases there may be some sort of external driving force on our system which depends on time, $F(t)$. This could be the case, for example, if our spring is being physically driven by a motor, or, in the case of the electronic circuit, there is an externally applied voltage. In this case, our equation becomes

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = \mathcal{L}_\beta y = f(t), \tag{5}$$

where

$$f(t) = F(t)/m. \tag{6}$$

For now, we will assume that F only depends on time, and not the particle's position.

In particular, this type of equation is known as an **in-homogeneous**, linear differential equation. It is linear because the term on the left is still described by a linear differential operator, but it is known an in-homogeneous equation, since the term on the right is no longer zero - it is some function which depends on time. How do we solve such an equation? The first thing to notice is that if we have a function y_h which is a solution to the *homogeneous* equation,

$$\mathcal{L}_\beta y_h = 0, \tag{7}$$

and another particular solution, y_p, which satisfies the full in-homogeneous equation,

$$\mathcal{L}_\beta y_p = f(t), \tag{8}$$

then the sum of the two will also satisfy the full in-homogeneous equation,

$$\mathcal{L}_\beta (y_h + y_p) = \mathcal{L}_\beta y_h + \mathcal{L}_\beta y_p = 0 + f(t) = f(t). \tag{9}$$

This tells us that if we can find one particular solution to this differential equation, the most general solution is given by adding a solution to the homogeneous equation. It is still true that in order to specify the full solution to our differential equation, we must supply two initial conditions, and these will determine the two constants in the solution y_h. Of course, we already know what y_h should look like from last lecture, and so the goal now is to find y_p.

While it may not seem obvious at first, it turns out that there are only three very special functions $f(t)$ which we need to consider, and every other case will follow from those solutions. Let's see exactly why this is the case.

Constant Forcing

Let's consider the simplest possible forcing function, in which $f(t)$ is a constant,

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = c, \tag{10}$$

so that

$$F(t) = mf(t) = mc. \tag{11}$$

Rearranging this equation slightly, we have

$$\ddot{y} + 2\beta\dot{y} + \omega^2 \left(y - \frac{c}{\omega^2}\right) = 0. \tag{12}$$

In this form, it's easy to see that if we propose the constant function,

$$y_p(t) = c/\omega^2, \tag{13}$$

then both sides of our differential equation are zero, and we indeed have a solution. So we find that in this case, there is a particular solution in which the oscillator does not move at all (can you see why this constant position does not depend on β?). Of course, we could have anticipated this on physical grounds. Notice that the force exerted by the spring, when it is held at this position, is given by

$$F_s = -ky = -kc/\omega^2 = -mc = -F(t). \tag{14}$$

This solution is the one in which the force of the spring precisely balances the applied force, and the entire motion is trivial.

The most general solution is now found by adding this particular solution to the solution from the homogeneous case,

$$y(t) = c/\omega^2 + y_h(t). \tag{15}$$

For example, in the case of weak damping, we would find

$$y(t) = c/\omega^2 + x_* + e^{-\beta t}\left[A\cos(\Omega t) + B\sin(\Omega t)\right], \tag{16}$$

written in terms of the original x coordinate, defined by

$$y = x - x_*. \tag{17}$$

In this form, it is clear to see that the effect of the constant external force is simply to change the equilibrium point of the spring,

$$x_* \to x_* + c/\omega^2. \tag{18}$$

In fact, this constant external force amounts to nothing other than a change in the potential energy of the spring,

$$\widetilde{U}(x) = \frac{1}{2}k(x - x_*)^2 - mcx. \tag{19}$$

So all of the effects of a constant driving force can be accounted for by simply modifying the equilibrium point of the problem.

Sinusoidal Forcing

A much more interesting forcing function, of course, is one which actually depends on time. As a first example, let's consider a sinusoidal driving force, so that

$$\ddot{y} + 2\beta\dot{y} + \omega^2 y = f_0 \cos(\gamma t), \tag{20}$$

where γ is some angular frequency (which is not necessarily related to ω), and f_0 is a constant. How do we now find a particular solution to this equation?

Once again, we can always resort to our guess-and-check method. In this case, as we will see shortly, the solution is not too complicated looking, so it's conceivable that we would have eventually guessed this function. However, today we want to start developing some more sophisticated tools, and so we're going to use a slightly different approach. The first new tool we will introduce is the **Fourier transform**.

The Fourier transform of a function $f(t)$ is defined by

$$\hat{f}(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\nu t}\, dt. \tag{21}$$

We can think of the Fourier transform as an "operator" that acts on one function and produces a new function, often denoted $\hat{f}(\nu)$, which depends on the variable ν. In fact, because of the linearity of integration, it is a linear operation, satisfying the sum rule,

$$\widehat{(f+g)}(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f(t) + g(t)) e^{-i\nu t}\, dt = \hat{f}(\nu) + \hat{g}(\nu), \tag{22}$$

and also the scalar multiplication rule,

$$\widehat{(\alpha f)}(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\alpha f(t)) e^{-i\nu t}\, dt = \alpha \hat{f}(\nu). \tag{23}$$

The Fourier transformation can be inverted, and the original function itself can be recovered from $\hat{f}(\nu)$ by writing

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\nu)\, e^{i\nu t}\ d\nu. \tag{24}$$

The fact that the transformation is invertible in this way is a result of the *Fourier inversion theorem*. You'll get some practice performing Fourier transforms of some simple functions in the homework. Another common notation for the Fourier transform is

$$\mathcal{F}[f](\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\nu t}\ dt. \tag{25}$$

For now, however, in addition to the linearity property, there is one very important property of the Fourier transform that will make it a valuable tool in solving our equation. If I Fourier transform the derivative of a function,

$$\widehat{(f')}(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(t)\, e^{-i\nu t}\ dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dt}\left(f(t)\right) e^{-i\nu t}\ dt, \tag{26}$$

then a short exercise (which you'll perform on the homework) shows that

$$\widehat{(f')}(\nu) = i\nu \hat{f}(\nu). \tag{27}$$

In other words, the Fourier transform of the derivative of a function is the same as the Fourier transform of the original function, but with an additional factor out front. In fact, for a derivative of an arbitrarily high order,

$$f^{(n)}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), \tag{28}$$

it is straight-forward to show that

$$\left(\widehat{f^{(n)}}\right)(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^n}{dt^n}\left(f(t)\right) e^{-i\nu t}\ dt = (i\nu)^n \hat{f}(\nu). \tag{29}$$

The fact that the Fourier transform acts in this manner on derivatives, combined with the linearity of the Fourier transform, means that if I act the Fourier transform on the left side of my differential equation, I have

$$\mathcal{F}\left[\ddot{y} + 2\beta\dot{y} + \omega^2 y\right] = -\nu^2\hat{y} + 2\beta i\nu\hat{y} + \omega^2\hat{y} = \left(-\nu^2 + 2\beta\nu i + \omega^2\right)\hat{y}. \tag{30}$$

Notice that after taking a Fourier transform, there are no derivatives left anywhere in this expression. If I Fourier transform both sides of my differential equation, for some arbitrary forcing function $f(t)$, I therefore find that

$$\left(-\nu^2 + 2\beta\nu i + \omega^2\right)\hat{y} = \hat{f}, \tag{31}$$

where $\hat{f}$ is the Fourier transform of the forcing function. However, this is just an *algebraic* equation for $\hat{y}$, which we can easily solve to find

$$\hat{y} = \frac{f_0}{\left(-\nu^2 + 2\beta\nu i + \omega^2\right)}\hat{f}. \tag{32}$$

Now that I know what $\hat{y}$ is, I can use the Fourier inversion theorem to write $y\left(t\right)$ as

$$y\left(t\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{y}\left(\nu\right)e^{i\nu t}\ d\nu = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{f}\left(\nu\right)}{\left(-\nu^2 + 2\beta\nu i + \omega^2\right)}e^{i\nu t}\ d\nu. \tag{33}$$

The punch line here is that by taking the Fourier transform of both sides, a linear differential equation for $y\left(t\right)$ was transformed into an algebraic equation for its Fourier transform, which I can then use to find the original function of time.

While this technique is very general, we still need to perform the integral involved in Fourier transforming the forcing function, and then perform the integral involved in recovering the original function of time. For most forcing functions, it is not possible to do this in closed form. However, our sinusoidal case is in fact one of the cases where we can perform these manipulations. In order to Fourier transform the cosine function, we can again use Euler's formula to write

$$\cos\left(\gamma t\right) = \frac{1}{2}\left(e^{i\gamma t} + e^{-i\gamma t}\right). \tag{34}$$

Using the expression for the Fourier transform, this means that

$$\mathcal{F}\left[f_0\cos\left(\gamma t\right)\right] = \frac{f_0}{2\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}e^{i\gamma t}e^{-i\nu t}\ dt + \int_{-\infty}^{\infty}e^{-i\gamma t}e^{-i\nu t}\ dt\right), \tag{35}$$

or,

$$\mathcal{F}\left[f_0\cos\left(\gamma t\right)\right] = \frac{f_0}{2\sqrt{2\pi}}\left(\int_{-\infty}^{\infty}e^{-i(\nu-\gamma)t}\ dt + \int_{-\infty}^{\infty}e^{-i(\nu+\gamma)t}\ dt\right). \tag{36}$$

At first glance, this expression does not seem like it is mathematically well-defined. We have the integral of a sinusoidally varying function, integrated over all time, which is something which seems like it should not converge. However, as good physicists, we never let issues of mathematical rigour stop us, and so we will make use of the formula

$$\int_{-\infty}^{\infty}e^{+ixt}\ dt = \int_{-\infty}^{\infty}e^{-ixt}\ dt = 2\pi\delta\left(x\right), \tag{37}$$

where $\delta\left(x\right)$ is the **Dirac delta function**. The Dirac delta function is a "function" which is equal to "infinity" when $x = 0$, and is equal to zero for all other values. The issues of mathematical rigour surrounding the Dirac delta function are very complicated, and it took many decades for Mathematicians to develop the appropriate tools (known as the "theory of distributions") to understand how to make something mathematically reasonable out of this vague idea. However, as physicists, the only thing we care about is that mathematicians have spent plenty of time developing these tools, and they assure us that we can get away with using this formula, along with several other properties. The main property we will need to know here, aside from the one above, is that

$$\int_a^b f(x)\,\delta(x)\,dx = f(0)\,, \tag{38}$$

as long as the region being integrated over contains the origin. You'll explore some properties of the delta function in more detail on the homework.

With this result, we can now see that the Fourier transform of our driving function is given by

$$\hat{f}(\nu) = \mathcal{F}[f_0\cos(\gamma t)] = \frac{\sqrt{2\pi}f_0}{2}\left(\delta(\nu-\gamma)+\delta(\nu+\gamma)\right). \tag{39}$$

Using this in our expression for the solution, we find

$$y(t) = \frac{f_0}{2}\int_{-\infty}^{\infty}\frac{\delta(\nu-\gamma)+\delta(\nu+\gamma)}{(-\nu^2+2\beta\nu i+\omega^2)}e^{i\nu t}\,d\nu, \tag{40}$$

or,

$$y(t) = \frac{f_0}{2}\left[\int_{-\infty}^{\infty}\frac{\delta(\nu-\gamma)}{(-\nu^2+2\beta\nu i+\omega^2)}e^{i\nu t}\,d\nu + \int_{-\infty}^{\infty}\frac{\delta(\nu+\gamma)}{(-\nu^2+2\beta\nu i+\omega^2)}e^{i\nu t}\,d\nu\right]. \tag{41}$$

The delta functions underneath the integrals make the integration simple to perform, and we find

$$y(t) = \frac{f_0}{2}\left[\frac{e^{i\gamma t}}{(\omega^2-\gamma^2+2\beta\gamma i)}+\frac{e^{-i\gamma t}}{(\omega^2-\gamma^2-2\beta\gamma i)}\right]. \tag{42}$$

While this is, in principle, the solution we are looking for, it pays to clean it up a little bit. After performing a little bit of algebra, we find

$$y(t) = \frac{f_0}{(\omega^2-\gamma^2)^2+4\beta^2\gamma^2}\left[\left(\omega^2-\gamma^2\right)\left(\frac{e^{i\gamma t}+e^{-i\gamma t}}{2}\right)+2\beta\gamma\left(\frac{e^{i\gamma t}-e^{-i\gamma t}}{2i}\right)\right]. \tag{43}$$

Using Euler's formula again, this becomes

$$y(t) = \frac{f_0}{(\omega^2-\gamma^2)^2+4\beta^2\gamma^2}\left[\left(\omega^2-\gamma^2\right)\cos(\gamma t)+2\beta\gamma\sin(\gamma t)\right]. \tag{44}$$

Lastly, using our formula to write two sinusoidal terms as one single cosine term, we find

$$y(t) = \frac{f_0}{\sqrt{(\omega^2-\gamma^2)^2+4\beta^2\gamma^2}}\cos(\gamma t-\delta)\,, \tag{45}$$

where

$$\delta = \arctan\left(\frac{2\beta\gamma}{\omega^2 - \gamma^2}\right). \tag{46}$$

Using the method of Fourier transform, we now have a solution to our differential equation. Notice that for this particular solution, there are **no** free constants to fix from initial conditions - everything in this solution is determined by the parameters of the differential equation. Specifying initial conditions will come in to play later, when we add the homogeneous solution. For now, let's think about what this particular solution tells us. First, we notice that we have a cosine term with the **same** frequency as the driving force - shaking the spring at some constant frequency results in an oscillatory response with the same frequency. However, the spring does not simply bow down to the external force without a fight - the parameters of the spring cause two important effects. First, there is a phase shift in our solution, given by δ, so that the response of the spring lags the applied driving force - the spring demonstrates a "resistance" against the external driving force.

Secondly, the **amplitude** of the response is controlled by the quantity

$$D = \frac{f_0}{\sqrt{\left(\omega^2 - \gamma^2\right)^2 + 4\beta^2\gamma^2}}. \tag{47}$$

The dependence of this amplitude on the parameters of the problem is non-trivial. In particular, different driving frequencies will cause different response amplitudes. A plot of this behaviour is shown in Figure 1, where the value of D is plotted against the driving frequency. In particular, the quantity D will be maximized when

$$\gamma = \omega_R = \sqrt{\omega^2 - 2\beta^2}, \tag{48}$$

where ω_R is the **resonant frequency** of the oscillator. When the amplitude is maximized as a result of the oscillator being driven at the resonant frequency, we say that the oscillator is in **resonance** with the driving force. This ability to elicit a dramatic amplitude response at a particular frequency is the underlying mechanism behind all radio tuners - incoming radio waves behave as time-dependent forcing functions which act on an RLC circuit, and by configuring the properties of the circuit correctly, only radio waves of a certain frequency will create a response in the radio. Resonance behaviour is also an important consideration when constructing buildings in earthquake zones - seismic waves tend to travel over a particular range of frequencies, and any building whose mechanical oscillations have a resonant frequency in this range will be especially susceptible to earthquake damage. Notice that a non-zero resonant frequency requires

$$\omega_R \neq 0 \;\Rightarrow\; \beta < \omega/\sqrt{2}. \tag{49}$$

That is, the oscillator must be sufficiently under-damped in order to exhibit a resonance response - damping values larger than this will result in a response

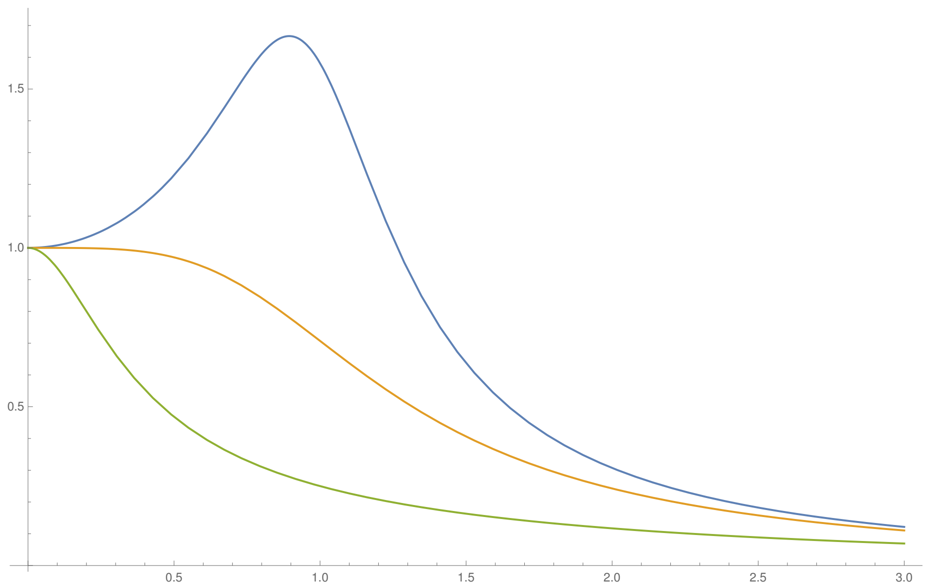

Figure 1: Resonance behaviour in the driven harmonic oscillator, for the case $\omega = 1$, and $\beta^2 = 0.1$ (blue curve), $\beta^2 = 0.5$ (orange curve), and $\beta^2 = 4$ (green curve). Notice that the first two cases correspond to weak damping, while the last corresponds to strong damping. A sharp resonance peak at a non-zero value of γ will only occur when $\beta^2 < \omega^2/2 < 0.5$, which is similar to, but not the same as, the criterion for critical damping.

function without a sharp peak, which is also shown in Figure 1. On the homework, you'll explore how the sharpness of this peak changes as we modify the parameters of the problem.

Now that we have the particular solution to our differential equation, we can easily construct the most general solution by adding to it the homogeneous solution,

$$y\left(t\right) = y_h\left(t\right) + \frac{f_0}{\sqrt{\left(\omega^2 - \gamma^2\right)^2 + 4\beta^2\gamma^2}} \cos\left(\gamma t - \delta\right). \tag{50}$$

For example, in the case of weak damping, we would have

$$y\left(t\right) = e^{-\beta t}\left[A\cos\left(\Omega t\right) + B\sin\left(\Omega t\right)\right] + \frac{f_0}{\sqrt{\left(\omega^2 - \gamma^2\right)^2 + 4\beta^2\gamma^2}} \cos\left(\gamma t - \delta\right), \tag{51}$$

where

$$\Omega = \sqrt{\omega^2 - \beta^2} \tag{52}$$

is NOT the resonant frequency. The constants A and B are again determined by initial conditions. Regardless of the damping regime, however, there is an important aspect of the homogeneous solution - its amplitude decays exponentially, whereas the amplitude of the particular solution does NOT. This implies that at large enough times, the only aspect of the solution which matters is the particular solution. At short times, the homogeneous solution induces what is known as "transient behaviour" - there are oscillations at frequency Ω which die out after the characteristic time. After this point, the oscillations at the driving frequency are the only substantial contribution to the motion, so that

$$y\left(t \to \infty\right) \approx \frac{f_0}{\sqrt{\left(\omega^2-\gamma^2\right)^2+4\beta^2\gamma^2}} \cos\left(\gamma t-\delta\right). \tag{53}$$

Notice that the long-time solution is *totally independent of the initial conditions*, since it does not involve the arbitrary constants A and B. For this reason, the particular solution is referred to as an "attractor" - all initial conditions approach this same solution. Needless to say, this means that studying the long-time behaviour of a driven oscillator is greatly simplified. In general, a linear differential equation subject to a driving force will only ever have one attractor. However, non-linear differential equations can have multiple attractors, which you will learn about if you take Physics 106.

Undamped Resonance

In the case of sinusoidal driving, we found that the response amplitude was given by

$$D = \frac{f_0}{\sqrt{\left(\omega^2-\gamma^2\right)^2+4\beta^2\gamma^2}}. \tag{54}$$

When β is very small, and when γ is very close to ω, all of the terms in the denominator of this expression are becoming very small. For this reason, the oscillations become very large in this limit, which you'll explore in more detail on the homework. For now, however, there is one aspect of this solution that should concern us - when there is no damping, and the driving frequency is the same as the natural frequency of the oscillator, we find

$$D\left(\beta=0,\gamma=\omega\right) \to \infty, \tag{55}$$

which does not make any physical sense - we know that if we take a spring and shake it at the right frequency, the position coordinate should not suddenly go to infinity. Also, there are a variety of theorems about the behaviour of differential equations which tell us that it does not make mathematical sense either. So something has clearly gone wrong with our solution technique. As it will turn out, our solution $y\left(t\right)$ in this case does not have a well-defined Fourier transform, and so our attempt at finding a solution this way was doomed from the beginning. So we need to find a new approach.

While it is clear that "infinity" is not the correct answer to our equation, it is plausible that in this situation, the amplitude of oscillation should be something other than a constant. We know that if we take a spring and give it a push whenever it is at a turning point, the oscillations should get bigger - for example, we know that when we were kids sitting on a swing, if our parents gave us a push at the top of the swing, we would keep swinging higher and higher. So we suspect that if our forcing function is in synch with the natural oscillations of the spring, our solution should take the form of an oscillation function with a growing amplitude. Let's therefore make the guess

$$y_p\left(t\right) = z\left(t\right)\cos\left(\omega t - \delta\right). \tag{56}$$

Remember that in this case, $\gamma = \omega$. If we set $\beta = 0$ and plug this guess into our differential equation, we find

$$-2\omega\sin\left(\omega t - \delta\right)\dot{z}\left(t\right) + \cos\left(\omega t - \delta\right)\ddot{z}\left(t\right) = f_0\cos\left(\omega t\right). \tag{57}$$

We now must choose $z\left(t\right)$ and δ so that both sides of this equation are equal. Our first attempt might be to set the cosine terms equal, by choosing

$$\delta = 0\ ;\ \ddot{z}\left(t\right) = f_0\ ??? \tag{58}$$

However, while this matches the two cosine terms together, we then have the sine function to deal with, and there is no way to cancel it out. A better approach is to notice that

$$\sin\left(\omega t - 3\pi/2\right) = \cos\left(\omega t\right). \tag{59}$$

Therefore, we should instead choose

$$\delta = 3\pi/2\ ;\ \dot{z}\left(t\right) = -f_0/2\omega. \tag{60}$$

Because $\dot{z}$ is a constant, this automatically eliminates the second term on the left, since $\ddot{z}$ is now zero. We now have both sides of our equation matched, since

$$-2\omega\sin\left(\omega t - 3\pi/2\right)\left(-f_0/2\omega\right) + \cos\left(\omega t - 3\pi/2\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(-f_0/2\omega\right) = f_0\cos\left(\omega t\right). \tag{61}$$

Therefore, the particular solution to our equation is given by

$$y_p\left(t\right) = \left(z_0 - \frac{f_0}{2\omega}t\right)\cos\left(\omega t - 3\pi/2\right) = \left(-z_0 + \frac{f_0}{2\omega}t\right)\sin\left(\omega t\right), \tag{62}$$

where we use another phase shift identity in the second equality. Notice that in this special case, the particular solution involves an arbitrary constant, which was not previously true. This is because in this situation, we have actually already captured part of the homogeneous solution, which we know from before is given by

$$y_h\left(t\right) = A\cos\left(\omega t\right) + B\sin\left(\omega t\right). \tag{63}$$

In this case, then, the most general solution is given by

$$y\left(t\right) = A\cos\left(\omega t\right) + \left(B + \frac{f_0}{2\omega}t\right)\sin\left(\omega t\right). \tag{64}$$

Rewriting this slightly, we find

$$y\left(t\right)=\sqrt{A^{2}+\left(B+\frac{f_{0}}{2\omega}t\right)^{2}}\cos\left(\omega t-\phi\right)\ ;\ \phi=\arctan\left(\frac{B}{A}+\frac{f_{0}}{2\omega A}t\right). \quad (65)$$

As expected, our solution has an amplitude that increases with time. Notice that in this form, the phase shift of the cosine term is time-dependent. At large enough times, it is again true that the particular solution dominates the homogeneous solution, and we find

$$y\left(t\rightarrow\infty\right)\approx\frac{f_{0}}{2\omega}t\sin\left(\omega t\right). \quad (66)$$

The oscillation amplitude grows without bound, which is what we would expect - if we constantly keep forcing an undamped oscillator in phase with its oscillations, we will keep adding more and more energy to the oscillator.

We can see now why our previous Fourier transform technique did not work - the solution we have found here does not have a well-defined Fourier transform, and so any attempt to find a solution this way was doomed to fail from the outset. However, the Fourier transform technique was not totally useless - by studying the limiting behaviour as $\beta\rightarrow 0$, we were able to gain some physical insight as to what was occurring in our system. Based on this insight, we were able to make a much better educated guess as to what the $\beta=0$ solution looked like. Again, a combination of mathematics and a physically motivated guess led to our solution.

While this solution is interesting from a mathematical sense, its applicability is actually quite limited from a practical standpoint. Almost any real system will have at least some damping, and so while the resonance peak in our system may be very large, there will typically be some finite amplitude of oscillation, which is constant and does not grow with time.

Arbitrary Periodic Forcing

We've now found the solution to two special cases - constant forcing, and sinusoidal forcing. While I promised you that we only needed to study three special cases, already with these two solutions we can actually make quite a bit of progress. To see why this is the case, let's notice another consequence of the linearity of our differential operator. Let's imagine that I have two forcing functions, each with a corresponding solution, so that

$$\mathcal{L}y_{1}=f_{1}\ ;\ \mathcal{L}y_{2}=f_{2} \quad (67)$$

From the linearity of my differential operator, we can see that

$$\mathcal{L}\left(y_{1}+y_{2}\right)=\mathcal{L}y_{1}+\mathcal{L}y_{2}=f_{1}+f_{2}. \quad (68)$$

In other words, if we add two driving forces together, the solution to the differential equation is just the sum of the two individual solutions. If we have a more general sum for the driving force,

$$f(t) = \sum_{n=0}^{\infty} a_n f_n(t), \tag{69}$$

then the solution will be given by

$$y(t) = \sum_{n=0}^{\infty} a_n y_n(t), \tag{70}$$

where

$$\mathcal{L} y_n = f_n. \tag{71}$$

Thus, each driving terms induces a certain response in the oscillator, which acts independently from the other driving forces.

This property is clearly very useful, since if I already know the solution to the differential equation for a collection of different driving forces, then I also know the solution for a forcing function which is an arbitrary linear combination of those driving forces. However, this property actually turns out to be even more powerful than we might have imagined. The reason for this is a result of **Dirichlet's Theorem**, which says that for any periodic function which satisfies a set of technical conditions known as Dirichlet conditions, we can always write the function as an infinite sum of sine and cosine terms,

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\lambda t) + b_n \sin(n\lambda t) \right], \tag{72}$$

where

$$\lambda = 2\pi / T, \tag{73}$$

is the angular frequency of the driving force, which has period T. This series expansion is known as the **Fourier Series** of the function. For any realistic function that we will consider as a driving force, the set of technical conditions behind Dirichlet's Theorem will always be true, and so we will generally assume that any periodic function we are interested in can be represented this way. If we have such a periodic function and we want to know what the coefficients a_n and b_n are, we can always find them according to the formulas

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t') \cos(n\lambda t') \, dt' \ ; \ b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t') \sin(n\lambda t') \, dt'. \tag{74}$$

A detailed derivation of these formulas, along with several examples, were given by Michael in discussion section last week. His notes are also posted online, for those of you who need a refresher on the subject of Fourier series. You'll also have some homework problems on the subject.

As a side note, notice that by using Euler's formula, we can write our Fourier series as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\lambda t}, \tag{75}$$

where the coefficients c_n can be determined from a_n and b_n. In this form, it becomes clear that the Fourier transform representation of a function is simply the continuous version of the Fourier series,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\nu) e^{i\nu t} \, d\nu, \tag{76}$$

in the limit that $T \to \infty$. In this way, the Fourier transform generalizes the Fourier series to non-periodic functions. In both cases, the summation coefficients (either $\hat{f}(\nu)$ or c_n) tell us "how much" of the frequency ν "contributes" to the function $f(t)$. This is why the Fourier transform of a cosine or sine function is the delta function - by definition, there is only one specific frequency which contributes to the sine or cosine function.

With this key property of periodic functions, along with the linearity of our differential equation, we actually already know the solution to our equation for an arbitrary periodic driving force. We've already derived

$$y_{\cos}(t) = \frac{1}{\sqrt{\left(\omega^2 - \gamma^2\right)^2 + 4\beta^2\gamma^2}} \cos(\gamma t - \delta) \;\; ; \;\; \delta = \arctan\left(\frac{2\beta\gamma}{\omega^2 - \gamma^2}\right) \tag{77}$$

as the solution to

$$\mathcal{L}y = \cos(\gamma t). \tag{78}$$

A similar calculation reveals that

$$y_{\sin}(t) = \frac{1}{\sqrt{\left(\omega^2 - \gamma^2\right)^2 + 4\beta^2\gamma^2}} \sin(\gamma t - \delta) \;\; ; \;\; \delta = \arctan\left(\frac{2\beta\gamma}{\omega^2 - \gamma^2}\right) \tag{79}$$

is the solution to

$$\mathcal{L}y = \sin(\gamma t). \tag{80}$$

Lastly, the solution in the presence of a constant force c is given by

$$y_c(t) = c/\omega^2. \tag{81}$$

If we now consider an arbitrary periodic forcing function

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\lambda t) + b_n \sin(n\lambda t)\right], \tag{82}$$

then the solution to our differential equation is given by

$$y(t) = \frac{1}{2}\frac{a_0}{\omega^2} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{\left(\omega^2 - n^2\lambda^2\right)^2 + 4\beta^2 n^2\lambda^2}} \left[a_n \cos(n\lambda t - \delta_n) + b_n \sin(n\lambda t - \delta_n)\right], \tag{83}$$

where

$$\delta_n = \arctan\left(\frac{2\beta n\lambda}{\omega^2 - n^2\lambda^2}\right). \tag{84}$$

This expression can also be rewritten slightly as

$$y(t) = \frac{1}{2}\frac{a_0}{\omega^2} + \sum_{n=1}^{\infty} \frac{\sqrt{a_n^2 + b_n^2}}{\sqrt{\left(\omega^2 - n^2\lambda^2\right)^2 + 4\beta^2 n^2\lambda^2}} \cos\left(n\lambda t - \delta_n - \phi_n\right)\ ;\ \phi_n = \arctan\left(b_n/a_n\right). \tag{85}$$

While perhaps not looking very pretty, this expression is our result for the most general periodic forcing function, written in terms of the forcing function's Fourier coefficients. In general, we can add a solution to the homogeneous equation to this result, but as we saw before, this will simply result in transient behaviour, which eventually decays. Of course, for the $\beta = 0$ case, we must modify this equation slightly in the event that any of the frequencies $\gamma_n = n\lambda$ become resonant with the natural frequency ω, in which case there will be a component of our solution whose amplitude grows without bound.

As an application of this result, let's consider the sawtooth wave, shown in Figure 2. This function increases linearly from -1 to $+1$ between the times $t = -\pi$ and $t = +\pi$, and then repeats this pattern indefinitely. It therefore is a periodic function with $\lambda = 2\pi/T = 1$, and a short calculation reveals that its Fourier coefficients are given by

$$a_n = 0\ ;\ b_n = \frac{2}{n\pi}(-1)^{n+1}. \tag{86}$$

Therefore, our solution is given by

$$y(t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi\sqrt{\left(\omega^2 - n^2\right)^2 + 4\beta^2 n^2}} \sin\left(nt - \delta_n\right), \tag{87}$$

where

$$\delta_n = \arctan\left(\frac{2\beta n}{\omega^2 - n^2}\right). \tag{88}$$

A plot of this solution is shown in Figure 3. Notice the unusual shape - this certainly doesn't look like any type of function we would have easily guessed.

Green's Functions

At this point, we have, in principle, seen how to solve our differential equation for any forcing function which is either periodic, or has a well-defined Fourier Transform. In the process of doing so we've studied two special cases of forcing functions, sinusoidal forcing and constant forcing. However, there are some situations in which these techniques fail us, when our function is not periodic, and when the Fourier Transform is either ill-defined, or difficult to work with. In this case, we must resort to the method of **Green's functions**.

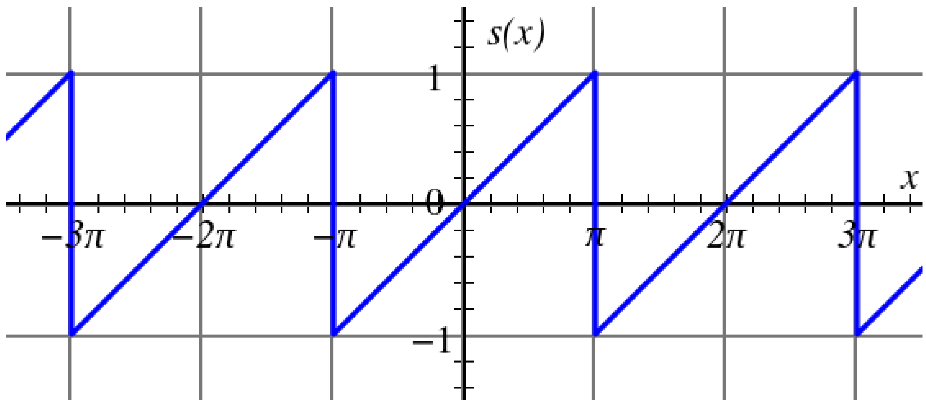

Figure 2: The sawtooth wave, an example of a periodic function with a Fourier series.

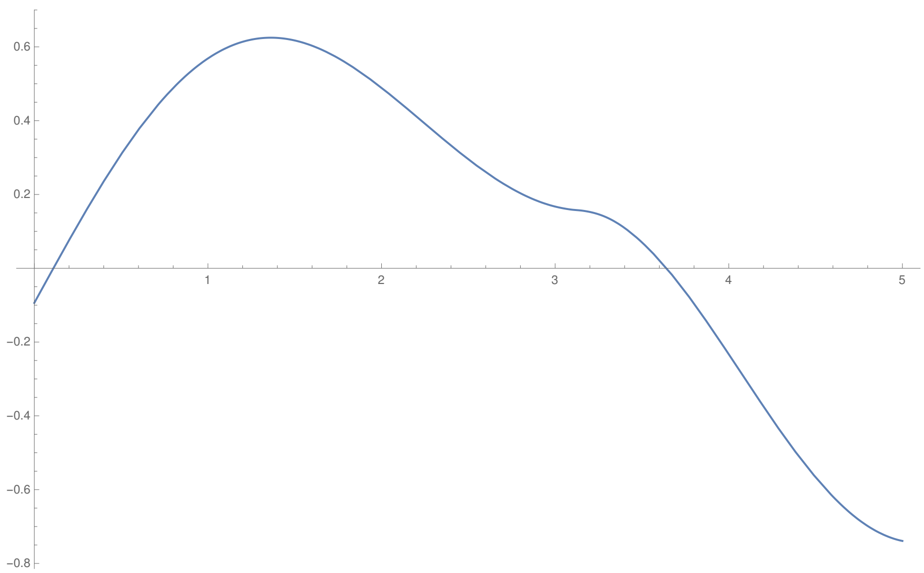

Figure 3: The solution to the differential equation with sawtooth forcing, for $\beta = 0.1$ and $\omega = \sqrt{2}$. Notice the non-trivial dependence on time. The plot was made by keeping the first 100 terms in the summation.

To motivate the method of Green's functions, let's consider our third and final special forcing function, the delta function, in which case our differential equation reads.

$$\mathcal{L}G\left(t;t_0\right) = \delta\left(t-t_0\right). \tag{89}$$

The solution to our differential equation in the presence of a delta function impulse is given a special name, and is referred to as the Green's function for the differential equation. The Green's function is the motion that results from hitting the oscillator very hard and very quickly at the time t_0 - for example, the motion of a spring after hitting it with a hammer. Before understanding how to actually solve for the Green's function, let's imagine that we somehow know what the Green's function is. What good does this do for us? My claim is that, in the presence of an absolutely arbitrary forcing function,

$$\mathcal{L}y\left(t\right)=f\left(t\right), \tag{90}$$

the solution to my equation is given by the integral expression

$$y\left(t\right)=\int_{-\infty}^{\infty}f\left(t'\right)G\left(t;t'\right)dt'. \tag{91}$$

To see that this is so, we can act the differential operator on both sides of the equation, so that

$$\mathcal{L}y\left(t\right)=\mathcal{L}\int_{-\infty}^{\infty}f\left(t'\right)G\left(t;t'\right)dt'. \tag{92}$$

Now, because the differential operator is linear, we can pull it inside of the integral, to find

$$\mathcal{L}y\left(t\right)=\int_{-\infty}^{\infty}\mathcal{L}\left[f\left(t'\right)G\left(t;t'\right)\right]dt'. \tag{93}$$

Because $\mathcal{L}$ is a differential operator with respect to t, and not t', then $f\left(t'\right)$ is simply a constant with respect to $\mathcal{L}$, and we find

$$\mathcal{L}y\left(t\right)=\int_{-\infty}^{\infty}f\left(t'\right)\mathcal{L}G\left(t;t'\right)dt'=\int_{-\infty}^{\infty}f\left(t'\right)\delta\left(t-t'\right)dt'=f\left(t\right), \tag{94}$$

which is precisely our differential equation. Thus, as long as we know the Green's function, then in principle we know the solution for an arbitrary forcing function.

In fact, you've already used Green's functions, although you may not have realized it. In electrostatics, the electric potential ϕ due to a configuration of charge ρ is given by Poisson's equation,

$$\nabla^2\phi=-\rho/\epsilon_0, \tag{95}$$

where ϵ_0 is the electric constant. To find the potential in practice, we typically perform the integral

$$\phi\left(\mathbf{r}\right)=\frac{1}{4\pi\epsilon_0}\int\frac{\rho\left(\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}d^3\mathbf{r}'. \tag{96}$$

Physically, we typically think of this expression as adding up all of the contributions from many different infinitesimal point charges. This is also what we are doing mathematically, since the Green's function for Poisson's equation is given by none other than

$$\nabla^2G\left(\mathbf{r},\mathbf{r}'\right)=-\delta^3\left(\mathbf{r}-\mathbf{r}'\right)/\epsilon_0\;\Rightarrow\;G\left(\mathbf{r},\mathbf{r}'\right)=\frac{1}{4\pi\epsilon_0}\frac{1}{\left|\mathbf{r}-\mathbf{r}'\right|}. \tag{97}$$

Of course, in this case, our variable is a three-dimensional position, as opposed to time, but the basic principle is still the same - knowing the response to a point-source allows us to reconstruct the solution to an arbitrary source.

With that said, we still need to know how to actually solve for the Green's function. To do this, we will make use of our old result

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\hat{f}(\nu)}{(-\nu^2 + 2\beta\nu i + \omega^2)} e^{i\nu t} \, d\nu. \tag{98}$$

In this case, the Fourier transform of the forcing function is

$$\hat{f}(\nu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(t - t_0) e^{-i\nu t} \, dt = \frac{e^{-i\nu t_0}}{\sqrt{2\pi}}. \tag{99}$$

Thus, our Green's function is given by

$$G(t; t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\nu(t-t_0)}}{(-\nu^2 + 2\beta\nu i + \omega^2)} \, d\nu. \tag{100}$$

Computing this integral is easy to do if we know how to use the residue method from the theory of complex analysis. Some of you may have already seen this in your math methods course. However, we will not assume that this is the case, and so I will simply quote the result of the integral, which is

$$G(t; t_0) = \frac{1}{2\Omega i} \left[e^{\alpha_+(t-t_0)} - e^{\alpha_-(t-t_0)} \right] \Theta(t - t_0), \tag{101}$$

where

$$\alpha_\pm = -\beta \pm \sqrt{\beta^2 - \omega^2} \; ; \; \Omega = \sqrt{\omega^2 - \beta^2}, \tag{102}$$

and Θ is the Heaviside Step Function,

$$\Theta(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} \tag{103}$$

On the homework, you'll check that this indeed satisfies the differential equation for the Green's function. Notice that taking a derivative of the step function yields a delta function, and so it's not surprising that the step function should appear as part of our answer. We can also rewrite our Green's function as

$$G(t; t_0) = \frac{e^{-\beta(t-t_0)}}{2\Omega i} \left[e^{+i\Omega(t-t_0)} - e^{-i\Omega(t-t_0)} \right] \Theta(t - t_0). \tag{104}$$

The solution to our equation for an arbitrary forcing function is thus

$$y(t) = \frac{1}{2\Omega i} \int_{-\infty}^{\infty} f(t') e^{-\beta(t-t')} \left[e^{+i\Omega(t-t')} - e^{-i\Omega(t-t')} \right] \Theta(t - t') \, dt'. \tag{105}$$

However, notice that because of the step function, only values of $t' < t$ will result in a non-zero integrand, and so we can rewrite this expression as

$$y(t) = \frac{1}{2\Omega i} \int_{-\infty}^{t} f(t') e^{-\beta(t-t')} \left[e^{+i\Omega(t-t')} - e^{-i\Omega(t-t')} \right] dt'. \tag{106}$$

I can interpret this expression as saying that at a given time, the motion of my oscillator is found by integrating up all of the forcing contributions from previous times, weighted by a factor which tells me how those contributions propagate over time. In particular, this result tells us that the value of our solution $y(t)$ at a given time t depends **only** on contributions from the forcing function that occur at times $t' < t$ - in other words, our system respects **causality**. Our system cannot have a behaviour at a given time t which involves forcing behaviour that has not happened yet, which is of course a very good behaviour for a physically realistic system to have. Also, notice that the presence of the decaying exponential means that forcing behaviour that occurred a long time ago does not have a large impact on the behaviour of the spring at the present moment - the effects of the forcing function have a tendency to dissipate away over time, as we might expect.

As an application of the Green's function technique, let's consider the case of week damping, along with a forcing function

$$f(t) = c\Theta(t). \tag{107}$$

That is, at time zero, I abruptly start applying a constant force to the spring. Physically, this is more realistic than my previous forcing functions, because we assume that any forcing function is probably "switched on" at some time, as opposed to being applied for all times infinitely far in the past. According to the Green's function method, I can write the particular solution in this case as

$$y_p(t) = c\int_{-\infty}^{\infty} \Theta(t')\, G(t;t')\, dt' = c\int_{-\infty}^{t} \Theta(t')\, \frac{e^{-\beta\left(t-t'\right)}}{\Omega} \sin\left[\Omega\left(t-t'\right)\right] dt', \tag{108}$$

where I've used Euler's formula to rewrite the Green's function slightly. Now, the presence of the step function means that the integrand will be zero for any $t' < 0$, and so we can write

$$y_p(t) = \frac{c}{\Omega}\int_{0}^{t} e^{-\beta\left(t-t'\right)} \sin\left[\Omega\left(t-t'\right)\right] dt'. \tag{109}$$

Performing this integration, we find

$$y_p(t) = \frac{c}{\omega^2} - \frac{c}{\Omega\omega^2} e^{-\beta t}\left[\Omega\cos(\Omega t) + \beta\sin(\Omega t)\right]. \tag{110}$$

This result tells me that the solution oscillates around the value c/ω^2, with the oscillations eventually decaying away exponentially. A plot of this behaviour is shown in Figure 4. Notice that the value c/ω^2 is exactly the same value we found in the case of constant forcing for all time - this tells us that if we suddenly apply a force to the oscillator, it will initially display some transient behaviour, before eventually settling down to the new equilibrium position. Interestingly, in this case, the transient behaviour of the particular solution takes the same form as the homogeneous equation - decaying oscillations with decay constant

β and frequency Ω. Therefore, the most general solution to the equation in this case is

$$y_p\left(t\right) = \frac{c}{\omega^2} - \frac{c}{\Omega\omega^2} e^{-\beta t}\left[\left(\Omega + A\right)\cos\left(\Omega t\right) + \left(\beta + B\right)\sin\left(\Omega t\right)\right]. \tag{111}$$

This tells us that any motion we can achieve by suddenly turning on the force at time zero can **also** be achieved with a force that has been acting for all time, with its initial conditions adjusted appropriately.

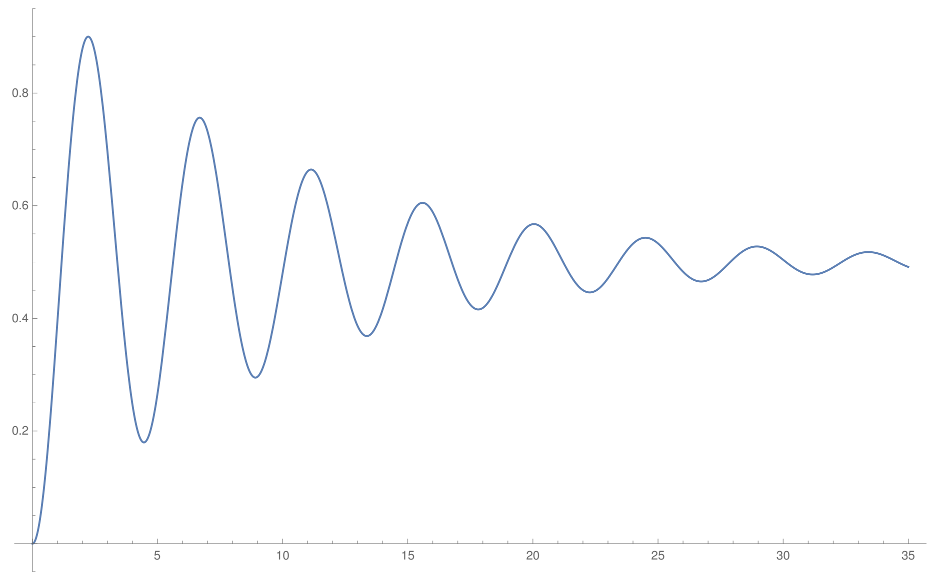

Figure 4: The particular solution to the differential equation with step function forcing, for $\beta = 0.1$, $c = 1$, and $\omega = \sqrt{2}$. Notice that the solution initially starts out at zero, and eventually decays to the value $1/2$, which is exactly the solution in the case of constant forcing for all time.

The real power of Green's functions comes into play when we have forcing functions that are not so simple. For example, instead of simply turning on a constant force at time zero, we could have ramped up the force more slowly, via a forcing function such as

$$f\left(t\right) = \frac{c}{2}\left(\tanh\left(\phi t\right) + 1\right). \tag{112}$$

This function is shown in Figure 5. While the force eventually reaches a steady value of 1, the amount of time it takes to do so is large compared with the natural frequency of the oscillator. The solution in this case is given by

$$y_p\left(t\right) = \frac{c}{2\Omega}\int_{-\infty}^{t}\left[\tanh\left(\phi t'\right) + 1\right] e^{-\beta\left(t - t'\right)}\sin\left[\Omega\left(t - t'\right)\right] dt'. \tag{113}$$

While we cannot do this integral in closed form, for any particular time t, it is easy to find the value of the function by numerically performing the integral with a calculator. The result of such a calculation is shown in Figure 6. Notice that while there are still transient oscillations which decay away to the steady value c/ω^2, the detailed nature of these oscillations is now significantly more complicated.

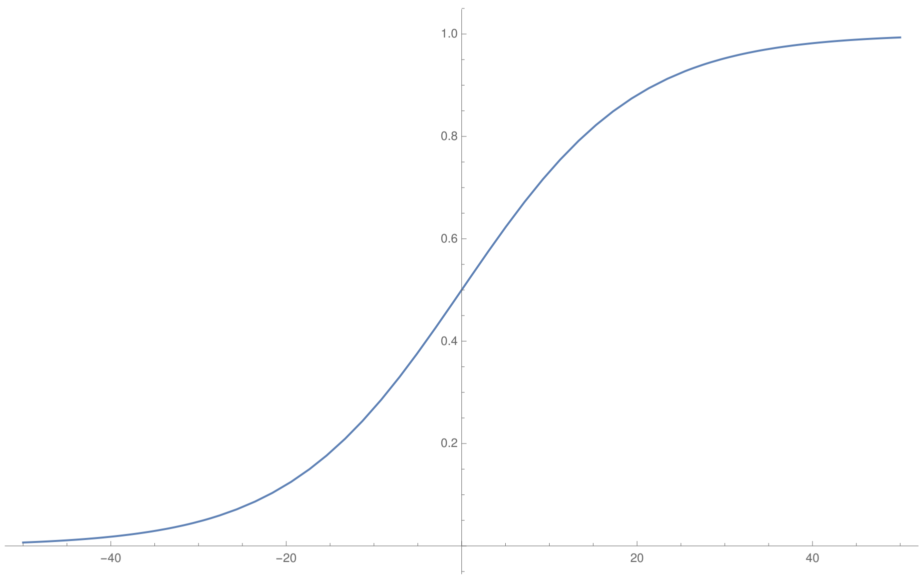

Figure 5: The forcing function $f(t) = \frac{c}{2}(\tanh(\phi t) + 1)$, for $c = 1$ and $\phi = 0.05$. Notice that while the force eventually reaches a steady value of 1, the amount of time it takes to do so is large compared with the natural frequency of the oscillator.

You may be wondering, if most forcing functions require the computation of a numerical integral, why didn't we just use a numerical algorithm to solve the differential equation from the beginning? Why didn't we just plug the original differential equation straight into Mathematica, and do away with this whole Green's function business? The reason for this is because computing an integral for a given time t is generally much easier for a computer to do, and involves significantly less numerical error, than numerically solving the differential equation from scratch. The rule of thumb in these situations is to **always** do as much work on pen and paper as you can before plugging things into a computer.

That concludes our study of the driven harmonic oscillator, and covers almost any case we could ever want to consider, when it comes to an oscillator described by a linear differential equation. On the homework, you'll spend some time learning how to apply all of these techniques to concrete examples. Tomorrow we'll spend a little bit of time talking about non-linear oscillators, before moving on to other subjects.

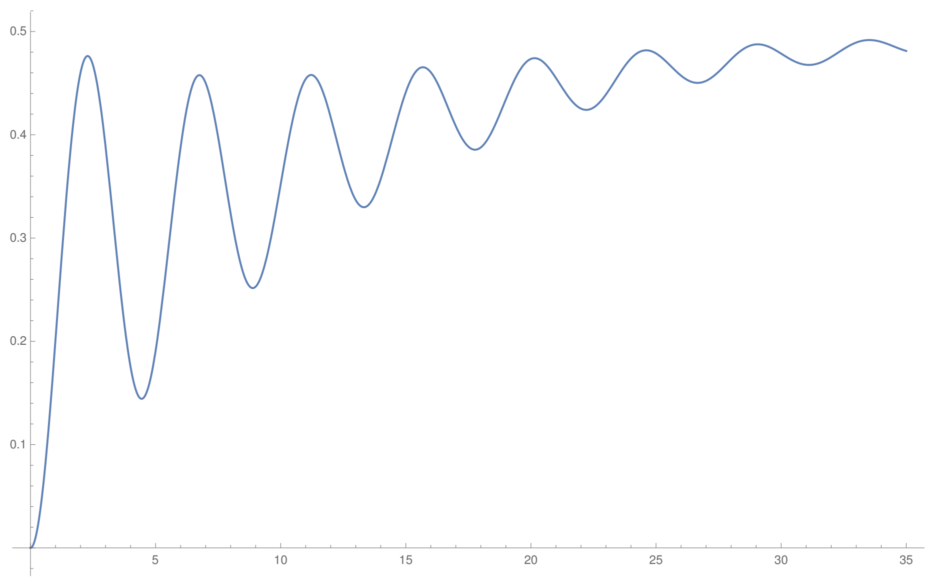

Figure 6: The particular solution to the differential equation with tanh forcing, for $\beta = 0.1$, $c = 1$, $\phi = 0.05$, and $\omega = \sqrt{2}$. Notice that the solution initially starts out at zero, eventually decays to the value $1/2$, but has complicated oscillatory behaviour in between.