Changing Coordinate Systems

In previous lectures, I've told you that when I do physics problems, it doesn't matter what choice of coordinate system I make. Today I'm going to explore this statement in a little more detail.

Let's imagine that I have two bodies in space, interacting gravitationally, shown in Figure 1. The force they experience depends on the distance between them, and their two masses - we'll have much more to say about Newton's law of gravitation in the coming lectures. I've included a choice of coordinates, and indicated the position vectors of each body.
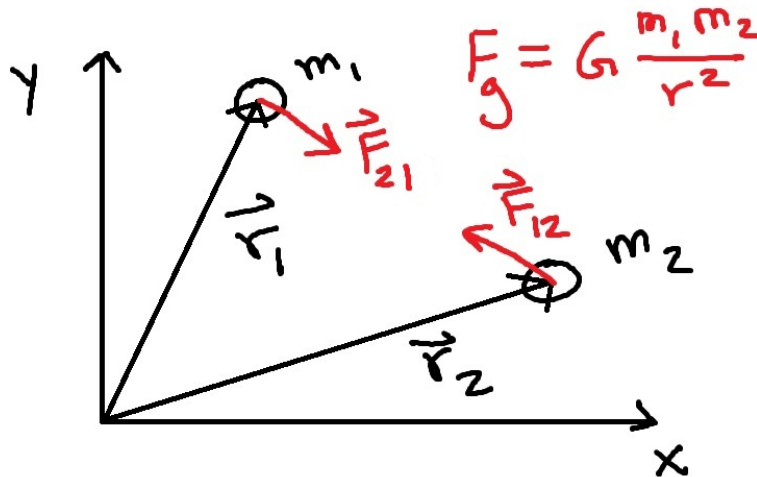

Figure 1: Two massive bodies will exert a gravitational force on each other, according to Newton's law of gravitation. We'll explore this law in more detail in coming lectures.

Now, what if I wanted to make another choice of coordinate system? I've indicated in Figure 2 the same physical system, but with another coordinate system chosen. This new coordinate system is drawn in blue, and I've labeled the coordinate axes with primes. Now, because the position vector of an object is defined with respect to the origin of some coordinate system, it becomes clear that with a new set of coordinates, the position vectors of my bodies will change. This is also shown.

Now, the question I want to ask is, how are the position vectors in the two coordinate systems related to each other? To answer this question, I've also drawn another vector, $\mathbf{d}$, which is the displacement vector from the origin of the first coordinate system to the origin of the second coordinate system. Using
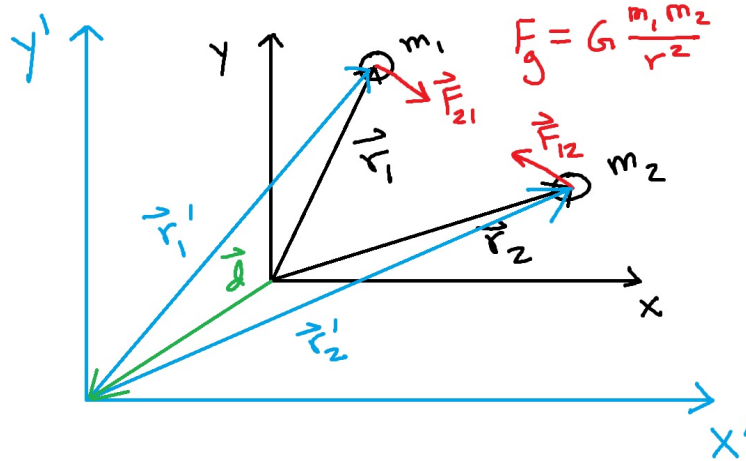
Figure 2: A second choice of coordinate system will, in general, lead to different position vectors.

the laws of vector addition, it becomes clear that

$$\mathbf{r}_1' = \mathbf{r}_1 - \mathbf{d}, \tag{1}$$

and likewise for the position of the second body. Alternatively, I could rearrange this to write

$$\mathbf{d} + \mathbf{r}_1' = \mathbf{r}_1. \tag{2}$$

This vector equation just says that if I start at the origin of the original coordinate system, move to the origin of the new coordinate system, and then move along the vector $\mathbf{r}_1'$, I'll end up at the location of the first body. This is what I should expect, since the net result of moving from the origin of the first coordinate system to the first body is described by the displacement vector $\mathbf{r}_1$.

Now, despite the fact that I can use two different coordinate systems to describe my physical problem, the forces experienced by the bodies should be the same regardless. *The forces are vector quantities which act on the bodies, and have an existence in their own right, independent of a choice of coordinate system.* This is made explicit by the formula for the force. It depends on the **physical distance** between the two bodies, and their masses, which are all quantities that have nothing to do with my choice of reference frame. So the forces are always the same, and so are the masses.

The fact that this is true means that for either of the bodies, the expression $\mathbf{F}/m$, or the force on that body divided by its mass, is always the same, no matter what choice of coordinates I pick, so that

$$\mathbf{F}/m = \mathbf{F}\,'/m. \tag{3}$$

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Now, I told you that the way that we actually determine the motion of a body is by using Newton's second law,

$$\mathbf{F}/m = \mathbf{a}. \tag{4}$$

If it is indeed true that I can use any choice of coordinate system to do a physics problem, then *Newton's law should be true no matter which coordinate system I choose.* If it is, and the quantity $\mathbf{F}/m$ is the same regardless of the choice of coordinates, then it better be true that the acceleration is also the same in both coordinate systems.

However, the acceleration is something which is defined in terms of the position vector. Since the position vector DOES depend on the choice of coordinate system, I had better be more careful in making sure that the acceleration vector is actually the same. In the original coordinate system, I have

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}. \tag{5}$$

Now, in the new coordinate system, I have

$$\mathbf{a}' = \frac{d^2\mathbf{r}'}{dt^2} = \frac{d^2}{dt^2}\left(\mathbf{r} - \mathbf{d}\right). \tag{6}$$

However, the vector $\mathbf{d}$ is just a constant displacement vector, and so its time derivative is zero. Therefore, we find

$$\mathbf{a}' = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}. \tag{7}$$

So everything checks out.

Galilean Relativity

But now let's imagine I do something a little different. Imagine I were to pick a new frame whose origin was not sitting still with respect to the old frame, but instead moving at a constant velocity with respect to the old frame. That is, imagine I had

$$\mathbf{d}\left(t\right) = \mathbf{d}_0 + t\mathbf{v}_{no}, \tag{8}$$

where $\mathbf{v}_{no}$ and $\mathbf{d}_0$ are some constant vectors. Notice that the vector $\mathbf{v}_{no}$ is *the velocity of the origin of the new frame with respect to the old frame.* We can see this because $\mathbf{d}$ is nothing other than the *position vector* of the origin of the new frame with respect to the old frame. If I therefore want to compute the velocity of the origin of the new frame, I need to compute the time derivative of $\mathbf{d}$,

$$\frac{d}{dt}\mathbf{d} = \frac{d}{dt}\left(\mathbf{d}_0 + \mathbf{v}_{no}t\right) = \mathbf{v}_{no}, \tag{9}$$

as claimed. Because this vector tells me the velocity of the origin of the new coordinate system with respect to the origin of the old coordinate system, we

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often just say that this is "the velocity of the new frame with respect to the old frame." The subscript "*no*" stands for "new with respect to old."

I now want to understand how this might change the physics of my problem. Again, so long as $t \neq 0$, the position vectors with respect to one frame will be different from the position vectors with respect to another frame. But, I can again show that the accelerations will stay the same, and so Newton's laws will still be valid. If I compute the acceleration in the new frame this time, I find

$$\mathbf{a}' = \frac{d^2\mathbf{r}'}{dt^2} = \frac{d^2}{dt^2}\left(\mathbf{r} - \mathbf{d}\right) = \frac{d^2}{dt^2}\left(\mathbf{r} - \mathbf{d}_0 - t\mathbf{v}_{no}\right). \tag{10}$$

However, the second derivative with respect to time will kill off the second and third terms, and so again

$$\mathbf{a}' = \frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}. \tag{11}$$

Thus, if Newton's laws hold in the first frame, they also hold in the new frame. This result is called the principle of *Galilean relativity*, and the change of coordinate system we have performed is called a *Galilean transformation*.

This result tells us that there is really no way to prefer one of these frames over the other. Newton's laws, which we believe to be the "laws of physics," hold the same way in both frames. So there is really no way we can say which one is the "correct frame," or which origin is the one sitting still.

However, while the accelerations are the same, the velocities will be different. We can see this by considering the velocity of body number one in the new frame. What we find is that

$$\mathbf{v}'_1 = \frac{d\mathbf{r}'_1}{dt} = \frac{d}{dt}\left(\mathbf{r}_1 - \mathbf{d}_0 - t\mathbf{v}_{no}\right) = \mathbf{v}_1 - \mathbf{v}_{no}. \tag{12}$$

We see that the velocity is shifted by the velocity of the new frame with respect to the old one. To make my notation a little more explicit, I can write this as

$$\mathbf{v}_{1n} = \mathbf{v}_{1o} - \mathbf{v}_{no}, \tag{13}$$

where the subscripts would be read as "body one with respect to new frame," "body one with respect to old frame," and "new frame with respect to old frame."

Now, velocity transformations can be a little annoying, because it's easy to accidentally get the subscripts backwards, or be dyslexic about which frame is which. There is a notational method I find very helpful to make sure you have the transformation correct. If I take the above equation and rearrange it, I have

$$\mathbf{v}_{1o} = \mathbf{v}_{1n} + \mathbf{v}_{no}. \tag{14}$$

Notice that on the right side, the letter "n" shows up twice, once on the outside of the two letters, and once on the inside. If I imagine these two copies of "n" "canceling", then they leave me with simply "1o,"

$$1n + no \rightarrow 1o. \tag{15}$$

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The result is the correct arrangement of subscripts on the left side.

I can also generalize this formula a little bit. Imagine that I have three objects or points in space, A, B, and C, which are moving with respect to each other. If I associate the origin of a coordinate system with two of them, and then consider the velocity of the third object, what I have is that

$$\mathbf{v}_{AC} = \mathbf{v}_{AB} + \mathbf{v}_{BC}. \tag{16}$$

This equation says that **the velocity of object A as observed from the location of C is the same as the vector sum of the velocity of object A with respect to the location of object B, plus the velocity of object B with respect to the location of object C**. I find this form of the velocity addition law to be the easiest one to remember, because I simply imagine "canceling out" the two Bs.

Another important property to remember is that for any two objects A and B,

$$\mathbf{v}_{AB} = -\mathbf{v}_{BA}. \tag{17}$$

Intuitively, this says that if I think you're moving to the left, you instead think that I'm moving to the right. I can verify this claim pretty easily from looking at Figure 2. If $\mathbf{d}$ is the position vector of the new frame with respect to the old frame, then certainly, $-\mathbf{d}$ must be the position vector of the old frame, as viewed by the new frame, since $-\mathbf{d}$ points in the *opposite* direction, from the origin of the new frame to the origin of the old frame. When we take a time derivative to get the velocity, the minus sign carries through.

These ideas are actually probably familiar to you already from freshman mechanics, but to refresh our memory, let's study an example of these ideas. Imagine I'm standing on the Earth and I fire a gun horizontally, as I've drawn in Figure 3. I've also set up a coordinate system centered around myself, indicated in blue, with the bullet moving along the x direction, with velocity $\mathbf{v}_b$ with respect to my coordinates. For the sake of simplicity, I'll ignore the effects of gravity in this example (I'll assume that I'm only worried about time periods short enough that the bullet hasn't started to fall noticeably).

I've also shown a car which I imagine happens to be driving by me as I fire the gun. Let's say that with respect to me, the car isn't moving in the y direction, but it is moving along the x direction with some velocity $\mathbf{v}_c$ with respect to me. Now, I've also drawn some coordinates centered around the car, labeled in red. If the driver of the car measures the velocity of the bullet, then we can use the velocity transformation rule to see that they will find

$$\mathbf{v}'_b = \mathbf{v}_b - \mathbf{v}_c. \tag{18}$$

If we want to be really careful that this is right, we can use the letter B for bullet, C for car, and G for ground (where I'm standing). Then the above equation says

$$\mathbf{v}_{BC} = \mathbf{v}_{BG} - \mathbf{v}_{CG} = \mathbf{v}_{BG} + \mathbf{v}_{GC}, \tag{19}$$

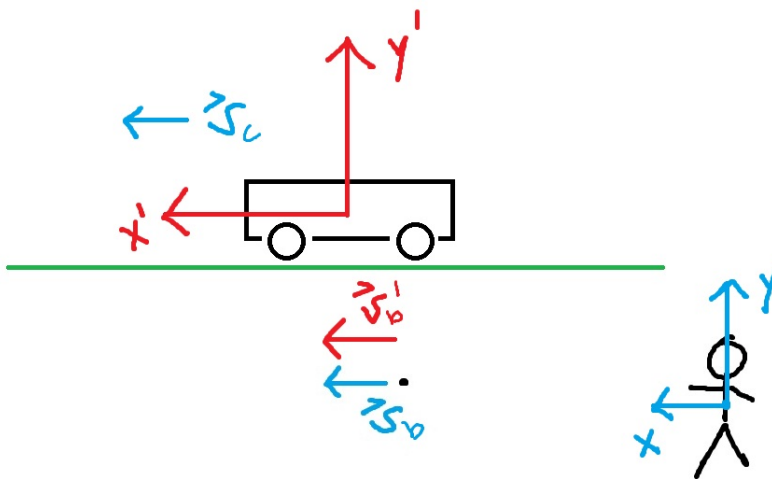which obeys the rule for the subscripts that I mentioned previously.

Figure 3: A demonstration of Galilean relativity, using the example of a bullet traveling through the air, as observed by the gunman and a driver passing by.

Now, if I notice that the car is passing by me at the same velocity that I fired the bullet, we have

$$\mathbf{v}_b' = \mathbf{v}_b - \mathbf{v}_b = 0, \tag{20}$$

which is to say that according to the driver of the car, the bullet is sitting still. Of course, this makes sense. If the car drives by me at the same velocity as the bullet, the driver of the car should see the bullet *travelling next to him*, sitting still with respect to his coordinates.

Now, it is tempting to say that I am the one who is "really sitting still," and that the car is actually moving. This is because I am on the ground, and the surface of the Earth is familiar to us, for obvious reasons. However, we know in reality that the Earth is actually moving through space, and doesn't really represent any sort of special location. In fact, there is really no difference between the car's choice of coordinates and mine. The driver of the car and I would both say that the bullet is moving at constant velocity, and so it is not accelerating (at least not in the x direction). We also both agree that aside from gravity (which would only affect the y component anyways), there is no force acting on the bullet (at least not in the x direction), since it is in free fall and not in contact with anything. Thus, we both agree that the force and the acceleration are zero, and so we both agree that Newton's second law is obeyed.

Inertial Reference Frames

Lastly, what if I consider two reference frames whose relative velocities are not constant? Let's go back to our example in Figure 1, and imagine that $\mathbf{d}$ is now

given by

$$\mathbf{d} = \mathbf{d}_0 + \mathbf{v}_{no}t + \frac{1}{2}\mathbf{g}t^2. \tag{21}$$

That is to say, the second frame is moving with respect to the first in a way that is quadratic in time, not linear - it is accelerating. If we now go to compute the acceleration of the first body, according to the new frame, we find

$$\mathbf{a}' = \frac{d^2\mathbf{r}'}{dt^2} = \frac{d^2}{dt^2}\left(\mathbf{r} - \mathbf{d}_0 - \mathbf{v}_{no}t - \frac{1}{2}\mathbf{g}t^2\right) = \mathbf{a} - \mathbf{g}, \tag{22}$$

and so the acceleration is NOT the same. However, we know that the forces **are** still the same in both frames. Therefore, *it cannot be true that Newton's second law is the same in both frames*. If it is true in one, then it cannot be true in the other.

The same idea can be examined in the case of the car and the bullet. If the car were accelerating with respect to me, then to the driver, *it would appear that the bullet had a net acceleration backwards, despite there not being any forces acting on it in the x direction.*

Assuming that Newton's law is correctly obeyed in the first frame, this frame is know as an *inertial reference frame*. The definition of an inertial reference frame is one in which Newton's laws are obeyed. We have seen that if I have an inertial frame, any other frame which is moving at *constant velocity* with respect to this frame will also obey Newton's laws, and so it is also an inertial reference frame. By starting with one inertial frame, we can find all other inertial frames by considering all of the frames moving with respect to it at constant velocity. However, any frames which are accelerating with respect to this frame will not obey Newton's laws, and we call them non-inertial reference frames.

When I first learned this fact, it seemed somewhat unsatisfying to me. The result we've found is that while the universe doesn't seem to "know" what speed you're moving at, it does seem to be able to make a distinction between which frames are accelerating, and which are not. Why should the first derivative of position be totally unimportant, while the second derivative is absolutely crucial?

An even more unsettling example involves what happens if you stand with your arms outstretched and spin around. Experience tells us that if I stand still, I feel nothing special. But if I start spinning around in a circle, I feel a tension in my arms. This is because my hands are now moving in a circle, and so there is a force required to keep them accelerating, which I feel in my arms. By the way, this effect has nothing to do with whether or not you are standing on the Earth - I could travel far out into intergalactic space, far away from any other bodies, and this would still be the case. But this simple fact actually reveals something profound about the way the universe works: there is a way to distinguish between a rotating and a non-rotating frame. I can *know* if I am "really" spinning or not. How can it be that if I go out into empty space, there's no way to tell what velocity I'm moving at, not even in principle, yet I *can* tell for sure whether or not I am really accelerating or rotating?

If we don't like these ideas, there are two possible ways we can think about getting around them. One idea might be to generalize Newton's laws. Maybe there is a more general law, which takes into account all possible types of frames, and the form of this equation always stays true, no matter what. The other idea is that perhaps if empty space seems to care about acceleration and rotation, perhaps it is actually not so empty as I thought. But it would have to be filled with something pretty weird, something that cares about acceleration, but not velocity.

It turns out that actually both of these ideas can be combined together to help us find a resolution to this problem, but it involves making very radical changes to the way we think about the universe. The resulting theory of physics is known as General Relativity, and I'll talk about it briefly at the very end of the course. But, for the time being, I want to show you a useful tool about accelerated frames which will not only help you do physics problems, but is actually, in disguise, the first step towards solving this puzzle.

The Equivalence Principle

Let's imagine a situation where I am in a very simple looking rocket traveling through empty space, shown in Figure 4. I've drawn a set of coordinate axes, which I know to be an inertial frame. In this frame, I happen to know that the rocket is about to start accelerating upwards, with an acceleration which we will call **g**. I'm standing in the rocket, and I'm holding a ball. I've aligned the axes so that at time zero, the floor of the rocket is at $y = 0$, and the ball is at $y = h$. The rocket starts accelerating right at time zero. This type of reference frame, aligned with an accelerating object in such a way that it is momentarily at rest with respect to it, is often called an *instantaneous rest frame* for that object at that time.

Now, at time zero, I let go of the ball, and I ask what happens. Because the ball is now in empty space with no forces acting on it, its acceleration will be zero. Because it initially had zero velocity, it will continue to not move, and so the ball will stay at $y = h$. The floor of the rocket, however, will accelerate from zero, and so the y coordinate of the floor will be given by

$$y_f = \frac{1}{2}gt^2.$$ (23)

Eventually, the floor of the rocket will move upwards far enough to hit the ball (which is sitting still). This time occurs when

$$y_f = h \Rightarrow t = \sqrt{\frac{2h}{g}}.$$ (24)

In general, before the floor hits, the distance between the ball and the floor will be
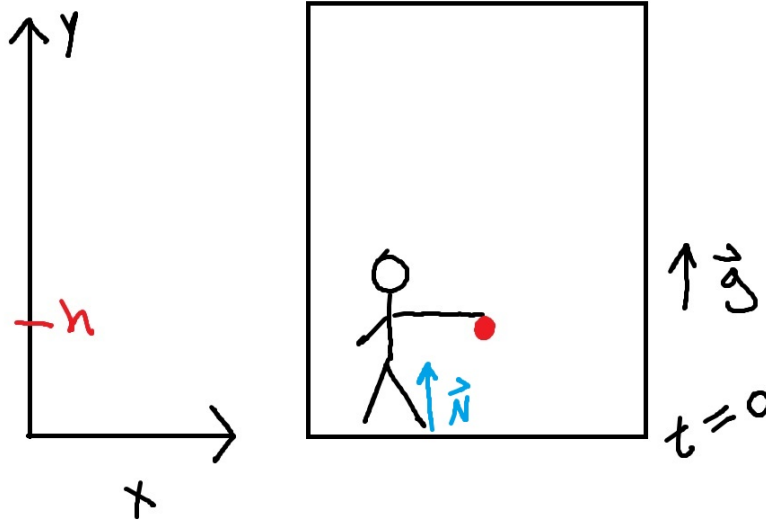
$$d = y_b - y_f = h - \frac{1}{2}gt^2.$$ (25)

Figure 4: Dropping a ball in a rocket... IN SPACE!

However, we know this equation looks very familiar. This is exactly the same expression for the height above the ground when I drop a ball on Earth. In fact, let's pretend that I don't know I'm in a rocket accelerating, and imagine my perspective from inside the rocket. As far as I know, I let go of a ball, and the distance between the ball decreased according to the above equation. Of course I'm now convinced I must be standing on the Earth, because clearly this ball is falling under the influence of a uniform gravitational field!

Not only would the motion of the ball look the same, but the sensation I would feel while standing would also be the same. If I were standing in a laboratory on Earth, I would feel a contact force with the floor. I would say that because I am not accelerating, there must be a normal force that the ground exerts on me to compensate gravity, whose magnitude is mg, where m is my mass. But someone watching me accelerate in the rocket would say I have it all wrong - I AM accelerating, at a magnitude of g. The rocket is providing a normal force to *cause* that acceleration, and it is the ONLY force acting on me. This is also shown in the figure.

This is a nice result, and you'll see when doing the homework that it can be incredibly useful in solving some problems. *I can always take an accelerating reference frame, and pretend that I am not actually accelerating, but instead I am subject to a gravitational field.* I can also do the reverse, and remove a gravitational field by assuming that I am in a laboratory in empty space, and that laboratory is accelerating.

Now, this certainly all makes sense in terms of the Newtonian mechanics

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we've been studying, although you might wonder if it's just a nice mathematical trick. Of course, I could also try seeing what happens to other objects in the rocket. I could shine a laser pen around inside, and the light beam would move along some path. If I were to go out on a limb and try to extend my previous ideas, my conclusion would be that the behavior of light in an accelerating reference frame is the same as its behavior in a gravitational field. In Figure 5, we've shown what it might look like if our rocket had a source of light on one side, and a beam of light passed across the length of the rocket. Now, the path of the light beam would appear to bend downwards, thus leading to the conclusion that light is affected by a gravitational field. But of course this is a silly idea, and it seems as though our astronaut in the rocket now has a way to determine, once and for all, that he is actually accelerating, and NOT in a uniform gravitational field.
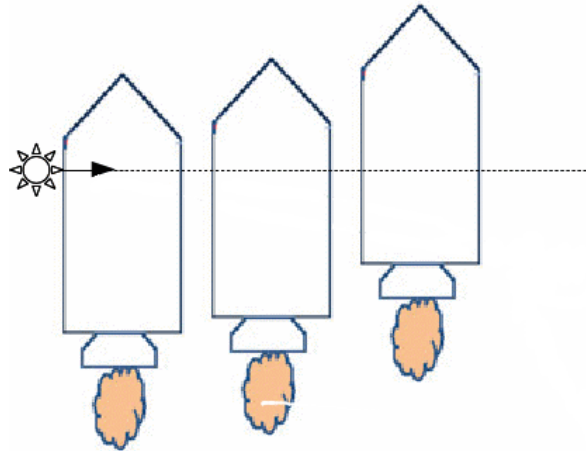
Figure 5: A light beam passing through a rocket.

Amazingly, it turns out that this equivalence of uniform gravity and acceleration is, in fact, always true. Believe it or not, light is affected by gravity - a light beam passing by a massive object will be deflected by it (although not by very much), and our astronaut cannot tell that he is accelerating. You might wonder how something like a massless wave of light could possibly be affected by gravity. There is a way to explain this phenomenon, but again, it requires a dramatically different way of thinking about the universe, which we'll discuss briefly at the end of the course.

Rotations

So far we've explored the result of shifting the origin of my coordinate system by some (possibly time-dependent) vector, and discussed the extent to which

Newton's laws are still valid. Of course, as we mentioned above, we can also ask about what happens to our system in a rotated reference frame - the use of a new coordinate system which differs from the old one by a rotation. In general, a rotation is described by a rotation operator $\mathcal{R}$, which acts on the basis vectors of my system in order to produce a new set of basis vectors, such that

$$\hat{x}' = \mathcal{R}\hat{x} \ ; \ \hat{y}' = \mathcal{R}\hat{y} \ ; \ \hat{z}' = \mathcal{R}\hat{z}. \tag{26}$$

This is shown in Figure 6. For example, we might have

$$\hat{x}' = \frac{1}{\sqrt{2}} \left(\hat{x} + \hat{y} \right) \ ; \ \hat{y}' = \frac{1}{\sqrt{2}} \left(-\hat{x} + \hat{y} \right) \ ; \ \hat{z}' = \hat{z}. \tag{27}$$

This would correspond to a rotation of the basis vectors around the z-axis by an angle of 45 degrees, which is shown in Figure 7. Notice something important about this transformation - the origin of my coordinate system is unchanged by the rotation. This means that the displacement vector of any point is the **same** in the two coordinate systems, unlike in the previous situation when we discussed translating the origin by a vector.

Of course, we know that the **components** of the vector will be different in the two systems. Before understanding how the two sets of coordinates are related to each other, it is convenient to adopt a slightly different notation for our basis vectors,

$$\hat{x} = \mathbf{e}_1 \ ; \ \hat{y} = \mathbf{e}_2 \ ; \ \hat{z} = \mathbf{e}_3. \tag{28}$$

With this notation, we can write the definition of our rotation more compactly as

$$\mathbf{e}_i' = \mathcal{R}\mathbf{e}_1 \ ; \ i = 1, 2, 3. \tag{29}$$

An arbitrary vector can then be written as

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 = \sum_{i=1}^{3} r_i\mathbf{e}_i = r_i\mathbf{e}_i, \tag{30}$$

where the last equality is an example of *Einstein summation notation* - whenever we see two repeated indices, we always assume that there is an implicit sum over them. While this may seem like a strange idea, it will turn out that almost all of the transformations we will study always involve summations where two instances of a given label are being summed over, and so in order to de-clutter our notation, we can simply drop the sum.

If we now also write the definition of our displacement vector in terms of the new coordinates, we have

$$\mathbf{r} = r_i'\mathbf{e}_i' \tag{31}$$

In order to find the new coordinates in terms of the old coordinates, we will make use of the fact that

$$r_i' = \mathbf{e}_i' \cdot \mathbf{r} = \mathbf{e}_i' \cdot (r_j\mathbf{e}_j) = r_j \left(\mathbf{e}_i' \cdot \mathbf{e}_j \right), \tag{32}$$
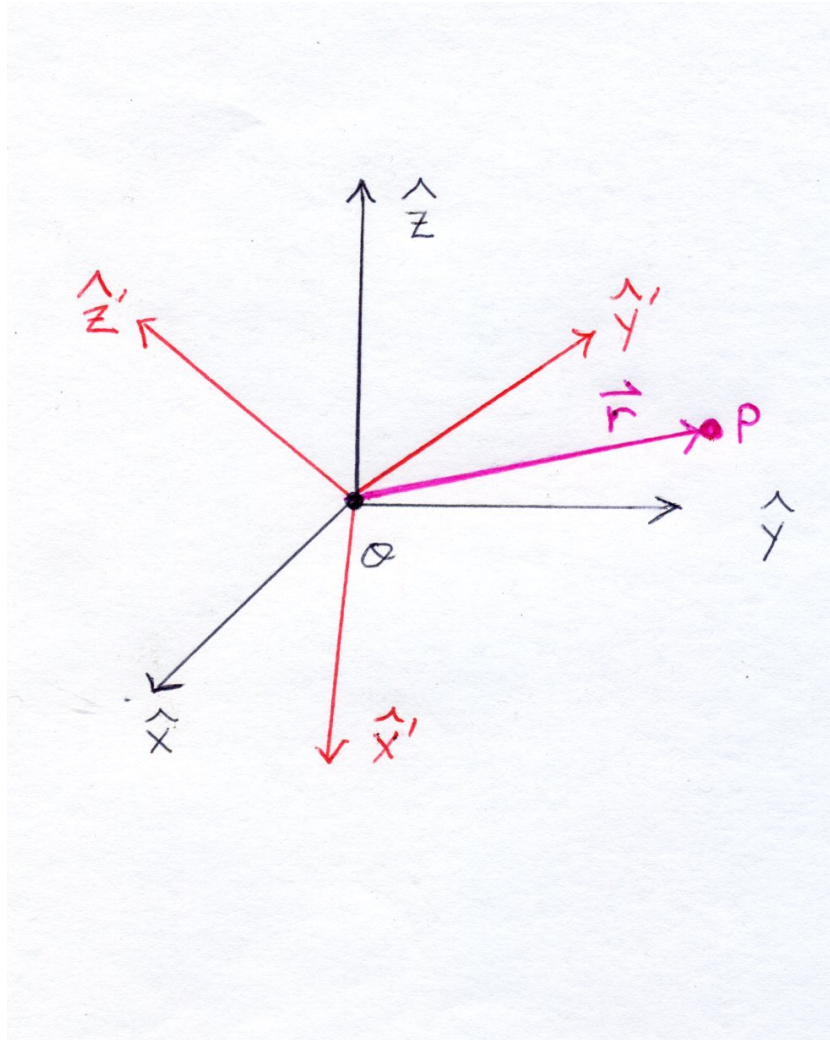
Figure 6: A rotated coordinate system.

where I have used another index j for the summation. With this result, we are motivated to define a **rotation matrix**, defined according to

$$R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j. \tag{33}$$

Our expression for the new coordinates in terms of the old can then be written more compactly as

$$r'_i = R_{ij} r_j. \tag{34}$$

Remember that the index j is being summed over! For example, the 45 degree rotation discussed above would correspond to a rotation matrix with the
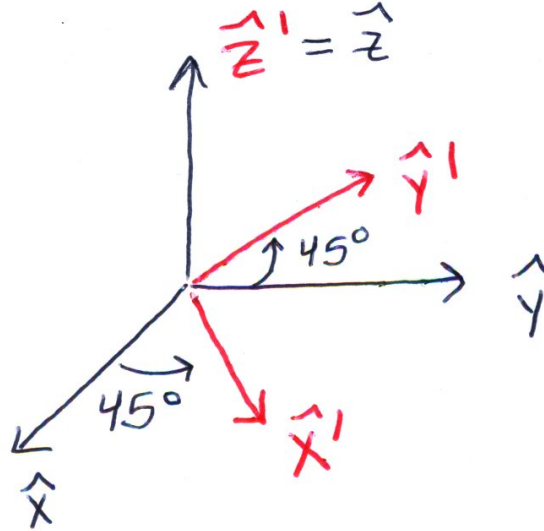
Figure 7: A new coordinate system found by rotating the old basis vectors around the z-axis by 45 degrees.

elements

$$R_{11} = R_{12} = R_{22} = \frac{1}{\sqrt{2}} \ ; \ R_{21} = -\frac{1}{\sqrt{2}} \ ; \ R_{33} = 1, \tag{35}$$

with all other entries being zero.

Before discussing how the physics of my system is affected by a rotation, we must notice that there is a slight restriction which we must place on the form that R_{ij} can take. We know that the inner product between any two vectors must be preserved by this change of coordinates. Therefore, it must be the case that

$$\mathbf{r} \cdot \mathbf{q} = r_i q_i = r_i' q_i' = (R_{ij} r_j)(R_{ik} q_k), \tag{36}$$

where we have used the definition of the rotation matrix. Now, remember from your linear algebra class that the definition of the **transpose** of a matrix is given by

$$\left(R^T\right)_{ji} = R_{ij}. \tag{37}$$

That is, we can find the transpose of a matrix by simply taking the original matrix and swapping rows and columns. Using this definition, we can rewrite our constraint as

$$r_i q_i = \left(R_{ji}^T R_{ik}\right) r_j q_k = \left(R^T R\right)_{jk} r_j q_k, \tag{38}$$

where in the second equality we've used the expression for matrix multiplication in terms of individual matrix elements. Now, in order for these two expression to be equal for two arbitrary vectors, we find that we must have

$$\left(R^T R\right)_{jk} = \delta_{jk}, \tag{39}$$

so that

$$r_i q_i = \delta_{jk} r_j q_k = r_j q_j. \tag{40}$$

The condition we have just derived above states that our rotation matrix must be an **orthogonal** matrix, which satisfies

$$R^T R = I \iff \left(R^T R\right)_{jk} = \delta_{jk}, \tag{41}$$

where I is the identity matrix. For any orthogonal matrix, it is easy to prove that

$$\det R = \pm 1, \tag{42}$$

or that the determinant of the matrix is plus or minus one. We often refer to this set of matrices as the group $O(3)$, which is the group of three by three orthogonal matrices (it is a group for reasons which you will learn if you take a course in group theory, which I highly recommend). However, there is actually one additional constraint that a rotation matrix must satisfy, which is not as obvious. Our additional constraint is that in fact the determinant must be equal to **positive** one in particular, and not negative one,

$$\det R = +1. \tag{43}$$

The reasons for this is that any rotation matrix that involves a smooth, continuous rotation cannot possibly have a determinant with negative one (you will learn how to prove such a thing in general if you take a course on group theory). This restricted set of matrices is sometimes known as the group $SO(3)$, or the group of special orthogonal transformations. The other matrices, those with a determinant of negative one, form the set of **parity transformations** - transformations which in some sense "invert" our coordinate system. An example of this is shown in Figure 8. There is no way to achieve this transformation through a continuous rotation of the coordinate axes. While we will not be focused on parity transformations here, they are important in physics nonetheless, especially in more advanced subjects such as quantum mechanics.

Additionally, we know that rotations must also preserve the form of the cross product. In order to see the restriction that this imposes on a rotation matrix, we will adopt a new notation for the cross product as well, and write

$$\mathbf{c} = \mathbf{r} \times \mathbf{q} \implies c_i = \varepsilon_{ijk} r_j q_k, \tag{44}$$

where the new term we have introduced is this **Levi-Civita** symbol,

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2) \text{ or } (2,1,3), \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases} \tag{45}$$
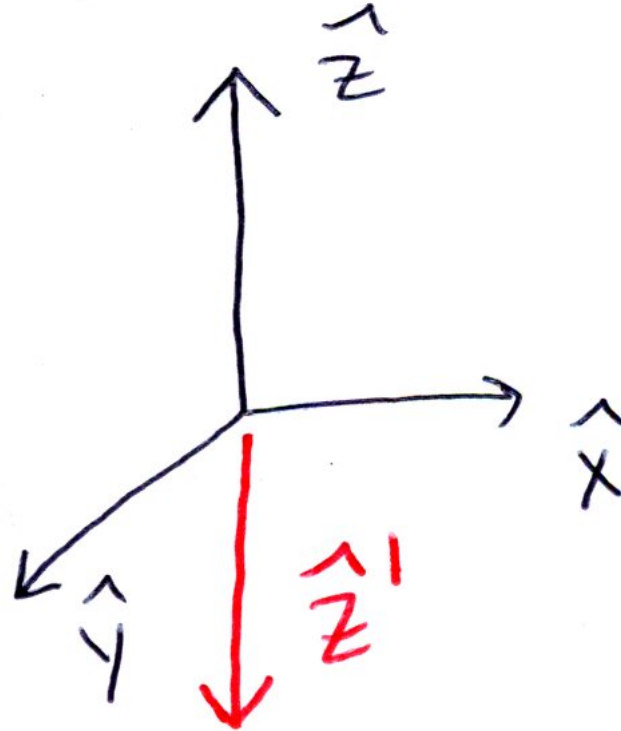
Figure 8: A parity transformation, which is a member of $O(3)$, but not $SO(3)$. This particular parity transformation corresponds to flipping the orientation of the z-axis.

It is also known as the permutation symbol, antisymmetric symbol, or alternating symbol. Some of these names reflect the important fact that the symbol is antisymmetric under the swap of any two indices. The expression for the cross product in another coordinate system will be

$$c_i' = R_{im}c_m = R_{im}\left(\varepsilon_{mjk}r_j q_k\right). \tag{46}$$

If we write the old coordinates in terms of the new ones via the **inverse** matrix,

$$r_j = R_{jn}^{-1}r_n', \tag{47}$$

then we find

$$c_i' = R_{im}\varepsilon_{mjk}R_{jn}^{-1}R_{kl}^{-1}r_n'q_l'. \tag{48}$$

Now, in order for the expression for the cross product to be the same in the new set of coordinates, we **want** to have

$$c_i' = \varepsilon_{inl}r_n'q_l', \tag{49}$$

since this new coordinate system should be just as valid as any other. This implies that in order for this relation to hold true for an arbitrary set of vectors, we must have

$$R_{im}\varepsilon_{mjk}R_{jn}^{-1}R_{kl}^{-1} = \varepsilon_{inl}. \tag{50}$$

Because we found that the transpose of a rotation matrix is the same as its inverse, we can also write this as

$$R_{im}\varepsilon_{mjk}R_{jn}^{T}R_{kl}^{T} = \varepsilon_{inl}, \tag{51}$$

or,

$$R_{im}R_{nj}R_{lk}\varepsilon_{mjk} = \varepsilon_{inl}. \tag{52}$$

The above condition we have derived tells us that the Levi-Civita symbol is an **invariant** symbol of the group $SO\,(3)$. This is a fancy way of saying that if we take the Levi-Civita symbol and act one rotation matrix on it for each index that it has, we end up with the Levi-Civita symbol again. In fact, our previous constraint that the rotation matrix preserve the inner product can be written as

$$R_{im}R_{nj}\delta_{mj} = \delta_{in}, \tag{53}$$

which tells us that the Kronecker delta is also an invariant symbol. In fact, these are the only two symbols which are "invariant" under the action of a rotation. For this reason, it turns out that any type of mathematical object which is invariant under a rotation can be written as an expression that somehow involves combining vectors together using the δ and ε symbols. In a deep sense, the fact that rotations leave these symbols invariant is the reason why the inner product and dot product are used so frequently in physics. We believe that the laws of our universe should not be altered depending on the orientation at which we view it, or that it possesses **rotational invariance**, and so it is natural that when we write down the laws of physics, we should do so in terms of things which do not change depending on the particular choice of coordinates we adopt. The dot product and cross product are in some sense the two most "basic" expressions that we can write down with this idea in mind, since their invariance corresponds to the above constraints we have derived involving rotation matrices. In more advanced physics courses, especially when you take a course in particle physics, you will learn more generally how a understanding of all of the symmetries of a system can allow you to determine the most "natural" mathematical objects with which to write down a physical theory.

However, for the time being, let us ask whether or not the use of a rotated coordinate basis affects the physics of my system in any way (although I suspect

that it should not). Let me imagine that I have an individual making observations and measurements using the old coordinate system, and another observer doing the same with the new system. I will assume that the observer who is at rest, at a fixed orientation with respect to the old coordinate system, is an inertial observer. Is the new observer also an inertial one? We can see that so long as the rotation matrix does not depend on time, then indeed, the new observer is an inertial one. In the old coordinates, we have

$$F_i \mathbf{e}_i = ma_i \mathbf{e}_i = m\ddot{r}_i e_i. \tag{54}$$

To find the acceleration in the coordinates of the new observer, we have

$$a_i' = \frac{d^2}{dt^2} r_i' = \frac{d^2}{dt^2} R_{ij} r_j = R_{ij} \frac{d^2}{dt^2} r_j = R_{ij} a_j. \tag{55}$$

Since the coordinates of the force should also transform according to the rotation matrix, we have

$$a_i' = R_{ij} F_j / m = F_j' / m \tag{56}$$

and so if we have an observer who has set up a coordinate system for himself which is at a fixed orientation with respect to the old frame, then his observations will indeed be inertial, as they satisfy Newton's laws.

However, if the matrix elements of the rotation are time-dependent, we find

$$a_j' = \frac{d^2}{dt^2} \left(R_{ij} r_j \right) = \ddot{R}_{ij} r_j + 2\dot{R}_{ij} v_j + R_{ij} a_j, \tag{57}$$

or,

$$a_j' = F_j' / m + \ddot{R}_{ij} r_j + 2\dot{R}_{ij} v_j \tag{58}$$

Thus, even in the absence of a force, we may find a non-zero acceleration with respect to the new basis vectors. Physically, this corresponds to the fact that if I am spinning around and describing the motion of objects with respect to a coordinate system that is moving along with me, I will observe otherwise stationary objects to be moving with respect to the coordinate systems. Make sure to keep in mind that **while the acceleration vector itself does not change, its components do change, and may become non-zero with respect to the rotating frame, even when they are zero with respect to the stationary frame**.

While using such a rotating frame of reference can be useful in some situations, the tools for doing so are more appropriately discussed next quarter, in Physics 104. For our purposes here, however, we will keep in mind that Newton's laws hold only with respect to an inertial set of coordinates, and a reference frame which is rotating with respect to an inertial one is **not** another inertial frame. We can, however, make use of a new reference frame which has been rotated by a static amount, which is a fact that we have already been making use of whenever we choose the most conveniently oriented frame of reference for a given problem.

Make sure to notice that our convention here has been to adopt a **passive** definition for rotations - we assume that the basis vectors have been changed, while the system is left in place. Other authors make use of active transformations, in which the object itself has been rotated, and the basis vectors describing the location of the object remain the same. While either of these approaches is valid, it is important to make sure you know which convention a given author is using!