Accelerated Observers

In the last few lectures, we've been discussing the implications that the postulates of special relativity have on the physics of our universe. We've seen how to compute proper times and change velocities, and also say something about how energy and momentum behave. However, so far, whenever we have done a calculation, we have used an inertial set of coordinates. We've argued that such observers are the only ones that are justified in using the results of special relativity.

However, while it is true that the postulates of special relativity, as we have been discussing them, only hold in an inertial frame, we know that in some situations, it is occasionally useful to work in an accelerated frame of reference. While we have not discussed this in our course in any sort of detail, an example of this might be to use the rotating frame of reference of the Earth, in order to understand the Coriolis effect. We might naturally ask whether there is a way we can perform the same calculations we have been performing in special relativity, except generalized to include the perspective of non-inertial observers.

Additionally, there is in fact another, not immediately obvious reason to ask this question. As we've discussed, maintaining causality in special relativity requires that no causal signal can travel faster than the speed of light. We've seen this by examining the barn and ladder paradox, and also by considering some more technical details of how light cones behave. As a result, whatever laws of physics we formulate had better respect this condition, if these laws of physics are to be consistent with special relativity. It turns out that Maxwell's equations, along with the Lorentz force acting on a particle,

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right), \tag{1}$$

are already consistent with the postulates of special relativity. While we won't prove this in denial, it certainly seems plausible - the behaviour of Maxwell's equations (in particular, the propagation of light) is what led us to the postulates of relativity in the first place. However, the interaction between two massive bodies according to Newton's Law of Gravitation,

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}, \tag{2}$$

is not consistent with the postulates of special relativity - the interaction between two bodies depends on the instantaneous distance between them. Thus, according to the Newtonian law of Gravitation, if I take a massive body and move it around, the force it exerts on a particle far away will instantly change. This certainly violates the speed limit we've set on the propagation of information. So, we need to find a way to modify our laws of Gravitation, although it's not immediately clear how we might do so.

However, as we discussed previously, the gravitational equivalence principle says that any observer in a uniform gravitational field can reach the same physical conclusions about the motion around him if he imagines that he is instead accelerating, and vice versa. We previously discussed this in the context of an observer out in space, accelerating inside of a rocket, as shown in Figure 1. Because of this close connection between gravitation and accelerated motion, we suspect that thinking about accelerated motion may help us understand how to generalize our theory of relativity to include gravity.

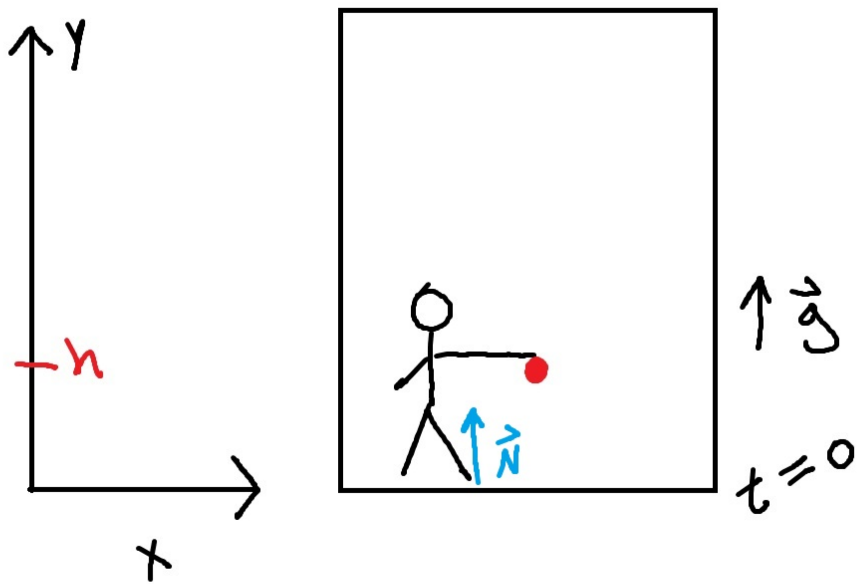

Figure 1: An accelerated observer out in empty space could come to the correct physical conclusions by assuming that he was not accelerating, but rather in a uniform gravitational field.

Rindler Observers

Let's consider again the space-time coordinates of an inertial observer, denoted as (T, X, Y, Z). We'll use capital letters to denote the coordinates of the inertial observer in this case. Let's now imagine that this observer witnesses a particle following the one-dimensional trajectory

$$X_p(T) = +c\sqrt{\frac{1}{\alpha^2} + T^2} \quad ; \quad Y = Z = 0. \tag{3}$$

The world-line of this particle is indicated in Figure 2. This trajectory corresponds to a particle which comes in from infinitely far away along the x-axis, comes to rest at a position $X = 1/\alpha$, and then travels back out to infinity. The velocity as a function of time is given by

$$V(T) = \frac{cT}{\sqrt{\frac{1}{\alpha^2} + T^2}}. \tag{4}$$

Notice that at large times,

$$V(T \to \infty) \to c. \tag{5}$$

This trajectory describes a particle which is accelerating towards, but not quite reaching, the speed of light. An observer following this particular trajectory is often known as a **Rindler observer**.

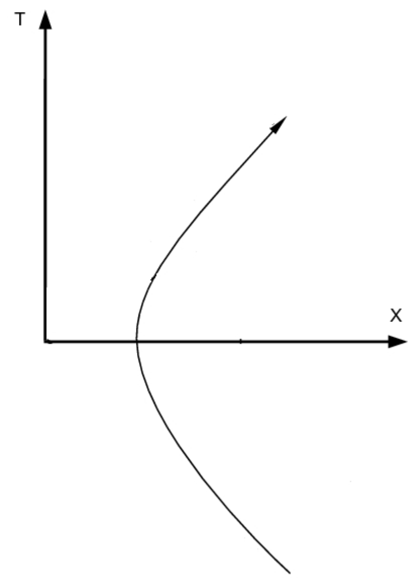

Figure 2: An accelerated particle (or observer) moving along a world-line.

We now want to understand how we can describe all of the physics of relativity not just from the perspective of the inertial observer, but also the perspective of the accelerated observer. To this end, let's start by finding the elapsed time, according to the accelerated observer. This is the natural "time coordinate" that he or she would use to describe events that they witness. From our time dilation formula, we know that for any infinitesimal time-step,

$$d\tau = \frac{dT}{\gamma} = \sqrt{1 - V^2(T^2)/c^2}\ dT = \frac{dT}{\sqrt{1 + \alpha^2 T^2}} \tag{6}$$

Note that we are now referring to the time coordinate of the accelerated observer as τ. If the observers synchronize their clocks so that $\tau = T = 0$ when the

Rindler observer comes to rest at $X = 1/\alpha$, then we can integrate the above expression to find

$$\tau = \int_0^\tau d\tau' = \int_0^T \frac{dT'}{\sqrt{1+\alpha^2 \left(T'\right)^2}} = \frac{1}{\alpha}\text{arcsinh}\left(\alpha T\right). \tag{7}$$

This tells us how much time elapses on the accelerated observer's clock, in terms of how much time has elapsed on the inertial observer's clock. Inverting this expression, we can write

$$T_p\left(\tau\right) = \frac{1}{\alpha}\sinh\left(\alpha\tau\right). \tag{8}$$

With this expression, we can also write the position of the body as a function of the proper time as

$$X_p\left(\tau\right) = \frac{1}{\alpha}\cosh\left(\alpha\tau\right). \tag{9}$$

Notice that we could think of τ as an "arc length parameter" along the particle's word line, with the above being parametric equations describing the particle's world-line through space-time.

Now, similar to when we derived the Lorentz transformations, we want to find an expression which relates the time and space coordinates of the inertial observer, T and X, to the time and space coordinates that the accelerated observer would use, which we will refer to as η and ξ. This transformation can be derived in a similar manner, although there are some subtleties which go beyond what we have time to discuss here. In essence, we need to imagine that at any momentum in time, we can perform a Lorentz transformation from the T and X coordinates to a new set of coordinates, which correspond to an inertial frame which instantaneously at rest with respect to the accelerated particle, right at that instance in time. There are some subtleties involves in making sure that the new time variable in this inertial frame agrees with the notion of time that the accelerated observer has. For now we will gloss over these details, and simply state the result, which is that

$$\eta = \frac{1}{\alpha}\text{arctanh}\left(\frac{T}{cX}\right) \quad ; \quad \xi = \sqrt{X^2 - c^2T^2} - \frac{c}{\alpha}. \tag{10}$$

This is certainly a more complicated looking expression than what we found for the usual Lorentz transformation, but to convince you that it is plausible, let's study an important special case. In the inertial coordinates, the trajectory of the particle was given by

$$X_p\left(T\right) = +c\sqrt{\frac{1}{\alpha^2} + T^2}, \tag{11}$$

which implies that

$$\frac{T}{cX} = \frac{\alpha T}{\sqrt{1+\alpha^2 T^2}} \quad ; \quad X^2 - c^2T^2 = \frac{c^2}{\alpha^2} \tag{12}$$

along the world-line of the particle. If we use this facts in our coordinate transformation rule, then according to the particle's own set of coordinates, its world-line is described by

$$\eta = \frac{1}{\alpha}\text{arctanh}\left(\frac{\alpha T}{\sqrt{1+\alpha^2 T^2}}\right) = \frac{1}{\alpha}\text{arcsinh}\left(\alpha T\right) = \tau, \tag{13}$$

and

$$\xi = \sqrt{\frac{c^2}{\alpha^2}} - \frac{c}{\alpha} = 0. \tag{14}$$

So in his own set of coordinates, the accelerated observer finds his own trajectory to be sitting still at $\xi = 0$, with a time η which is just the proper time he experiences, τ. This is certainly what we should find. Verifying that the transformation works for other points in space-time is a trickier matter, which we will not discuss here.

Notice that if we like, we can also invert these transformations, in order to find

$$cT = \left(\xi + \frac{c}{\alpha}\right)\sinh\left(\alpha\eta\right) \quad ; \quad X = \left(\xi + \frac{c}{\alpha}\right)\cosh\left(\alpha\eta\right). \tag{15}$$

The Metric

Now that we have found how to transform between the coordinates of the two observers, a natural question would be, how do we rewrite the "laws of physics" in terms of the coordinates of our new observer. Well, first, we need to decide what the laws of physics are. While for an arbitrary particle experiencing all sorts of forces and interactions, the equations of motion can be quite complicated, and we won't attempt a discussion of that here. But one very simple law we can address - we know that a particle which is free-floating, and experiences no forces, should move with constant velocity,

$$X = vT, \tag{16}$$

according to the coordinates of an inertial observer. Now, in the interest of formulating a space-time version of this statement, notice that on a space-time diagram, particles moving with constant velocity move along **straight lines**. We also know that between any two space-time events, the world-line connecting them which has the **longest** proper time is an inertial observer, following a straight line. We've seen a few examples of this, and did an explicit calculation last lecture, for the world-line shown in Figure 3.

With this idea in mind, we suspect that perhaps the geometric way of describing our law of physics is that "particles which feel no forces move along world lines in space-time which **maximize** the proper time." In other words, the world-line which a freely floating particle will travel between any two space-time points is the one with the longest space-time length. This is indeed a good way to formulate our law of physics, because as we have seen, the space-time length, ds^2, between any two events in space-time is an invariant quantity, so

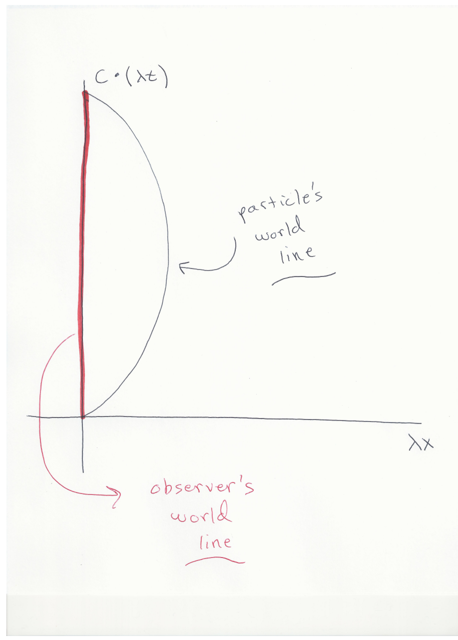

Figure 3: An accelerated particle (or observer) moving along a world-line.

describing the motion of particles in terms of the space-time length seems like a good way to generalize our laws of physics to all possible observers.

Now, for two inertial observers, we know that we can write the infinitesimal amount of space-time length as

$$ds^2 = -c^2 dT^2 + dX^2 = -c^2 \left(dT'\right)^2 + \left(dX'\right)^2, \tag{17}$$

which is to say that the expression takes the same form for any two observers. For our accelerated observer, however, we expect that the expression for ds^2 between any two events in space-time will look somewhat different, when written in terms of his or her coordinates. Finding this new expression, though, is not too difficult. In fact, we can use the chain rule to write the old infinitesimals in terms of the new ones. For example,

$$dT = \frac{dT}{d\eta} d\eta + \frac{dT}{d\xi} d\xi = \left(\frac{\alpha\xi}{c} + 1\right) \cosh\left(\alpha\eta\right) d\eta + \frac{1}{c} \sinh\left(\alpha\eta\right) d\xi. \tag{18}$$

We can also write

$$dX = \frac{dX}{d\eta} d\eta + \frac{dX}{d\xi} d\xi = \left(\alpha\xi + 1\right) \sinh\left(\alpha\eta\right) d\eta + \cosh\left(\alpha\eta\right) d\xi. \tag{19}$$

If we now substitute these into our expression for ds^2, we find, after some algebra

$$ds^2 = -\alpha^2 \xi^2 d\eta^2 + d\xi^2. \tag{20}$$

In terms of our new coordinates, this gives us an expression for the infinitesimal amount of length between two space-time events.

Notice that indeed, the form of the expression is somewhat different from the one for inertial observers. In particular, the coefficients which appear on the differentials are **not** constant. For example, the $d\eta^2$, which measures how much the space-time length changes as we move in the η direction, has a coefficient which depends on ξ. However, this is not too surprising, and we have seen something like this before. Remember that for regular two-dimensional geometry, the infinitesimal amount of length in the plane, in Cartesian coordinates, could be given by

$$dl^2 = dx^2 + dy^2. \tag{21}$$

A rotation of our coordinates, which would be a linear transformation between the two coordinate system, would result in the modified expression,

$$dl^2 = \left(dx'\right)^2 + \left(dy'\right)^2, \tag{22}$$

which has the same form as the first. Alternatively, however, if we were describing our plane in terms of polar coordinates, we know the expression would instead take the form

$$dl^2 = dr^2 + r^2 d\theta^2. \tag{23}$$

Thus, our transformation to the space and time coordinates of an accelerated observer is tantamount to performing a non-linear coordinate transformation, which results in a different form for the infinitesimal amount of space-time length.

Regardless of what form ds^2 takes, however, we will stick with our geometric formulation of the laws of physics, and say that, according to any observer, using whatever coordinates they like, a freely floating particle will travel on a path through space-time in such a way that the quantity

$$\tau = \int \sqrt{-ds^2} \tag{24}$$

is **maximized**. This is now a statement which makes sense in **any** set of coordinates. In this more general context, the quantity ds^2 is often known as the **metric** of space-time - it is the object which tells us how to measure distances in space-time, in terms of coordinates.

Now, if we actually want to perform calculations in our new coordinates, we need to know how to look at the expression for the metric, and figure out, for all possible curves through space-time (described in our new coordinates η and ξ), which curve maximizes the proper time? In the usual inertial coordinates, this was easy to find, since the result was just a straight line. In terms of our new expression, it is not so obvious what the answer to this question is. In general, answering this question requires a subject known as **differential geometry**, and in particular, a result known as the **geodesic equation**. These tools allows us to essentially solve for the motion of a "straight line," using whatever coordinates we like. While developing these tools goes beyond the scope of our course, it turns out that in our new coordinates, the resulting equation of motion for a free particle takes the form

$$\ddot{\eta} + \frac{2}{\xi}\dot{\xi}\dot{\eta} = 0 \quad ; \quad \ddot{\xi} + \xi\dot{\eta}^2 = 0. \tag{25}$$

Here, the dots refer to derivatives with respect to the proper time of the particle. Solving these equations results in a set of parametric equations for the world-line of the particle, described by $\eta(\tau)$ and $\xi(\tau)$. If we rearrange this to find $\xi(\eta)$, and then convert back to $X(T)$, we will indeed find an expression of the form $X = vT$, which simply described motion at constant velocity (in an inertial frame).

Curvature

Now that we've learned how to describe the motion of freely moving particles using different sets of (accelerated) coordinates, we want to return to the question of Gravity, and see how what we have learned so far might help us understand how to develop a relativistic theory of gravity. Now, because of the striking similarity between accelerated observers and gravitational fields which is encoded in the equivalence principle, we want to seek inspiration from the results we found for an accelerated observer. The general conclusion we came to is that *while the expression for the metric might change, freely floating observers are described by world lines that maximize proper time.* This was true for any set of coordinates we chose.

Now, at the end of the day however, the actual physical result we found was the same - using either coordinate system, the answer we ultimately found, after transforming back to the coordinates X and T, was that the particles follow some constant velocity trajectory. This certainly does not describe the types of motion that would occur in the presence of a gravitational field. So we need to understand how to generalize our result even further, in order to actually find a qualitatively different behaviour for the motion of free particles.

Taking another hint form the geometry of regular two-dimensional space, we saw that the "metric" of a plane could be written as

$$dl^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2. \tag{26}$$

The infinitesimal length is the same, while the expression in terms of coordinates changes. In fact, there are plenty of different coordinate transformations we could make, all of them resulting in slightly different expressions for the line element. However, what if I proposed a "new" metric, defined in terms of two new variables θ and ϕ, which I claimed was given by

$$dl^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2. \tag{27}$$

The question I want to ask now is, what coordinate transformation did I have to perform to get from Cartesian coordinates to some new set of coordinates, so that this the resulting expression for the metric.

The answer is that there is no such transformation, and the reason is because the line element above no longer describes a plane - this line element describes the infinitesimal amount of length on the surface of a **sphere** of radius R, which is shown in Figure 4. is. A sphere is a two-dimensional object, just like a flat plane, but it is fundamentally different, in that it is a curved surface. In the study of differential geometry, it is possible to give a more precise meaning to this statement, with a mathematical definition for the notion of curvature. We would say that the first metric, defined in either Cartesian or polar coordinates, has **no** curvature, while the second metric, no matter how we transform to another set of coordinates, will always possess a non-zero curvature.

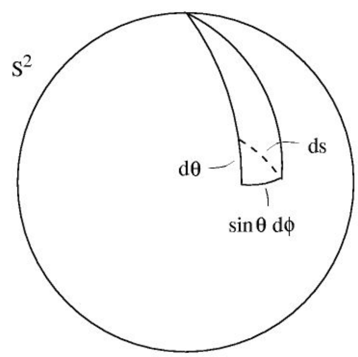

Figure 4: The line element on a sphere.

Now, without knowing all of the details of the mathematical theory of curvature, we still have some intuitive sense of what it means for a sphere to be curved, since we typically think of the sphere as curving "within" the three-dimensional space around it. However, it was one of the most profound discoveries in mathematics that there is a way to write the expression for the curvature of the sphere (and all other types of surfaces) in a way that only made reference to the two coordinates that live on the sphere, without any reference to the surrounding space, or how the surface is embedded in the space. Thus, *there is a mathematical sense in which a surface can "curve" without curving "inside" or anything else.*

The last key insight we need before making our generalization is one that Einstein had while formulating a relativistic theory of gravity, which allowed him to come to the theory of General Relativity. He noticed that a freely falling observer moving in the presence of a gravitational field, which is sufficiently uniform that there are no major tidal forces, should feel **no** forces as he falls through space. Because everything around him, and all of the parts of his body, accelerate uniformly, he will feel as though he is just freely floating through space. Thus, in General Relativity, we want to think of observers being affected by gravity as simply "floating through space."

Thus, taking all of these ideas together, here is the generalization we will make, in order to come upon a relativistic theory of gravity. We will now assume that instead of the usual metric of Minkowski space, it could be possible for our space-time to have a totally different metric, one which cannot be found from simply performing a coordinate transformation. This new metric could possess a non-zero curvature. Then, in the presence of this new metric, *the motion of observers moving freely under the influence of gravity will be curves though space-time with the longest possible proper time, or **geodesics***. This is indeed the fundamental idea underpinning General Relativity.

The Schwarzschild Solution

Of course, after having this key insight, it took Einstein (and many others) several years to fully put together all of the pieces of General Relativity. The reason is that while we have a broad notion of what General Relativity should be, we actually need a detailed theory of **how** space-time should actually curved. While the details of this go well beyond the scope of our course, roughly speaking, in General Relativity, the amount of curvature at a point in space-time is given by the **energy density**, or loosely speaking, the amount of matter at that point. Other particles then follow geodesics in space-time. The full set of equations which describe this are known as **Einstein's Equations**. They are in fact quite formidable to solve in most cases, and only a few exactly solvable cases exist.

To give an example of one of the few simple solutions, let us introduce the Schwarzschild metric, which is the metric

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \tag{28}$$

where

$$r_s = 2GM/c^2. \tag{29}$$

Here, G is the usual Newton's constant of gravitation, while c is again the speed of light. M can be thought of as a parameter of the metric. It turns out, however, that this metric describes the gravitational behaviour outside of a massive, spherical body, similar to the Earth for example. The total mass of this body is M.

Now, notice that for very large values of r, we have

$$ds^2 \approx -c^2 dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right). \tag{30}$$

This is just the line element in regular Minkowski space. So we see that this metric is describing some sort of four-dimensional space-time where the behaviour of the metric at very large distances is "flat" (it reduces to the Minkowski metric), but as r becomes smaller, it starts to develop a noticeable curvature. This certainly seems like the correct behaviour for the gravitational field of the Earth - close to the Earth, there is some noticeable gravitational attraction, while far away, the effects are negligible. Now, we could use these coordinates to set up an equation of motion for particles moving through this space-time. However, the details of this calculation would take much more time than we have today. However, it is the case that the shapes and behaviour of two-body orbits become much more interesting in General Relativity, so it is a well-studied phenomenon.

As one closing calculation, let's see how much time elapses for an observer which is "stationary" in the Schwarzschild Solution, which is to say an observer who is maintaining a constant r and angular value. This computation is straightforward, since we have

$$d\tau = \sqrt{-ds^2} = c\sqrt{\left(1 - \frac{r_s}{R}\right)}dt. \tag{31}$$

This result tells us how much proper time is experienced by a stationary observer, in terms of whatever dt is. But notice that when r is very large, our metric becomes approximately equal to that of Minkowski, at which point the coordinate t corresponds to the time coordinate of an inertial observer. So our interpretation of the quantity dt is the time measurements made by someone very far away from the Earth. The above expression then says that someone close to the Earth experiences **less time** than the person far away. This effect is known as **gravitational time dilation**. Notice again the relationship between gravitational effects and accelerated motion - while an accelerated observer measures less time than an inertial observer, like-wise, an observer sitting still on the surface of the Earth measures less time than an inertial observer far away at infinity.

While we have only barely touched upon the ideas of General Relativity, entire textbooks can (and have been written) on all of the amazing aspects of this. Before we come to a close, I will just mention some of the most bizarre results of General Relativity:

1. In General Relativity, in a more general context, it no longer becomes meaningful to talk about the relative velocities of any two objects which are not in the same place

2. Space-time itself can be thought of as a dynamical object, which can bend and move.

3. Energy, or at least energy as we usually define it, will no longer be conserved in general. There is a much more general equation which takes the place of energy conservation, although interpreting its physical meaning is a little trickier.

4. If a region of space has a curvature large enough, it can create a black hole, in which not even light can escape.

All of these facts (and many other interesting phenomenon) would be discussed in an introductory course on General Relativity, which I highly recommend all of you take, if you ever have the chance.