Some Perspective

The language of Kinematics provides us with an efficient method for describing the motion of material objects, and we'll continue to make refinements to it as we introduce additional types of coordinate systems later on. However, kinematics tells us nothing about how or why material objects move in the way they do - in order to answer those questions, we need some sort of physical principle which tells us how materials bodies interact with each other and how this affects their motion.

Even today we don't understand the full extent of these physical principles - science is always a work in progress, and physics is no exception. Every day, physics experiments (including ones happening right here at UCSB!) make further and further refinements to our understanding of these principles. Because of this, the "laws" of physics which we will discuss in our course, in particular Newtonian mechanics, will necessarily be approximations.

However, they are in fact incredibly good approximations. For hundreds of years, it was believed that Newtonian mechanics was THE correct model of the universe, capable of describing everything in the material world around us. While more efficient versions of Newtonian mechanics were later invented (Lagrangian and Hamiltonian mechanics, which you will learn about next quarter), these were only mathematical modifications which made no fundamental changes to the underlying physics. Of course, the reason people believed that Newtonian mechanics was correct was because the predictions it made were so accurate, no one could perform a sensitive enough experiment which disagreed with them! The extent to which Newtonian mechanics makes incorrect predictions could only be determined once experiments were sensitive enough to be able to detect the detailed structure of the atom, at which point it became clear that Quantum mechanics was ultimately a more accurate description of the universe.

The punchline is that so long as we're not trying to make detailed predictions about the behaviour of things like atoms, we can safely use Newtonian mechanics as if it were correct, because we will never know the difference anyways. And there is a huge advantage to doing just that - the laws of Newtonian Mechanics are significantly less complicated than those of Quantum Mechanics! There are also still plenty of applications for Newtonian mechanics - everyday mechanical objects are still an important part of our lives, and we need efficient tools for describing how they behave. So with these caveats in mind, let's start learning about Newtonian Mechanics.

Forces and Newton's Laws

As we know from freshman mechanics, Newton's Laws tell us how an object's motion is determined by the forces acting on it. Intuitively, a force on an object is a sort of "push" or "pull" that an object experiences, and based on what is currently happening to the object, we have rules that tell us what the force acting on the object is. For example, if the object is in a gravitational field, we have rules which tell us what the force acting on the object should be (Newton's Law of gravitation), or if the object happens to be an electron moving in a magnetic field, we have an equation describing the resulting force in that case (the Lorenz force law). Ultimately, all of the forces that act on a body are a result of interactions it experiences with other bodies, and so the idea of forces is really just a succinct mathematical way to describe how bodies interact with each other. From your freshman mechanics class, you probably have a pretty decent intuitive grasp of what a force is, and for the most part, this intuitive notion will be sufficient for our purposes. Entire books could (and have been) written on the philosophy of exactly what a "force" is, but this would be overkill for our course, so we won't dwell on it now (although Taylor discusses this matter in slightly more detail than I have, for those of you interested in reading more).

In particular, the force acting on a body is a **vector** quantity. It has a magnitude (roughly speaking, how much we're pushing on the body), along with a direction (roughly speaking, where we are trying to push it). When more than one type of force acts on a body, we say that the *net force* acting on it is the *vector sum* of all of the individual forces. **Newton's Second Law** tells us that the total force acting on a body causes that body to accelerate according to

$$\vec{F} = m\vec{a}, \tag{1}$$

where m is the mass of the body. The mass of an object is another quantity which you probably have an intuitive sense of from freshman mechanics, and we know it is a rough measure of how much an object resists acceleration. It turns out there are also some philosophical subtleties in defining the mass of an object, which we won't dwell on either, but for those that are interested, Kibble's textbook has a good discussion of how to handle these issues (section 1.3). Remember that the mass of an object is different from its weight. The weight of an object is the force that object experiences due to the effects of gravity. If I take a ball on Earth and move it to the Moon, where the effects of gravity are weaker, then the weight of the ball will be reduced. But its *mass* will stay the same. If I were to take a magnetic ball out into empty space far from any other bodies, where any gravitational effects are negligible, and study its reaction to a magnetic force, I would be learning something about its mass.

When there is no force acting on an object, the above equation tells us that its acceleration is zero, and therefore it will travel with a constant velocity. This fact, despite following as an obvious consequence of Newton's Second Law, is given its own name - Newton's **First** Law.

Newton's second law isn't very useful unless I start telling you something about what types of forces can act on an object and how they behave. But before I start giving examples of forces, Newton's **third** law tells us that there are certain conditions that any valid force must obey. In particular, whenever two bodies interact, the *magnitude* of the force exerted on each body is the same, while the *directions* of the forces are opposite. This is often stated by saying that bodies exert equal and opposite forces on each other. For example, two bodies with mass will be attracted to each other through gravity, and the force they exert on each other will be the same in magnitude. The direction of the force on one body is such that the force points towards the other body, so that the two forces are opposite in direction. This is sketched in Figure 1. We'll have plenty more to say about Newton's law of gravitation as the course progresses.

To clarify the difference between the two forces in a force pair, we often develop a subscript notation. If I have two bodies which I will call A and B, then the force from body A acting on body B is written $\vec{F}_{\text{A on B}}$. Newton's third law then reads

$$\vec{F}_{\text{A on B}} = -\vec{F}_{\text{B on A}}. \tag{2}$$

From a practical standpoint, Newton's second law is the one we will make use of the most in this course, although from a fundamental standpoint, Newton's third law is absolutely crucial. It turns out that (as discussed in section 1.3 of Kibble's textbook) Newton's third law is necessary in order to be able to define the mass of an object, and later in the course, Newton's third law will play an important role in the concept of center of mass motion and conservation of momentum.

Projectile Motion with Air Resistance

To see a concrete example of Newton's laws in action, let's revisit the projectile motion problem we recently studied, but with a slight modification - this time we will include the effects of air resistance. We're all familiar with the fact that if we move very quickly through the air, we feel a force pushing on our bodies. While this force is typically a result of complicated microscopic interactions between the molecules in the air and our bodies, it is usually possible to model the force in a simple form. In many realistic situations, it is accurate to model the force acting on a body due to air resistance as

$$\vec{F}_R = -kv^n\hat{v}, \tag{3}$$

where v is the magnitude of the velocity, $\hat{v}$ is a unit vector pointing in the direction of the velocity, n is some integer power, and k is some numerical constant, referred to as the *drag coefficient*. The constant k is something which in principle could be calculated from first principles, and typically depends on the size and shape of the body in question. As a practical matter, it is often easiest to simply measure the value of k by doing an experiment on the body. In either case, we will assume that the parameter k is something that we know the value of. A free-body diagram for this modified situation is shown in Figure 2.

As we usually do, we're going to make the approximation that the cup is a point-like object, which, despite sounding somewhat silly, turns out to be a surprisingly reasonable assumption, provided that the shape and overall

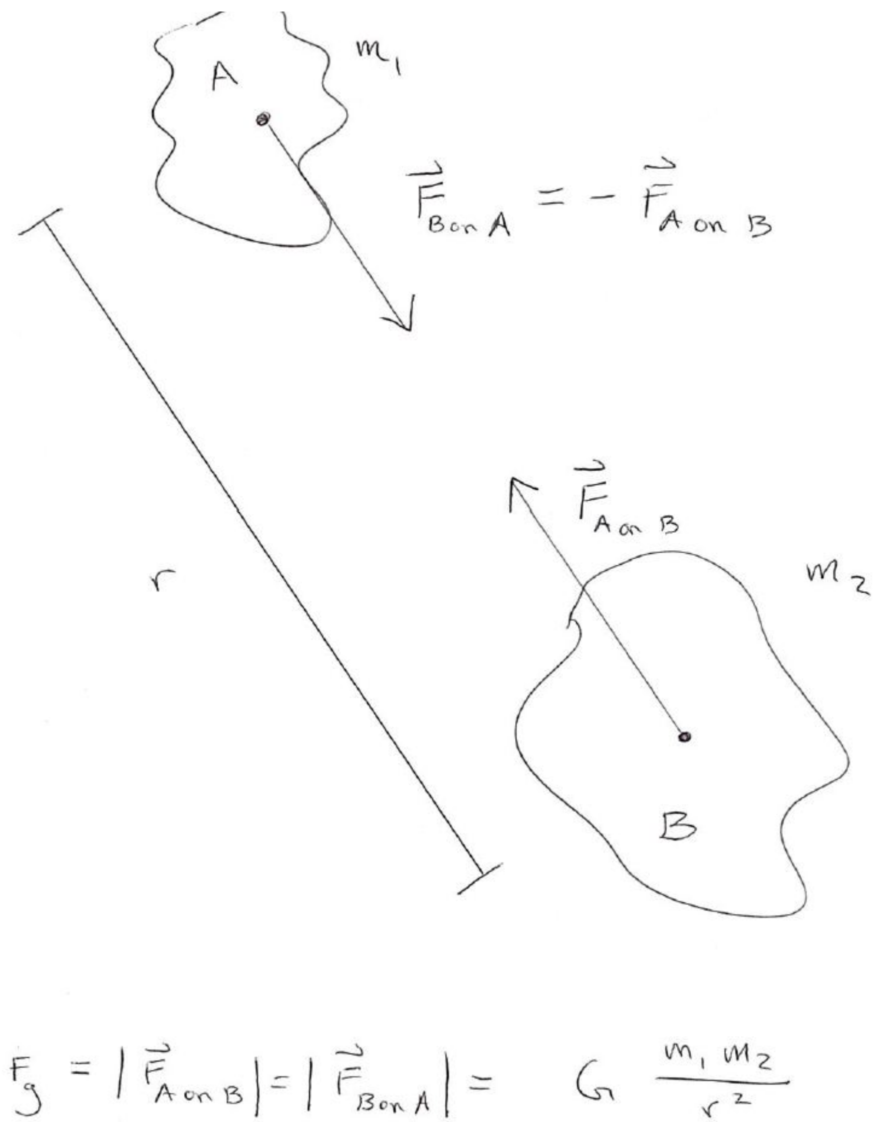

Figure 1: Two massive bodies will exert a gravitational force on each other, and it will obey Newton's third law. Newton's law of gravitation is shown at the bottom.

structure of the block doesn't change much (later in the course we'll understand why this approximation works so well when we discuss the notion of center of mass).

Let's consider an example with $n = 1$, when the drag force can be written

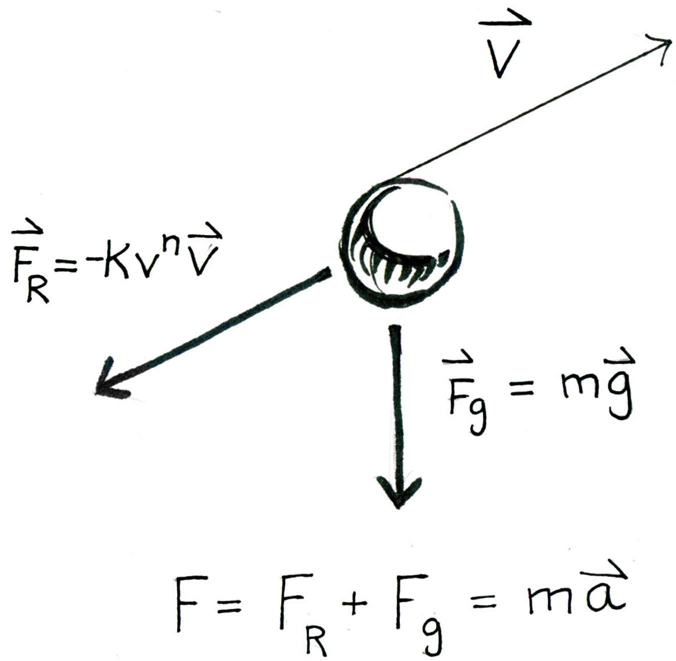

Figure 2: The various forces acting on a projectile in motion through the air. Image credit: Kristen Moore

as

$$\vec{F}_R = -kv\hat{v} = -k\vec{v}. \tag{4}$$

If we revisit our projectile motion problem from our discussion of kinematics, where the material body in question is now subject to the force of gravity and air resistance, the total force will be

$$\vec{F} = \vec{F}_g + \vec{F}_R = m\vec{g} - k\vec{v}, \tag{5}$$

where m is the mass of the body, and $\vec{g}$ is the local acceleration due to gravity. Again, we'll set up a coordinate system whose x axis is parallel with the ground and whose y axis points vertically upwards away from the ground. We'll place the origin so that it is on the ground, directly below the muzzle of the gun. This is shown in Figure 3. In our coordinates, the initial location of the bullet is specified by

$$\vec{r}_0 = \begin{pmatrix} 0 \\ h \end{pmatrix}, \tag{6}$$

where h is the height of the nozzle of the gun off of the ground. If the gun makes an angle of θ with respect to the horizontal, our initial velocity will be

$$\vec{v}_0 = \begin{pmatrix} v_0 \cos\theta \\ v_0 \sin\theta \end{pmatrix}, \tag{7}$$

where v_0 is the initial speed of the bullet. The acceleration due to gravity is given by

$$\vec{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix}, \tag{8}$$

where g is approximately 9.8 meters per second squared.

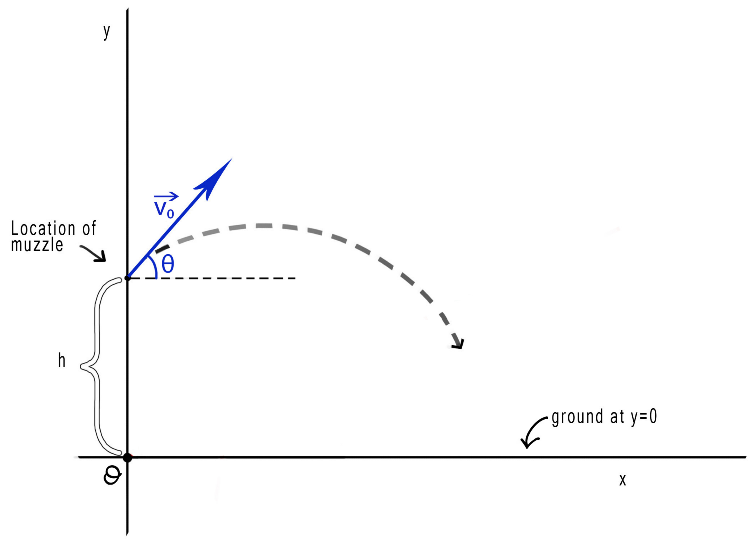

Figure 3: The initial set up of our projectile motion problem, the same as it was in the previous lecture. Image credit: Kristen Moore

If we now write Newton's Second Law, $\vec{F} = m\vec{a}$, in terms of components, we find

$$\begin{pmatrix} ma_x \\ ma_y \end{pmatrix} = \begin{pmatrix} 0 \\ -mg \end{pmatrix} + \begin{pmatrix} -kv_x \\ -kv_y \end{pmatrix}, \tag{9}$$

or,

$$\begin{pmatrix} m\ddot{r}_x \\ m\ddot{r}_y \end{pmatrix} = \begin{pmatrix} -k\dot{r}_x \\ -mg - k\dot{r}_y \end{pmatrix}. \tag{10}$$

The above vector equation is a *system of differential equations* - the two component equations define the coordinates $r_x(t)$ and $r_y(t)$ in terms of their own derivatives. In particular, the above system of differential equations is *decoupled*

- one equation involves terms that only depend on the x coordinate, while the other equation involves terms that only depend on the y coordinate. This type of system is especially easy to solve, since we can treat each equation separately, without worrying about the other. If we had chosen a drag force with $n \neq 1$, this would **not** have been the case - the higher power of the velocity would have mixed the two coordinates together in a more complicated way.

At this point in your physics career, you should have encountered some basic differential equations at least once before, but we'll review the basic steps here. Let's start with the equation for the x coordinate, which reads

$$\ddot{r}_x = -\frac{k}{m}\dot{r}_x. \tag{11}$$

If we make use of the basic definitions of velocity as the derivative of position, and acceleration as the derivative of velocity, then we can rewrite this equation in terms of the velocity and its derivative,

$$\dot{v}_x = -\frac{k}{m}v_x. \tag{12}$$

In this form, the differential equation in the x direction is a *first-order* differential equation for the x coordinate of the velocity. We say that it is a first-order differential equation because the highest derivative that appears is a first-order one.

In order to proceed in solving the differential equation, we use a technique known as *separation of variables*. The idea is to first rearrange the equation so that it reads

$$\frac{\dot{v}_x}{v_x} = -\frac{k}{m}, \tag{13}$$

with the velocity and its derivative appearing on the left, and the constant terms appearing on the right. Now, we integrate each side of the differential equation with respect to time,

$$\int_{t_0}^{t} \frac{\dot{v}_x}{v_x}\ dt' = -\frac{k}{m}\int_{t_0}^{t}\ dt', \tag{14}$$

where t_0 is the initial time (often taken to be zero), and t is some later time at which we want to know the location (and velocity) of the projectile. The integral on the right side is trivial to perform, and we find

$$\int_{t_0}^{t} \frac{\dot{v}_x}{v_x}\ dt' = -\frac{k}{m}\left(t - t_0\right). \tag{15}$$

Now, without prior knowledge of what $v_x\left(t\right)$ is, we can't immediately compute the integral on the left side. However, we have a valuable tool at our disposal - the substitution rule for definite integrals. The substitution rule says that

$$\dot{v}_x\ dt' = \frac{dv_x}{dt'}\ dt' = dv_x, \tag{16}$$

which allows us to perform a change of variables in an integral. Using this fact, we find

$$\int_{v_x(t_0)}^{v_x(t)} \frac{dv_x}{v_x} = -\frac{k}{m}(t - t_0). \tag{17}$$

We can now perform the integral over the left side, to find

$$\ln\left(\frac{v_x(t)}{v_x(t_0)}\right) = -\frac{k}{m}(t - t_0). \tag{18}$$

If we now take the exponential of both sides, and do a little bit of algebraic rearrangement, we find that

$$v_x(t) = v_x(t_0)\exp\left[-\frac{k}{m}(t - t_0)\right]. \tag{19}$$

This equation tells us that the velocity in the x direction decays exponentially over time, which makes sense, since we expect the retarding force to decrease the speed of the projectile. Notice that when $k = 0$, the velocity is constant for all time, which is exactly what we would expect when there is no drag. We can rewrite the equation slightly,

$$v_x(t) = v_x(t_0)\exp\left[-(t - t_0)/\tau\right], \tag{20}$$

where

$$\tau \equiv m/k \tag{21}$$

is known as the *characteristic time*. The characteristic time gives a rough sense of how long it takes for the velocity to decay appreciably (more precisely, it is the time it takes for the magnitude of the velocity to decrease by a factor of $e \approx 2.718$). Notice that when $k = 0$, the characteristic time is infinity.

The above solution of the differential equation for the x coordinate should be familiar to you from previous courses. If any of this seems unfamiliar, you should review ASAP! Notice that if the terms on the right had depended on time in some specified way, the integral on the right still would have been straightforward to perform. For example, if the drag coefficient k were a function of time, so that $k \equiv k(t)$, the integral over time could have still been computed, so long as we knew the functional form for $k(t)$. This is why this method is known as separation of variables - all of the explicit dependence on time is separated to one side, where all of the explicit dependence on the velocity (and its derivative) is separated to the other side. While this solution technique is a valuable tool, it unfortunately only works for first-order equations. We'll learn how to solve more general equations later on in the course.

The differential equation for the y coordinate can similarly be written in terms of the velocity, and, after some rearrangement, takes the form

$$\frac{\dot{v}_y}{g + \frac{k}{m}v_y} = -1. \tag{22}$$

We can again proceed by integrating both sides with respect to time, and then performing a change of variables on the left. This results in the equation

$$\int_{v_y(t_0)}^{v_y(t)} \frac{dv_y}{g + \frac{k}{m} v_y} = -\left(t - t_0\right). \tag{23}$$

Solving the integral on the left, we find

$$\frac{m}{k} \ln \left[\frac{mg + kv_y\left(t\right)}{mg + kv_y\left(t_0\right)} \right] = -\left(t - t_0\right). \tag{24}$$

With a little bit of algebraic rearrangement, this becomes

$$v_y\left(t\right) = \left(\frac{m}{k} g + v_y\left(t_0\right) \right) \exp \left[-\frac{k}{m}\left(t - t_0\right) \right] - \frac{m}{k} g, \tag{25}$$

or, in terms of the characteristic time,

$$v_y\left(t\right) = \left(g\tau + v_y\left(t_0\right)\right) \exp\left[-\left(t - t_0\right)/\tau\right] - g\tau. \tag{26}$$

The above equation for the y component of the velocity tells us that at the initial time, when $t = t_0$,

$$v_y\left(t = t_0\right) = \left(g\tau + v_y\left(t_0\right)\right) e^0 - g\tau = g\tau + v_y\left(t_0\right) - g\tau = v_y\left(t_0\right), \tag{27}$$

exactly as it should. The more interesting case occurs at infinitely long times, when

$$v_y\left(t = \infty\right) = \left(g\tau + v_y\left(t_0\right)\right) e^{-\infty} - g\tau = -g\tau. \tag{28}$$

This tells us that in the limit of infinitely long time, the velocity approaches a **constant** value, which is known as the *terminal velocity.* The terminal velocity is the velocity which is reached when the gravitational force on the falling projectile precisely balances out the force due to air resistance. To see that this is so, simply plug the expression for the terminal velocity into the equation for the force due to drag,

$$\vec{F}_{Ry} = -kv_y = k\frac{m}{k} g = g = -\vec{F}_{gy}. \tag{29}$$

Now that we have the two velocity components, one property of the projectile's motion that we can compute is its acceleration. The acceleration is found by differentiating the velocity, and we find

$$a_x\left(t\right) = -\frac{v_x\left(t_0\right)}{\tau} \exp\left[-\left(t - t_0\right)/\tau\right] = -k\frac{v_x\left(t_0\right)}{m} \exp\left[-\left(t - t_0\right)/\tau\right] \tag{30}$$

for the x component. Notice that at $t = t_0$, this expression tells us that

$$a_x\left(t = t_0\right) = -k\frac{v_x\left(t_0\right)}{m}, \tag{31}$$

which is simply the force due to drag right at the moment that the projectile is fired. At infinitely long times, the acceleration decays to zero, as it should, since the velocity is approaching the constant value of zero. For the y component, we find

$$a_y(t) = -\frac{(g\tau + v_y(t_0))}{\tau} \exp[-(t-t_0)/\tau] = -\left(g + \frac{k}{m} v_y(t_0)\right) \exp[-(t-t_0)/\tau]. \tag{32}$$

At the initial time, we find

$$a_y(t = t_0) = -g - \frac{k}{m} v_y(t_0), \tag{33}$$

which is simply the acceleration due to gravity and the acceleration due to the initial drag force. At infinitely long times, the acceleration in the y direction also decays to zero, since the gravitational and drag forces eventually balance each other, leading to zero net acceleration. The magnitude of the acceleration is also a simple calculation, and after a little algebra, we find

$$a(t) = \sqrt{a_x^2(t) + a_y^2(t)} = \sqrt{\left(\frac{k}{m} v_x(t_0)\right)^2 + \left(g + \frac{k}{m} v_y(t_0)\right)^2} e^{-(t-t_0)/\tau}. \tag{34}$$

The magnitude of the acceleration also decays exponentially, approaching zero at long times.

Projectile Range with Air Resistance

Now that we know the velocity of the projectile as a function of time, we can find the position of the projectile as a function of time simply by integrating the velocity. For the x component, we find

$$r_x(t) = r_x(t_0) + \int_{t_0}^{t} v_x(t')\ dt' = v_x(t_0) \int_{t_0}^{t} \exp[-(t'-t_0)/\tau]\ dt', \tag{35}$$

since the initial value of the x component is zero. Computing the integral, we have

$$r_x(t) = v_x(t_0)\,\tau \left[1 - e^{-(t-t_0)/\tau}\right]. \tag{36}$$

Written in terms of the angle at which our projectile is initially pointed,

$$r_x(t) = v_0 \tau \cos(\theta) \left[1 - e^{-(t-t_0)/\tau}\right]. \tag{37}$$

At large times, this becomes

$$r_x(t = \infty) = v_0 \tau \cos(\theta) \left[1 - e^{-\infty}\right] = v_0 \tau \cos(\theta). \tag{38}$$

This tells us that even if the projectile were to travel through the air for infinitely long times, it would never reach a horizontal position larger than the above constant. Intuitively, this makes sense - just like a block sliding on a table eventually comes to rest, a projectile's horizontal motion is eventually impeded by air resistance, and since there are no other forces in the horizontal direction, it will stay at rest along this direction.

Of course, in a realistic situation, a projectile will generally run into the ground in an amount of time which is less than infinity, so the horizontal range of the projectile will be less than the above constant. To find out exactly how far the projectile will travel, we need to know how long it takes for the vertical component of the position to reach zero, which indicates that the projectile has hit the ground. The y component of the position is also found by integrating, and we have

$$r_y(t) = r_y(t_0) + \int_{t_0}^{t} v_y(t')\, dt' = h + \int_{t_0}^{t} \left[\left(g\tau + v_y(t_0)\right) e^{-(t-t_0)/\tau} - g\tau \right] dt', \tag{39}$$

which, after integration, yields

$$r_y(t) = h + \left[g\tau^2 + v_y(t_0)\,\tau\right]\left[1 - e^{-(t-t_0)/\tau}\right] - g\tau\,(t - t_0)\,. \tag{40}$$

Notice that as the time becomes large and the exponential term becomes small, we have

$$r_y(t \to \infty) = \left[h + g\tau^2 + v_y(t_0)\,\tau\right] - g\tau\,(t - t_0)\,, \tag{41}$$

which is just motion with a constant velocity of $g\tau$, as we found earlier. In order to find the time at which the projectile hits the ground, we need to set the vertical coordinate of the position equal to zero. Taking $t_0 = 0$ for simplicity, and writing the initial velocity in terms of the angle of the projectile, this means that we need to solve the equation

$$r_y(t_R) = 0 \;\Rightarrow\; h + \left[g\tau^2 + v_0\tau\sin(\theta)\right]\left[1 - e^{-t_R/\tau}\right] = g\tau t_R. \tag{42}$$

Unfortunately, it is not possible to solve the above equation in a simple closed form. No clever combination of taking logarithms or exponentials will result in a simple expression for t_R. Because of the presence of the exponential term, this type of equation is known as a *transcendental equation*, since the exponential is an example of a transcendental function (a transcendental function is a function which is not a simple algebraic function of its arguments). However, there are several ways to find the value of t_R, depending on what our needs are.

If we are trying to calculate the range of a specific projectile in a real-world application (say, for calculating the range of a missile on some navy boat), then a *numerical approach* is often the best approach. If we specify a particular numerical value for all of the parameters in the problem (the mass of the projectile, the drag coefficient, the acceleration due to gravity, and the initial position and velocity), then there are many computer programs which can compute an approximate value for t_R. If we only need to know the range of one specific projectile, but we need it to a high level of precision, this is often the best approach - modern computers can perform billions of calculations per second, and so can attain very accurate numerical solutions in a short period of time. You'll explore how some of these algorithms work in the homework (although I promise I won't make you do a billion calculations).

However, in many cases we want to gain a better intuitive understanding of how the details of the problem vary as we modify or change its parameters. This might be important, for example, if we were trying to understand how to modify the parameters of a projectile to ensure that it had the best possible performance under different circumstances. In this case, a **perturbative approach** is typically a better route. In fact, perturbative approaches are so commonplace in physics that it pays to set aside some time and understand how exactly they work. We'll use a perturbative method here to solve our current projectile problem, although we will see perturbative approaches appear several more times before the course is through.

When we use a perturbative approach, we first identify some parameter in our problem which we can think of as being sufficiently "small." Usually we choose this parameter so that when the parameter is equal to zero, we know how to solve the problem in closed form. In our case, the drag coefficient k is a natural choice for this parameter, since we've already seen how to calculate t_R when there is no drag. Next, we assume that the quantity we are trying to find, in this case t_R, is a function of k, $t_R \equiv t_R(k)$, which has a well-defined Taylor series expansion in terms of k,

$$t_R(k) = \sum_{n=0}^{\infty} t_{Rn} k^n = t_{R0} + t_{R1} k + t_{R2} k^2 + \ldots \tag{43}$$

This is almost always the case in real-world problems, and we expect it to be true here - as we slowly add air resistance to the projectile, the range of the projectile should decrease slightly, due to the effects of drag. The goal of the perturbative approach is then to identify each of the coefficients which appear in the Taylor series expansion, one at a time.

To see how this method works, let's assume that the drag coefficient, k, is indeed suitably "small" in some sense. Exactly what constitutes "small" can sometimes be a subtle question, the answer to which is sometimes only clear after performing the perturbative method. However, it is often the case that the approach works very well even for fairly large parameter values. So let's start by taking our original equation and rewriting it slightly,

$$h + \left[g\frac{m^2}{k^2} + v_0 \sin(\theta)\frac{m}{k}\right]\left[1 - e^{-kt_R/m}\right] = g\frac{m}{k}t_R, \tag{44}$$

so that the dependence on k is explicit. We'll also define

$$\phi \equiv v_0 \sin(\theta), \tag{45}$$

and multiply both sides by k^2, in order to get

$$hk^2 + \left[gm^2 + \phi mk\right]\left[1 - e^{-kt_R/m}\right] = gmkt_R. \tag{46}$$

Now, we insert the expansion of t_R in terms of k into this equation, to find

$$hk^2 + \left(gm^2 + \phi mk\right)\left(1 - \exp\left[-\frac{k}{m}\left\{\sum_{n=0}^{\infty} t_{Rn}k^n\right\}\right]\right) = gmk\left\{\sum_{n=0}^{\infty} t_{Rn}k^n\right\}. \quad (47)$$

This expression may not immediately seem like an improvement, since we still need to solve for a bunch of terms which are in the exponential. However, the trick is to realize that the exponential function itself also has a Taylor series expansion, which can be written as

$$e^x = \sum_{p=0}^{\infty} \frac{x^p}{p!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ... \quad (48)$$

Because the small parameter k also shows up inside of the exponential, we can use this expansion in our equation, to find

$$hk^2 + \left(gm^2 + \phi mk\right)\left(1 - \sum_{p=0}^{\infty}\frac{1}{p!}\left[-\frac{k}{m}\left\{\sum_{n=0}^{\infty} t_{Rn}k^n\right\}\right]^p\right) = gmk\left\{\sum_{n=0}^{\infty} t_{Rn}k^n\right\}. \quad (49)$$

We now have an equation which involves an expansion only in powers of k, which thus eliminates the issue of the exponential function.

Of course, this expression is still somewhat intimidating looking, especially with one infinite sum nested inside of the other. The trick to being able to actually do something with this equation is to remember a theorem about Taylor series: if two power series expansions are equal to each other, then it must be true that the individual terms are equal. In other words, if we have

$$a_0 + a_1x + a_2x^2 + ... = b_0 + b_1x + b_2x^2 + ... \quad (50)$$

then it must be the case that

$$a_0 = b_0\ ,\ a_1 = b_1\ ,\ a_2 = b_2\ ,\ ... \quad (51)$$

This means that if we expand out both sides of our equation, we can match powers of k to determine the coefficients. If we only want a small number of coefficients, we only need to expand out a few terms. Let's see exactly how expanding out both sides of our equation in this way tells us something about the coefficients.

The right side of our equation is easy to expand - it simply becomes

$$gmk\left\{\sum_{n=0}^{\infty} t_{Rn}k^n\right\} = gmt_{R0}k + gmt_{R1}k^2 + gmt_{R2}k^3 + ... \quad (52)$$

For now, let's expand each side out to third order in k. We'll see later why this was a good choice. As for the left side of our equation, a little more care is required. We want to expand the sums on the left so that we keep all of the powers of k that appear up through k^3, so that we can match powers on both sides of the equation. To begin this expansion, let's notice that

$$1-e^x = 1-\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+...\right) = -\sum_{p=1}^{\infty}\frac{x^p}{p!}, \tag{53}$$

since the first term in the sum cancels. Using this, our equation becomes

$$hk^2-\left(gm^2+\phi mk\right)\left(\sum_{p=1}^{\infty}\frac{1}{p!}\left[-\frac{k}{m}\left\{\sum_{n=0}^{\infty}t_{Rn}k^n\right\}\right]^p\right) = gmt_{R0}k+gmt_{R1}k^2+gmt_{R2}k^3+... \tag{54}$$

This means that the quantity on the left we need to expand is

$$\sum_{p=1}^{\infty}\frac{1}{p!}\left[-\frac{k}{m}\left\{\sum_{n=0}^{\infty}t_{Rn}k^n\right\}\right]^p = \sum_{p=1}^{\infty}\frac{(-1)^p}{p!}\frac{k^p}{m^p}\left\{\sum_{n=0}^{\infty}t_{Rn}k^n\right\}^p \tag{55}$$

keeping terms as high as k^3.

In order to perform the expansion, we'll consider the sum over p, one term at a time. Let's start with the $p=1$ term, which gives

$$-\frac{k}{m}\left\{\sum_{n=0}^{\infty}t_{Rn}k^n\right\} \quad = \quad -\frac{t_{R0}}{m}k-\frac{t_{R1}}{m}k^2-\frac{t_{R2}}{m}k^3+... \tag{56}$$

Despite the fact that the inside sum over n has infinitely many terms, we only need to keep the first three, since we are only expanding both sides through k^3. Any additional terms contribute a factor of k^4 or higher, which contributes to a power that we are not interested in matching.

Continuing with the expansion over p, the $p=2$ term is

$$\frac{1}{2}\frac{k^2}{m^2}\left\{\sum_{n=0}^{\infty}t_{Rn}k^n\right\}^2 = \frac{1}{2}\frac{k^2}{m^2}\left(\,t_{R0}+t_{R1}k+...\,\right)\times\left(\,t_{R0}+t_{R1}k+...\,\right). \tag{57}$$

The sums inside the parentheses contain infinitely many terms, which then must be multiplied. While it may seem like carrying out this multiplication is impossible, the fact that we only want an expansion through k^3 means that this is actually a relatively simple task. First, we notice that because the expression already contains an overall factor of k^2, we only need to expand multiplication of the parentheses out to order one. If we start to perform the multiplication by expanding one term at a time, we find

$$\begin{aligned}&\left(\,t_{R0}+t_{R1}k+...\,\right)\times\left(\,t_{R0}+t_{R1}k+...\,\right)= \\ &t_{R0}\left(\,t_{R0}+t_{R1}k+...\,\right)\ +\ t_{R1}k\left(\,t_{R0}+t_{R1}k+...\,\right)\ +\ ...\end{aligned} \tag{58}$$

Despite the fact that there are infinitely many terms in this sum, each term contributes increasingly higher powers of k. If we only want to expand this multiplication to first order in k, then we simply take

$$\begin{aligned}&\left(\ t_{R0} + t_{R1}k + ...\ \right) \times \left(\ t_{R0} + t_{R1}k + ...\ \right) = \\ &t_{R0} \left(\ t_{R0} + t_{R1}k + ...\ \right) + t_{R1}k \left(\ t_{R0} + ...\ \right) + ... = t_{R0}^2 + 2t_{R0}t_{R1}k + ...\end{aligned} \tag{59}$$

Therefore, if we only want to expand our expression through k^3, the only part of the $p = 2$ term that we need is

$$\frac{1}{2}\frac{k^2}{m^2}\left\{\sum_{n=0}^{\infty} t_{Rn}k^n\right\}^2 = \frac{1}{2}\frac{k^2}{m^2}\left(t_{R0}^2 + 2t_{R0}t_{R1}k + ...\right) = \frac{1}{2}\frac{t_{R0}^2}{m^2}k^2 + \frac{t_{R0}t_{R1}}{m^2}k^3 + ... \tag{60}$$

Every additional term is at least as large as k^4.

The $p = 3$ term in the sum is given by

$$-\frac{1}{6}\frac{k^3}{m^3}\left\{\sum_{n=0}^{\infty} t_{Rn}k^n\right\}^2 = -\frac{1}{6}\frac{k^3}{m^3}\left(\ t_{R0} + t_{R1}k + ...\ \right)^3 . \tag{61}$$

Because the overall pre-factor on this term is already third order in k, the term in parentheses only needs to be expanded to zero order in k, which simply gives the constant value t_{R0}^3. Therefore, the relevant contribution from the $p = 3$ term is

$$-\frac{1}{6}\frac{k^3}{m^3}\left(\ t_{R0} + t_{R1}k + ...\ \right)^3 = -\frac{1}{6}\frac{t_{R0}^3}{m^3}k^3 + ... \tag{62}$$

Continuing further, the $p = 4$ term in the sum is given by

$$\frac{1}{24}\frac{k^4}{m^4}\left\{\sum_{n=0}^{\infty} t_{Rn}k^n\right\}^2 = \frac{1}{24}\frac{k^4}{m^4}\left(\ t_{R0} + t_{R1}k + ...\ \right)^4 . \tag{63}$$

However, this term does not contribute any factors of k which are less than fourth order, due to the overall factor of k^4. Of course, this will also be the case for every higher power of p. Thus, the only two powers of p which are important are the first three, which is much less than infinity!

Taking our expressions for the $p = 1$, $p = 2$ and $p = 3$ terms and combining them together, we find that the infinite sum on the left side of our equation can be expressed as

$$\begin{aligned}&\sum_{p=1}^{\infty}\frac{(-1)^p}{p!}\frac{k^p}{m^p}\left\{\sum_{n=0}^{\infty} t_{Rn}k^n\right\}^p \\ &= -\frac{t_{R0}}{m}k - \frac{t_{R1}}{m}k^2 - \frac{t_{R2}}{m}k^3 + \frac{1}{2}\frac{t_{R0}^2}{m^2}k^2 + \frac{t_{R0}t_{R1}}{m^2}k^3 - \frac{1}{6}\frac{t_{R0}^3}{m^3}k^3 + ... \\ &= -\frac{t_{R0}}{m}k + \left(\frac{1}{2}\frac{t_{R0}^2}{m^2} - \frac{t_{R1}}{m}\right)k^2 + \left(\frac{t_{R0}t_{R1}}{m^2} - \frac{1}{6}\frac{t_{R0}^3}{m^3} - \frac{t_{R2}}{m}\right)k^3 + ...\end{aligned} \tag{64}$$

All of the terms that appear after the ellipses are at least fourth order in k. Despite the appearance of an infinite sum, the number of terms we need to worry about is actually fairly small, all things considered. Using this expression for the sum, our original equation now reads

$$hk^2 - \left(gm^2 + \phi mk\right)\left(-\frac{t_{R0}}{m}k + \left(\frac{1}{2}\frac{t_{R0}^2}{m^2} - \frac{t_{R1}}{m}\right)k^2 + \left(\frac{t_{R0}t_{R1}}{m^2} - \frac{1}{6}\frac{t_{R0}^3}{m^3} - \frac{t_{R2}}{m}\right)k^3\right) + ... \tag{65}$$
$$= gmt_{R0}k + gmt_{R1}k^2 + gmt_{R2}k^3 + ...$$

Finally, our original equation is *starting* to look simpler. The last thing we need to do is perform the multiplication on the left side, and then match powers of k. When we multiply out the terms on the left side, there will be some powers of k^4 which appear. However, we can also ignore these, since we are only matching the first few powers of k. If we expand out these terms and ignore the k^4 contributions, we find

$$\begin{aligned} hk^2 - gm^2\left(-\frac{t_{R0}}{m}k + \left(\frac{1}{2}\frac{t_{R0}^2}{m^2} - \frac{t_{R1}}{m}\right)k^2 + \left(\frac{t_{R0}t_{R1}}{m^2} - \frac{1}{6}\frac{t_{R0}^3}{m^3} - \frac{t_{R2}}{m}\right)k^3\right) \\ - \phi m\left(-\frac{t_{R0}}{m}k^2 + \left(\frac{1}{2}\frac{t_{R0}^2}{m^2} - \frac{t_{R1}}{m}\right)k^3\right) + ... \\ = gmt_{R0}k + gmt_{R1}k^2 + gmt_{R2}k^3 + ... \end{aligned} \tag{66}$$

At last, we now have an expansion of our equation, on **both** sides, which is valid through k^3.

Matching powers of k now gives us three equations. The first one comes from matching first-order powers on both sides, and it reads

$$gmt_{R0} = gmt_{R0}. \tag{67}$$

This equation is certainly consistent, although it isn't exactly very interesting. So let's move on to the second equation, which is more interesting.

Matching powers of k^2, we find

$$h + gmt_{R1} - \frac{1}{2}gt_{R0}^2 + \phi t_{R0} = gmt_{R1}, \tag{68}$$

or simply

$$h - \frac{1}{2}gt_{R0}^2 + \phi t_{R0} = 0. \tag{69}$$

This is a quadratic equation for t_{R0}, and in fact it is ***exactly the same equation that we found when we neglected air resistance***. This is exactly what we want, because our expansion in powers of k tells us that when there is no air resistance, we should have

$$t_R\left(k=0\right) = \sum_{n=0}^{\infty} t_{Rn}\left(0\right)^n = t_{R0}. \tag{70}$$

In other words, t_{R0} is precisely the result we get when there is no drag. Solving this quadratic equation gives the same result as before,

$$t_{R0} = \frac{\phi + \sqrt{\phi^2 + 2gh}}{g}. \tag{71}$$

The third equation is the one which yields new information about the effects of air resistance. Matching powers of k^3, we find

$$-gm^2 \left(\frac{t_{R0} t_{R1}}{m^2} - \frac{1}{6} \frac{t_{R0}^3}{m^3} - \frac{t_{R2}}{m} \right) - \phi m \left(\frac{1}{2} \frac{t_{R0}^2}{m^2} - \frac{t_{R1}}{m} \right) = gmt_{R2}, \tag{72}$$

or, after a little rearrangement,

$$\left(\phi - gt_{R0} \right) t_{R1} + \frac{g}{6m} t_{R0}^3 - \frac{\phi}{2m} t_{R0}^2 = 0. \tag{73}$$

Since the term t_{R1} only appears linearly in this equation, some simple algebra immediately yields

$$t_{R1} = \frac{\left(3\phi - gt_{R0} \right)}{\left(\phi - gt_{R0} \right)} \frac{t_{R0}^2}{6m}. \tag{74}$$

We now have an explicit expression for t_{R1} in terms of the parameters of the problem (remember that t_{R0} has its own expression in terms of the parameters of the problem, which we found previously). With this knowledge, we can now write

$$t_R \left(k \right) = t_{R0} + t_{R1} k + ... = t_{R0} + \frac{\left(3\phi - gt_{R0} \right)}{\left(\phi - gt_{R0} \right)} \frac{t_{R0}^2}{6m} k + ... \tag{75}$$

where t_{R0} is given by

$$t_{R0} = \frac{\phi + \sqrt{\phi^2 + 2gh}}{g}. \tag{76}$$

Despite looking slightly complicated, we have now achieved our goal of finding an explicit expression for t_R, which is valid to first order in the drag coefficient k. If the drag coefficient is small enough, so that the force due to air resistance is not too strong, this should be a reasonably good approximation for the value of t_R.

Of course, with this value for t_R, we can finally return to our original goal of finding the range of the projectile. The horizontal range of the projectile is found by evaluating the x coordinate at the time t_R,

$$R = r_x \left(t = t_R \right) = v_0 \tau \cos \left(\theta \right) \left[1 - e^{-t_R / \tau} \right]. \tag{77}$$

Now, before we rush to plug in our value of t_R, let's remember that our original constraint equation was given by

$$r_y \left(t_R \right) = 0 \;\Rightarrow\; h + \left[g\tau^2 + \phi\tau \right] \left[1 - e^{-t_R / \tau} \right] = g\tau t_R. \tag{78}$$

If we rearrange this constraint equation slightly, it becomes

$$1 - e^{-t_R/\tau} = \frac{g\tau t_R - h}{g\tau^2 + \phi\tau}. \tag{79}$$

Using this in the expression for the horizontal range, we find

$$R = v_0 \tau \cos(\theta) \left[\frac{g\tau t_R - h}{g\tau^2 + \phi\tau}\right] = v_0 \cos(\theta) \left[\frac{gmt_R - hk}{gm + \phi k}\right]. \tag{80}$$

Using our result for t_R, we finally arrive at

$$R = v_0 \cos(\theta) \left\{gm + \phi k\right\}^{-1} \left\{gmt_{R0} - hk + g\frac{(3\phi - gt_{R0})}{(\phi - gt_{R0})}\frac{t_{R0}^2}{6}k\right\}. \tag{81}$$

We now have an approximate expression for the range of the projectile, in terms of all of the parameters of the problem. However, there is actually one more simplification we can make. Notice that our final answer contains the term

$$\left\{gm + \phi k\right\}^{-1} = \frac{1}{gm + \phi k}. \tag{82}$$

This is not a simple algebraic function of k, and in fact the Taylor series expansion of this expression contains infinitely many powers of k,

$$\frac{1}{gm + \phi k} = \frac{1}{gm} - \frac{\phi}{g^2m^2}k + \frac{\phi^2}{g^3m^3}k^2 + ... \tag{83}$$

This means that in fact, we can write our expression for the range as

$$R = v_0 \cos(\theta) \left\{\frac{1}{gm} - \frac{\phi}{g^2m^2}k + \frac{\phi^2}{g^3m^3}k^2 + ...\right\} \left\{gmt_{R0} - hk + g\frac{(3\phi - gt_{R0})}{(\phi - gt_{R0})}\frac{t_{R0}^2}{6}k\right\}. \tag{84}$$

Since the second bracket term, the one involving t_R, is only expanded out to first order in k, it doesn't really make sense to keep the first bracket term to a higher order either. If we multiply the two bracket terms together, and ignore any term which is higher than first order in k, we find

$$R = v_0 \cos(\theta) \left\{t_{R0} + \left(\frac{(3v_0 \sin(\theta) - gt_{R0})}{(v_0 \sin(\theta) - gt_{R0})}\frac{t_{R0}^2}{6m} - \frac{v_0 \sin(\theta)}{gm}t_{R0} - \frac{h}{gm}\right)k\right\}, \tag{85}$$

where, as before,

$$t_{R0} = \frac{v_0 \sin(\theta) + \sqrt{v_0^2 \sin^2(\theta) + 2gh}}{g}. \tag{86}$$

This is our final expression for the horizontal range of the projectile, as an approximation to first order in k.

The last thing we should do before wrapping up this subject is make sure that our perturbative answer makes sense. When $k = 0$, we have

$$R\left(k=0\right) = v_0 \cos\left(\theta\right) t_{R0} = v_0 \cos\left(\theta\right) \frac{v_0 \sin\left(\theta\right) + \sqrt{v_0^2 \sin^2\left(\theta\right) + 2gh}}{g}, \quad (87)$$

which is exactly what we found when we neglected air resistance. We can also get an intuitive sense for how the drag affects the motion of the projectile by considering the slightly simpler case when $h = 0$ (when the projectile is fired from the ground). In this case,

$$t_{R0} = \frac{2\phi}{g} = \frac{2v_0 \sin\left(\theta\right)}{g}, \quad (88)$$

and thus

$$t_{R1} = \frac{\left(3\phi - gt_{R0}\right)}{\left(\phi - gt_{R0}\right)} \frac{t_{R0}^2}{6m} = -\frac{2}{3}\frac{\phi^2}{mg^2} = -\frac{2}{3}\frac{v_0^2 \sin^2}{mg^2}. \quad (89)$$

The expression for the range in this case simplifies to

$$R\left(h=0\right) = \frac{2v_0^2 \sin\left(\theta\right) \cos\left(\theta\right)}{g} \left\{1 - \left(\frac{4v_0 \sin\left(\theta\right)}{3mg}\right) k\right\}. \quad (90)$$

When there is no drag, the overall term in brackets is simply equal to one. We see now that the effect of drag is to reduce the range slightly, since when $k \neq 0$, the term in brackets is slightly less than one. While this may have been obvious to us, we can now see in more detail exactly how the range is reduced. Perhaps the most striking effect we see is that once air resistance is introduced, the range of the projectile is no longer independent of the mass - the smaller the mass, the more important the first-order correction is. Of course, this agrees with our common intuition that, for a given shape and size, less massive objects are affected by air resistance more strongly. We also see that the reduction in range increases as the initial velocity increases, which makes sense, as the drag force increases with increasing velocity. Similarly, the effects of air resistance also become more important as gravity becomes weaker. Interestingly, the reduction in the range also depends on the initial firing angle.

It may seem as though this was a lot of work to get an answer which is only approximately correct. However, if you become a professional physicist, this type of calculation will become very familiar to you. Despite what some textbook homework problems may lead you to believe, the vast majority of problems which show up in physics cannot be solved in a simple, closed form. In order to make any kind of progress, it is almost always the case that we need to make some sort of approximation, like the one we have made here. In fact, this type of approach is so common in physics that it might be accurate to say that Taylor series are **THE** most important tool in all of physics (or, as some people say, the most important physicist was a mathematician, and his name was Taylor). With this in mind, it's good to get some practice with these methods as soon a possible, because they will show up over and over again in your future physics courses, especially if you go to graduate school.