

Hooke's Law

Let's imagine that we are studying a particle moving in one dimension, subject to a conservative force with a corresponding potential energy function $U(x)$. We'll assume that the potential energy function is totally arbitrary, aside from one key fact: the potential has a local minimum at a point x_* (there may or may not be other minima). This is sketched in Figure 1. Additionally, let's consider the case that the particle has a total energy $E = K + U$ which is only slightly larger than $U(x_*)$. If the particle moves in the vicinity of the point x_* , then based on the ideas we discussed in the previous lecture, we know that the particle can not travel very far from this point - the particle is constrained to move only in regions such that $U(x) < E$. This is indicated by the red line in the figure. Does the fact that the particle only moves within a small region help us simplify the details of this problem at all? It turns out that the answer is yes, and that the simplification is quite dramatic indeed.

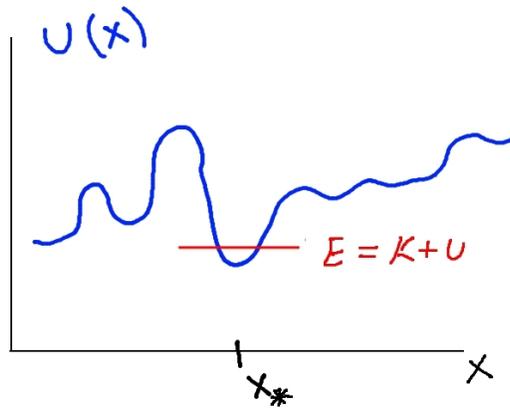


Figure 1: A potential energy function, with a local minimum at the point $x = x_*$.

To see why this is the case, let's make one further assumption about the potential energy function - it is reasonably smooth enough that we can Taylor expand it around the point x_* ,

$$U(x) = \sum_{n=0}^{\infty} U_n (x - x_*)^n = U_0 + U_1 (x - x_*) + U_2 (x - x_*)^2 + \dots \quad (1)$$

This assumption should be valid for virtually any physically realistic system. Because the particle only moves within a small region surrounding the point x_* , our Taylor series approximation of the potential only needs to be reasonably accurate within this small region. So long as this is the case, we can replace

the full potential energy function with a Taylor series approximation containing only a small number of terms, and any calculations we perform regarding the particle's motion should be approximately correct. This is because the particle only explores the region of space where the difference between the two functions is negligible. Figure 2 illustrates this basic idea.

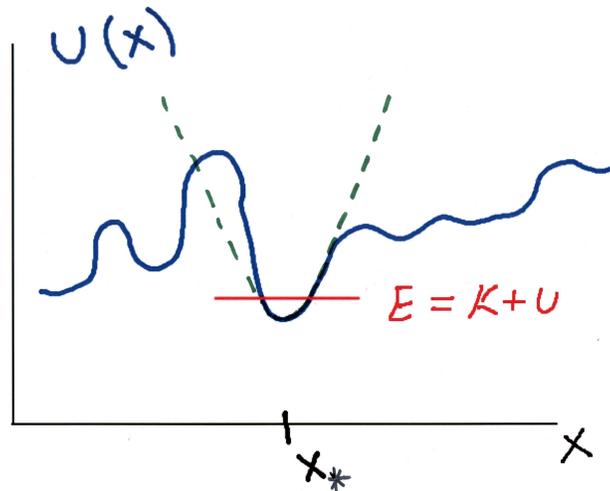


Figure 2: A hypothetical Taylor expansion of our potential energy function, centered around the point $x = x_*$, is shown in green. While a Taylor expansion with a finite number of terms will typically deviate badly from the true function at values of x far away from x_* , so long as the agreement is good in the region in which the particle moves, any calculations performed using the approximate function will still yield accurate answers.

What exactly are the terms in this Taylor expansion? Well, the first term is simply the constant value

$$U_0 = U(x_*). \quad (2)$$

However, we know from the previous lecture that we can shift our potential energy function by a constant without altering any of the actual physics of the problem. So, without any loss of generality, let's simply assume that

$$U_0 = U(x_*) = 0, \quad (3)$$

that is, we'll set the zero of the potential energy to be at the point $x = x_*$.

The next term in the expansion is simply given by the first derivative, evaluated at the point around which we are expanding,

$$U_1 = U'(x_*). \quad (4)$$

However, the point $x = x_*$ corresponds to a potential minimum, and we know from calculus that the derivative of a function must vanish at a minimum. Thus, we must have

$$U_1 = 0. \quad (5)$$

The third term is given by

$$U_2 = \frac{1}{2}U''(x_*), \quad (6)$$

which involves the second derivative. In general, there is no reason for this term to be zero. In some special cases, say for example a potential energy described by $U(x) = x^4$, the second derivative could vanish at the potential minimum. But, for a totally arbitrary potential energy function, it will typically be the case that

$$U_2 = \frac{1}{2}U''(x_*) \equiv \frac{1}{2}k > 0, \quad (7)$$

where K is some positive constant (positive because we have a minimum, and not a maximum).

If we assume that the region around x_* which the particle explores is sufficiently small, then it will be a reasonably good approximation to only expand the potential to second order. In this case, based on the results we have found, our potential energy function will be approximately

$$U(x) \approx \frac{1}{2}k(x - x_*)^2. \quad (8)$$

Thus, almost any particle which is moving in the vicinity of a potential minimum can effectively be described by a potential energy function of this form. But this potential energy should be familiar to you as nothing other than the potential energy of a spring! If we write the expression for the force on the particle, we find

$$F = -\frac{dU}{dx} = -k(x - x_*), \quad (9)$$

which is nothing other than Hooke's Law.

Hooke's Law is often presented as some sort of "fundamental" law of springs when it is introduced in freshman mechanics, but we can see here that it is nothing other than Taylor's theorem in disguise. Almost any oscillating mechanical system will behave like a vibrating spring, so long as the oscillations are small enough. For this reason, the vibrating spring, or **simple harmonic oscillator** (SHO) as it is often called, is one of the most important mechanical systems in all of physics. Because of this, the SHO will be the primary focus of our study for the next few lectures.

Simple Harmonic Motion

The equation of motion for the simple harmonic oscillator is given, as usual, by Newton's second law,

$$F = ma \Rightarrow m\ddot{x} = -k(x - x_*). \quad (10)$$

If we make the change of variables,

$$y = x - x_*, \quad (11)$$

which is just a shift by a constant, then our differential equation simply becomes

$$\ddot{y} = -\frac{K}{m}y = -\omega^2 y, \quad (12)$$

where

$$\omega \equiv \sqrt{k/m} \quad (13)$$

is the angular frequency of the harmonic oscillator. This differential equation tells us that the second derivative of the position is the same as the position itself, aside from an overall negative constant. Of course, from our calculus class, we're all familiar with two functions that satisfy this property, sine and cosine,

$$y_1(t) = \cos(\omega t) ; y_2(t) = \sin(\omega t). \quad (14)$$

Are there any other solutions to this equation?

To answer this, let's first rewrite our differential equation as

$$\ddot{y} + \omega^2 y = \mathcal{L}y = 0, \quad (15)$$

where

$$\mathcal{L} = \frac{d^2}{dt^2} + \omega^2 \quad (16)$$

is a **linear operator** acting on the space of functions of time. We say it is an operator, because it takes a function y , "operates" on it, and then returns the function

$$\mathcal{L}y = \ddot{y} + \omega^2 y. \quad (17)$$

It is "linear" because it satisfies the property of linearity, which is to say that for any two functions f and g ,

$$\mathcal{L}(f + g) = \mathcal{L}f + \mathcal{L}g, \quad (18)$$

and for any constant number α ,

$$\mathcal{L}(\alpha f) = \alpha \mathcal{L}f. \quad (19)$$

In particular, we say that this linear operator is a second-order differential operator, since it is an operator that involves a derivative, and the highest order of derivative which appears is second order. We also say that it is a

homogeneous equation, which simply means that the right side of the equation is zero.

Now, a theorem from linear algebra tells us that for a differential equation which can be written as

$$\mathcal{L}y = 0, \quad (20)$$

where \mathcal{L} is a linear, n^{th} -order differential operator, there are precisely n linearly independent solutions to the differential equation. Linear independence means that none of the n solutions can be written as a simple linear combination of the other ones. The most general solution to the differential equation is then given as a linear combination of these n “basis” solutions. In our case, it is clear that neither sine nor cosine can be written as a simple multiple of the other, and so these two solutions are linearly independent. Because our differential equation is second order, this exhausts the list of linearly independent solutions. Therefore, the most general possible solution to our differential equation is

$$y(t) = A \cos(\omega t) + B \sin(\omega t), \quad (21)$$

or, changing variables back to x ,

$$x(t) = x_* + A \cos(\omega t) + B \sin(\omega t). \quad (22)$$

As we can see, the particle oscillates sinusoidally around the equilibrium point x_* , as we are all probably familiar with. In order to fix the constants A and B , we must impose a set of initial conditions. For simplicity, we’ll assume the initial time is $t_0 = 0$. If we define the initial position to be

$$x_0 \equiv x(t = t_0 = 0), \quad (23)$$

then

$$x_0 = x_* + A \cos(0) + B \sin(0) = x_* + A \Rightarrow A = x_0 - x_*. \quad (24)$$

Differentiating the position as a function of time, we find the velocity to be

$$v(t) = -\omega A \sin(\omega t) + \omega B \cos(\omega t), \quad (25)$$

so that

$$v_0 \equiv v(t = 0) = -\omega A \sin(0) + \omega B \cos(0) \Rightarrow B = v_0/\omega. \quad (26)$$

Thus, our final expression for the most general motion of a harmonic oscillator is

$$\Delta x(t) = \Delta x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t), \quad (27)$$

where

$$\Delta x(t) \equiv x(t) - x_* \quad (28)$$

A slightly more compact expression for the position can be found by defining

$$C = \sqrt{A^2 + B^2} = \sqrt{\Delta x_0^2 + v_0^2/\omega^2}, \quad (29)$$

and writing

$$\Delta x(t) = C \left(\frac{\Delta x_0}{C} \cos(\omega t) + \frac{v_0}{\omega C} \sin(\omega t) \right). \quad (30)$$

Now, because

$$\left(\frac{\Delta x_0}{C} \right)^2 + \left(\frac{v_0}{\omega C} \right)^2 = \frac{\Delta x_0^2 + v_0^2/\omega^2}{\Delta x_0^2 + v_0^2/\omega^2} = 1, \quad (31)$$

we can define an angle δ such that

$$\cos \delta = \frac{\Delta x_0}{C}; \quad \sin \delta = \frac{v_0}{\omega C}; \quad \delta = \arctan \left(\frac{v_0}{\omega \Delta x_0} \right). \quad (32)$$

The position now takes the form

$$\Delta x(t) = C [\cos \delta \cos(\omega t) + \sin \delta \sin(\omega t)]. \quad (33)$$

Using a standard trigonometric identity, this expression can finally be rewritten as

$$\Delta x(t) = C \cos(\omega t - \delta). \quad (34)$$

Differentiating, we see that the velocity is given by

$$v(t) = -C\omega \sin(\omega t - \delta). \quad (35)$$

In this form, we can see that no matter what the initial position and velocity of the particle are, the motion is always a sine wave, with some overall amplitude and phase shift. The period of oscillation is

$$T = 2\pi/\omega = 2\pi\sqrt{m/k}, \quad (36)$$

a result which we found previously from energy considerations.

Speaking of energy, with our explicit expressions for the position and velocity, we can now see that the potential energy as a function of time is given according to

$$U(t) = \frac{1}{2}k\Delta x^2 = \frac{1}{2}kC^2 \cos^2(\omega t - \delta), \quad (37)$$

while the Kinetic energy is given by

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2C^2 \sin^2(\omega t - \delta) = \frac{1}{2}kC^2 \sin^2(\omega t - \delta) \quad (38)$$

Thus, the total energy is

$$E = K + U = \frac{1}{2}kC^2 (\sin^2(\omega t - \delta) + \cos^2(\omega t - \delta)) = \frac{1}{2}kC^2, \quad (39)$$

or,

$$E = \frac{1}{2}k(\Delta x_0^2 + v_0^2/\omega^2) = \frac{1}{2}(k\Delta x_0^2 + mv_0^2), \quad (40)$$

which is a constant, exactly as it should be. In fact, in this form, it is manifestly obvious that the total energy is simply the sum of the initial kinetic energy and

the initial potential energy. With this result, we can see that the maximum displacement of the spring is defined by

$$\frac{1}{2}k\Delta x_M^2 = \frac{1}{2}(k\Delta x_0^2 + mv_0^2) \Rightarrow \Delta x_M = \pm\sqrt{\Delta x_0^2 + v_0^2/\omega^2}, \quad (41)$$

since this is the condition that all of the energy in the system has been converted to potential energy. Similarly, the maximum velocity that the particle obtains during its motion is defined by

$$\frac{1}{2}mv_M^2 = \frac{1}{2}(k\Delta x_0^2 + mv_0^2) \Rightarrow v_M = \pm\sqrt{\omega^2\Delta x_0^2 + v_0^2} \quad (42)$$

since this corresponds to all of the energy in the system being converted into kinetic energy.

Damped Harmonic Motion

The results of the previous section tell us pretty much everything we could ever want to know about simple harmonic motion. As we've seen, almost any object which is oscillating around a potential minimum will exhibit simple harmonic motion, so long as the amplitude of its oscillation is small enough. However, we can generalize this problem a little bit - let's assume that in addition to a conservative force, there is also a drag force acting on the particle. As we've mentioned before, there are many different forms that a drag force can take, depending on the particular physical system. However, we'll focus again on the simplest possible case, linear drag. This means that in addition to being subject to the potential energy of the conservative force, there is some additional force

$$F_d = -b\dot{x}, \quad (43)$$

where b is some constant parameter. This expression is a reasonable approximation for small objects moving very slowly through a fluid, and is also the simplest case to consider from a mathematical standpoint.

In the presence of this additional force, the full differential equation for the particle's position is now given by

$$F = ma \Rightarrow m\ddot{x} = -k(x - x_*) - b\dot{x}. \quad (44)$$

Again changing variables to

$$y = x - x_*, \quad (45)$$

our differential equation becomes, after a little rearrangement,

$$\ddot{y} + 2\beta\dot{y} + \omega^2y = \mathcal{L}_\beta y = 0, \quad (46)$$

where

$$\beta \equiv b/2m, \quad (47)$$

and

$$\mathcal{L}_\beta \equiv \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega^2 \quad (48)$$

is again a linear, second-order differential operator (the factor of two in the definition of β will simplify some of the results later). This system is known as the **damped harmonic oscillator**.

What are the solutions to this differential equation? It will still be true that there are two linearly independent basis solutions, although they will be different from the un-damped solutions. In the presence of drag, we should expect that the particle will not oscillate forever. In fact, from our previous discussion of air resistance, we might expect an exponential decay behaviour instead, although it is not immediately clear to us what exact form the solution should take. Because of this, we will make an educated guess for the form of the solution,

$$y_g(t) = e^{\alpha t}, \quad (49)$$

where α is some constant number we don't know (yet). If we plug this educated guess into our differential equation, we find

$$\left(\frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega^2 \right) e^{\alpha t} = \alpha^2 e^{\alpha t} + 2\beta \alpha e^{\alpha t} + \omega^2 e^{\alpha t} = 0. \quad (50)$$

Now, we will have a solution to our differential equation so long as we can satisfy the above constraint. If we cancel the exponential factors, then the constraint we need to solve is

$$\alpha^2 + 2\beta\alpha + \omega^2 = 0. \quad (51)$$

This is simply a quadratic equation for α , which has the two solutions

$$\alpha_{\pm} = -\beta \pm \sqrt{\beta^2 - \omega^2}. \quad (52)$$

When $\beta > \omega$, we have two real solutions for α , and thus two exponentially decaying solutions to our differential equation,

$$y_{\pm}(t) = e^{-\beta t} e^{\pm \sqrt{\beta^2 - \omega^2} t}. \quad (53)$$

Notice that both α_+ and α_- are negative numbers, and so indeed, both of these solutions decay exponentially. The most general solution to our equation is thus

$$y(t) = e^{-\beta t} \left[C_+ e^{+\sqrt{\beta^2 - \omega^2} t} + C_- e^{-\sqrt{\beta^2 - \omega^2} t} \right], \quad (54)$$

or, in terms of the original coordinate x ,

$$x(t) = x_* + e^{-\beta t} \left[C_+ e^{+\sqrt{\beta^2 - \omega^2} t} + C_- e^{-\sqrt{\beta^2 - \omega^2} t} \right], \quad (55)$$

where C_+ and C_- are determined by initial conditions. In fact, it is a straightforward exercise to verify that

$$C_{\pm} = \frac{v_0 - \alpha_{\mp} (x_0 - x_*)}{\pm (\alpha_+ - \alpha_-)} = \frac{v_0 + \left(\beta \pm \sqrt{\beta^2 - \omega^2} \right) (x_0 - x_*)}{\pm 2\sqrt{\beta^2 - \omega^2}}. \quad (56)$$

Damping Regimes

The case in which $\beta > \omega$ is known as the over-damped, or **strongly** damped, regime of the oscillator. In this case, the damping due to drag is so strong that there is no longer any hint of oscillatory behaviour - the oscillator's position decays exponentially to the point x_* , the equilibrium position of the spring. In contrast, when $\beta < \omega$, in the regime typically referred to as **weak** damping, or under-damping, our expressions

$$\alpha_{\pm} = -\beta \pm \sqrt{\beta^2 - \omega^2} \quad (57)$$

no longer have real solutions. This is not surprising, since below a certain threshold value of β , we would expect that the effects of air-resistance become less important, and the particle should still display some oscillating motion. In this case, the motion of the particle should not simply be a decaying exponential. Faced with this fact, it may seem as though we need to come up with a new educated guess for the solution to our differential equation.

However, this is in fact not the case, due to a result we are all probably familiar with, Euler's formula. Euler's formula famously states that

$$e^{ix} = \cos x + i \sin x \quad (58)$$

for any x . Not only is the exponential of a complex number a well-defined mathematical object, but it is in fact equal to a simple trigonometric function (in the homework you'll explore the proof of Euler's formula). How exactly does this help us? Well, let's imagine that instead of just real values for $y(t)$, I consider the possibility of solutions which are more general, complex numbers,

$$y(t) = y_R(t) + iy_I(t), \quad (59)$$

with a real and imaginary part. If I plug this into my differential equation, I find that

$$(\ddot{y}_R + 2\beta\dot{y}_R + \omega^2 y_R) + i(\ddot{y}_I + 2\beta\dot{y}_I + \omega^2 y_I) = 0, \quad (60)$$

since the number i is simply a constant. Now, since two complex numbers are equal if and only if their real and imaginary parts are equal, this equation implies that

$$\ddot{y}_R + 2\beta\dot{y}_R + \omega^2 y_R = 0 ; \quad \ddot{y}_I + 2\beta\dot{y}_I + \omega^2 y_I = 0. \quad (61)$$

Thus, it makes sense to consider a more general complex solution to our differential equation, and the real and imaginary parts will both satisfy the same differential equation independently. In light of this sort of thinking, we can reinterpret our previous two solutions as corresponding to complex solutions,

$$\alpha_{\pm} = -\beta \pm \sqrt{\beta^2 - \omega^2} = -\beta \pm i\sqrt{\omega^2 - \beta^2}, \quad (62)$$

which yields

$$y_{\pm}(t) = e^{-\beta t} e^{\pm i\sqrt{\omega^2 - \beta^2} t}. \quad (63)$$

Using Euler's formula, we can rewrite this as

$$y_{\pm}(t) = e^{-\beta t} \left[\cos\left(\sqrt{\omega^2 - \beta^2}t\right) \pm i \sin\left(\sqrt{\omega^2 - \beta^2}t\right) \right]. \quad (64)$$

In particular, let's focus on the solution with the + sign. In this form, it becomes clear that what we have found by using this technique is a solution which is **already** a complex linear combination of

$$y_1(t) = e^{-\beta t} \cos\left(\sqrt{\omega^2 - \beta^2}t\right) \quad (65)$$

as the real part, and

$$y_2(t) = e^{-\beta t} \sin\left(\sqrt{\omega^2 - \beta^2}t\right) \quad (66)$$

as the imaginary part. However, remember that the real and imaginary parts of y both satisfy the **same** differential equation. Therefore, we see now that y_1 is one possible solution to this equation, and y_2 is another possible solution. Because they are clearly linearly independent, they must in fact be the two basis solutions we are looking for. Thus, the most general solution for the real part is,

$$y_R(t) = e^{-\beta t} \left[A_R \cos\left(\sqrt{\omega^2 - \beta^2}t\right) + B_R \sin\left(\sqrt{\omega^2 - \beta^2}t\right) \right], \quad (67)$$

or, in a more compact form

$$y_R(t) = e^{-\beta t} [A_R \cos(\Omega t) + B_R \sin(\Omega t)], \quad (68)$$

where

$$\Omega = \sqrt{\omega^2 - \beta^2}. \quad (69)$$

Similarly, the most general solution for the imaginary part is

$$y_I(t) = e^{-\beta t} [A_I \cos(\Omega t) + B_I \sin(\Omega t)]. \quad (70)$$

Of course, in our system we are studying, there is no imaginary component of the particle's position, and so we should simply have

$$y_I(t) = 0 \quad (71)$$

for all time. You should convince yourself that if we set the initial conditions of the imaginary part to be

$$y_I(0) = v_I(0) = 0, \quad (72)$$

then the only solution for the imaginary part is zero for all time. Therefore, if there is no imaginary component at time $t = 0$, this will stay the case for all time (as it must). So even though we allowed the possibility of a more general, complex solution to our differential equation, it was really only a mathematical trick for easily finding the motion of the real part. So at this point, we will simply drop the real and imaginary subscripts on our solutions, and consider

only real solutions, now that we have succeeded in finding the two solutions we were looking for. However, this general approach of simplifying a problem by allowing the possibility of complex numbers is a technique that will come up over and over again in your physics courses, especially if you take a course on complex analysis.

Writing our solution in terms of the original x coordinate, we find that

$$x(t) = x_* + e^{-\beta t} [A \cos(\Omega t) + B \sin(\Omega t)], \quad (73)$$

for some constants A and B which are determined by initial conditions. In particular, a short calculation shows that

$$A = x_0 - x_* ; B = \frac{v_0 - \beta(x_0 - x_*)}{\Omega}. \quad (74)$$

We see now that in this under-damped case, there is still oscillatory behaviour around the equilibrium point x_* . However, the frequency of this oscillation has been decreased, since

$$\Omega = \sqrt{\omega^2 - \beta^2} < \omega. \quad (75)$$

This seems intuitive, since we might expect the effects of drag to “slow down” the particle. Additionally, these oscillations now have an exponentially decaying amplitude - both the sine and cosine terms are weighted by an overall exponential. This overall exponential factor provides a sort of “envelope” underneath which the oscillatory behaviour happens. Figure 3 shows in more detail what such a motion would look like. The blue curve indicates the motion of the damped oscillator, while the yellow curve shows the overall exponential factor determining the decay of the oscillations. In particular, the characteristic time for the decay of the oscillation amplitude is given by

$$\tau = 1/\beta = 2m/b, \quad (76)$$

which has a form similar to the case of projectile motion with air resistance. You should verify for yourself that plugging $\beta = 0$ into this solution yields the usual result for oscillation without any damping force.

Exploring this solution a little more, we can ask about the total energy of the damped oscillator. As before, we can rewrite our solution slightly, so that it reads

$$x(t) = x_* + C e^{-\beta t} \cos(\Omega t - \delta), \quad (77)$$

where, as before

$$C = \sqrt{A^2 + B^2} ; \delta = \arctan(B/A). \quad (78)$$

Taking a derivative, the velocity is

$$v(t) = -C e^{-\beta t} [\beta \cos(\Omega t - \delta) + \Omega \sin(\Omega t - \delta)] \quad (79)$$

With these expressions, it is easy to see that the potential energy is

$$U(t) = \frac{1}{2} k (x - x_*)^2 = \frac{1}{2} k C^2 e^{-2\beta t} \cos^2(\Omega t - \delta). \quad (80)$$

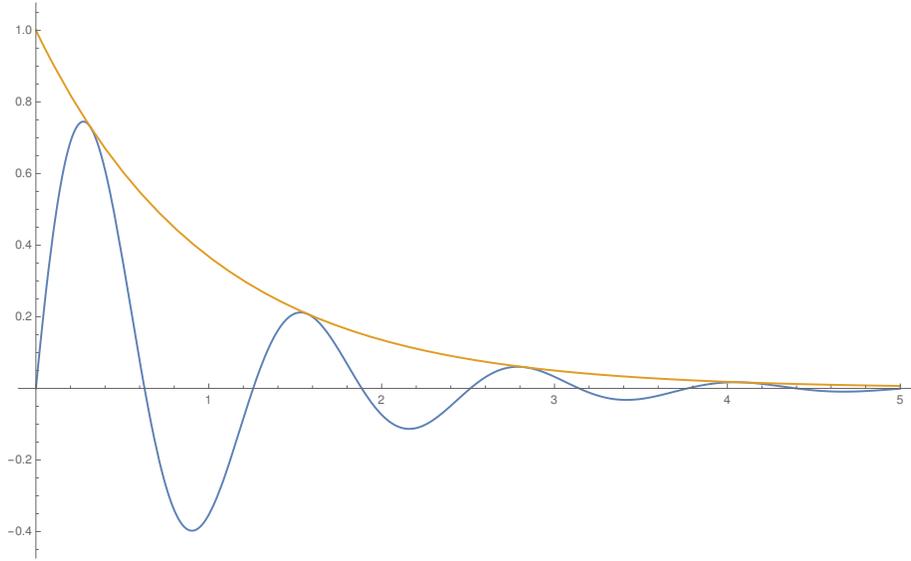


Figure 3: The motion of the damped harmonic oscillator, shown in blue, in the case that $\beta = 1$, $\Omega = 5$, $x_* = 0$, $A = 1$, and $B = 0$. Notice that the overall amplitude of the oscillations decays exponentially, given by an overall factor that is indicated by the yellow curve.

while the kinetic energy is

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}mC^2e^{-2\beta t} [\beta^2 \cos^2(\Omega t - \delta) + \Omega^2 \sin^2(\Omega t - \delta) + 2\beta\Omega \cos(\Omega t - \delta) \sin(\Omega t - \delta)]. \quad (81)$$

If we use a trigonometric identity, the kinetic energy can be simplified to

$$K(t) = \frac{1}{2}mC^2e^{-2\beta t} [\beta^2 \cos^2(\Omega t - \delta) + \Omega^2 \sin^2(\Omega t - \delta) + \beta\Omega \sin(2\Omega t - 2\delta)]. \quad (82)$$

Adding together the kinetic and potential energies, and performing some algebra, we find the total energy to be

$$E(t) = \frac{1}{2}mC^2e^{-2\beta t} [\omega^2 + \beta^2 \{\cos^2(\Omega t - \delta) - \sin^2(\Omega t - \delta)\} + \beta\Omega \sin(2\Omega t - 2\delta)]. \quad (83)$$

Using another trigonometric identity, this becomes

$$E(t) = \frac{1}{2}mC^2e^{-2\beta t} [\omega^2 + \beta^2 \cos(2\Omega t - 2\delta) + \beta\Omega \sin(2\Omega t - 2\delta)]. \quad (84)$$

When $\beta \neq 0$, we can see that the energy is a function of time - it is *not* a constant. As we would expect, the presence of a non-conservative force results

in a lack of energy conservation. In fact, we can explicitly compute

$$\frac{dE}{dt} = -\beta m C^2 e^{-2\beta t} [\omega^2 + (\beta^2 - \Omega^2) \cos(2\Omega t - 2\delta) + 2\beta\Omega \sin(2\Omega t - 2\delta)]. \quad (85)$$

With a little bit of algebra and a trigonometric identity, this can be rewritten as

$$\frac{dE}{dt} = -\beta m \omega^2 C^2 e^{-2\beta t} [1 + \cos(2\Omega t - 2\delta - \phi)] < 0, \quad (86)$$

where

$$\phi \equiv \arctan\left(\frac{2\beta\Omega}{\beta^2 - \Omega^2}\right). \quad (87)$$

We can see from the explicit factor of β out front that in the case where there is no drag, the rate of change of the energy is zero, as it should be. Notice that the derivative of the energy is **always** negative. The energy transfer due to drag is a one-way street - energy is lost to air resistance, and never comes back. This is true in general for dissipative systems. Also, notice that the rate of energy loss is not a constant - energy is lost more quickly at certain points in time. In fact, we would typically expect the energy loss to be largest when the velocity is at a maximum. Figure 4 shows that this is indeed the case. You'll explore this fact more quantitatively in the homework.

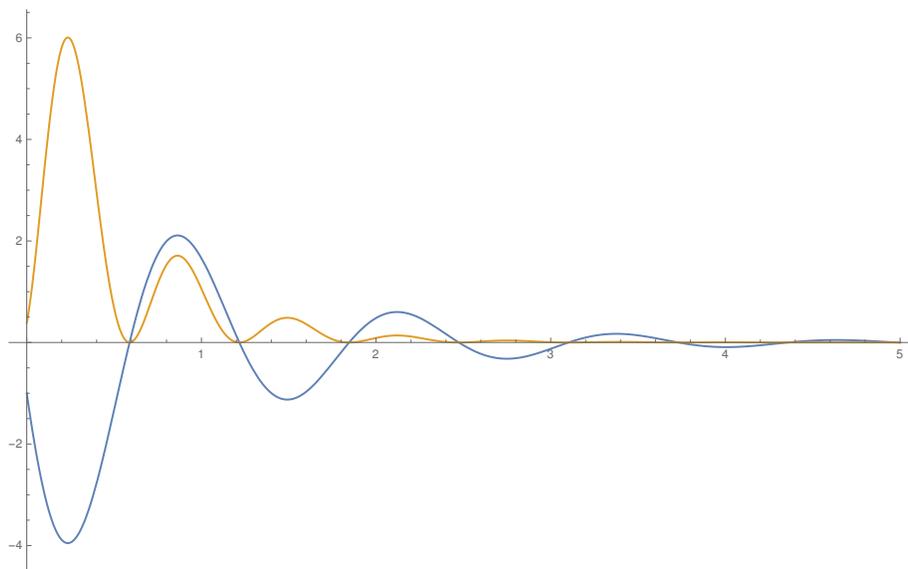


Figure 4: The velocity of the damped harmonic oscillator, shown in blue, in the case that $\beta = 1$, $m = 1$, $\Omega = 5$, $A = 1$, and $B = 0$. The rate at which energy is being lost to dissipation is shown by the yellow curve. Notice that the energy loss is the greatest when the particle has the largest velocity.

Having thoroughly analysed the cases of weak damping and strong damping, there is one more case which we must consider. In the event that our damping parameter is precisely equal to the frequency of undamped oscillations,

$$\beta = \omega, \quad (88)$$

which is known as **critical** damping, our solution technique yields the result

$$\alpha_{\pm} = -\beta \pm \sqrt{\beta^2 - \omega^2} = -\beta. \quad (89)$$

Thus, our method only yields **one** solution, given by

$$y_1(t) = e^{-\beta t}. \quad (90)$$

However, we know that our differential equation should have two linearly independent solutions. It is not clear what form this second solution should take. However, it is reasonable to assume that it probably also has some exponentially decaying behaviour - we just don't know what the function multiplying the exponential is. So, let's quantify this state of affairs by writing our guess for the second solution as

$$y_2(t) = f(t) e^{-\beta t}, \quad (91)$$

where $f(t)$ is some function we don't know.

However, assuming this guess is correct, we can learn something about $f(t)$ by plugging this guess into our original differential equation. Using the chain rule, we have

$$\dot{y}_2(t) = [f'(t) - \beta f(t)] e^{-\beta t}, \quad (92)$$

along with

$$\ddot{y}_2(t) = [f''(t) - 2\beta f'(t) + \beta^2 f(t)] e^{-\beta t}, \quad (93)$$

and so inserting our guess into our differential equation yields

$$\ddot{y}_2 + 2\beta\dot{y}_2 + \beta^2 y_2 = f''(t) e^{-\beta t} = 0. \quad (94)$$

Cancelling the exponential factor, this simply yields

$$f''(t) = 0. \quad (95)$$

This (incredibly simple) differential equation has the general solution

$$f(t) = A + Bt, \quad (96)$$

and so the most general solution to this critically damped oscillator is

$$y_2(t) = (A + Bt) e^{-\beta t}, \quad (97)$$

where A and B are to be determined by initial conditions. Once again, an educated guess about the long-time behaviour of the system helped us arrive at the correct solution to the differential equation. Of course, the nice thing

about solving differential equations is that it is usually quite easy to verify that a proposed solution satisfies the equation, and we also know exactly how many solutions we need to find. Thus, we can be as non-rigorous and cavalier as we want when finding (read: guessing) a solution, so long as we check afterwards that it does indeed satisfy the differential equation.

To understand why this damping case is known as critical damping, we can compare the characteristic time for the various values of β . For small values of β , we know we have the weakly damped solution

$$x(t) = x_* + e^{-\beta t} [A \cos(\Omega t) + B \sin(\Omega t)], \quad (98)$$

with an oscillation amplitude that decays according to the characteristic time

$$\tau_{\beta < \omega} = 1/\beta. \quad (99)$$

For the critically damped case, we can see that the overall exponential decay factor takes the same form, and so the characteristic time in this case is also given by $1/\beta$. For the strongly damped case, our solution takes the form

$$x(t) = x_* + e^{-\beta t} [C_+ e^{+\sqrt{\beta^2 - \omega^2} t} + C_- e^{-\sqrt{\beta^2 - \omega^2} t}]. \quad (100)$$

The second exponential factor is the one which decays the most quickly, and so at large times, the first term is what dominates the time evolution. Thus, the characteristic time in this case is given by

$$\tau_{\beta > \omega} = \frac{1}{\beta - \sqrt{\beta^2 - \omega^2}}. \quad (101)$$

A plot of the characteristic time is shown in Figure 5. We can see that the characteristic time for decay decreases up until the case of critical damping at $\beta = \omega$, at which point it begins to increase again. Thus, perhaps somewhat counter intuitively, increasing the drag parameter beyond the critical value actually *increases* the amount of time it takes for the oscillator to return to its equilibrium position. This fact is an important consideration in designing physical systems which we want to return to their starting positions as quickly as possible (for example, shock absorbers in a car).

These results we have derived here constitute a fairly thorough analysis of the (damped) harmonic oscillator. In the coming lectures we'll understand how these results change as we add a variety of complicating factors to our system.

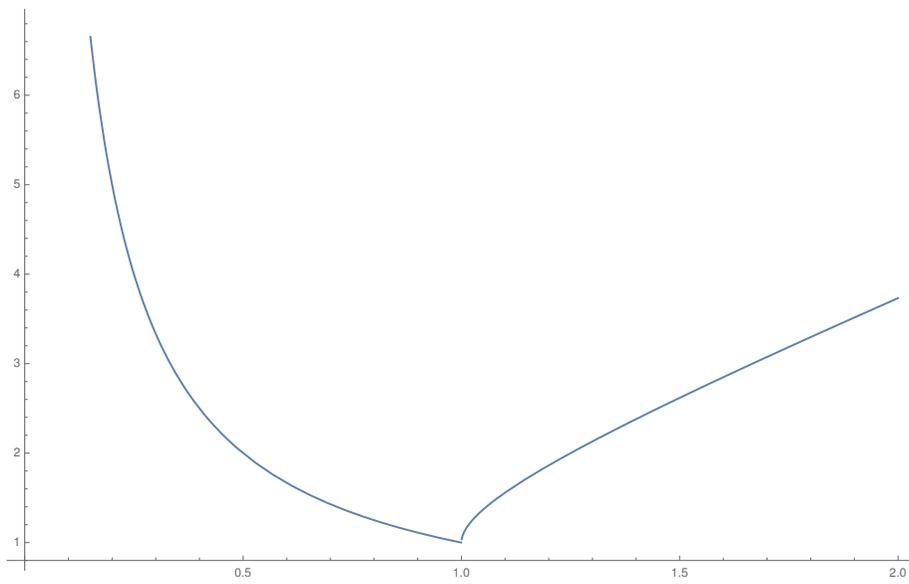


Figure 5: The characteristic time as a function of the damping parameter β , for the case $\omega = 1$. Notice that the characteristic time decreases up until the point of critical damping at $\beta = \omega$.