The Lorentz Transformation

During the fourth week of the course, we spent some time discussing how the coordinates of two different reference frames were related to each other. Now that we know about the existence of time dilation and length contraction, we might suspect that we need to modify the results we found when discussing Galilean coordinate transformations. Indeed, we will find out that this is the case, and the resulting coordinate transformations we will derive are often known as the **Lorentz transformations**.

To derive the Lorentz Transformations, we will again consider two inertial observers, moving with respect to each other at a velocity v. This is illustrated in Figure 1. This time, we will refer to the coordinates of the train-bound observer with primed quantities. We will assume that the two observers have synchronized their clocks so that $t = t' = 0$, and at this point in time, the two origins coincide. We will also assume that the two coordinate systems move with respect to each other along the x-direction - in particular, the ground-based observer witnesses the train moving in the positive x-direction. We now imagine that some event occurs inside of the train (or anywhere else in space, for that matter), which is described by the train-bound observer using the coordinates (x', y', z', t'). A common example might be that a firecracker explodes at some location in space, at some point in time. How do these coordinates relate to the ones that the ground-based observer would use to describe the event, (x, y, z, t)?

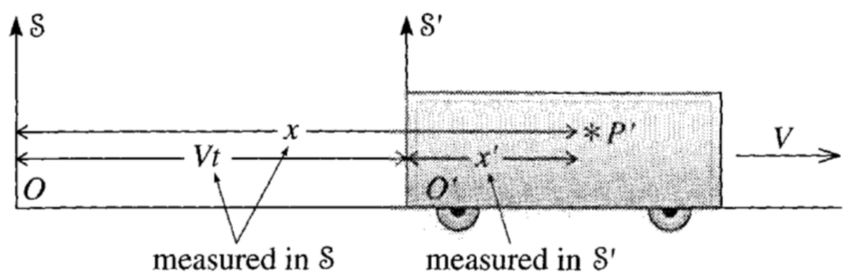

Figure 1: Taylor Figure 15.4: Two inertial observers experiencing the same event.

First, we notice that we must have

$$y' = y \ ; \ z' = z. \tag{1}$$

This is because lengths perpendicular to the direction of motion cannot contract without leading to a variety of physical paradoxes. To understand why, imagine that as the train is sitting still at a station on the ground, its height is just barely taller than a tunnel which is further down the track. After the train leaves the station and is moving at some speed v, if the ground-based observer were to witness a contraction of the train's height, he might believe that the train is now shorter than the tunnel, and can make it through the tunnel without crashing. However, the train-based observer will believe that the tunnel is now even shorter than it was before, and still cannot accommodate the train. Thus, the two observers disagree as to whether the train does or does not crash, which is nonsensical.

We do know, however, that the direction parallel to the motion will contract. Now, the coordinate x' is the *distance* between the location of the event, as measured by the train-based observer, and the origin O' that the train-based observer has set up for his coordinates. This **same** distance, according to the ground-based observer, is $x - vt$. The reason for this is that according to the ground-based observer, the distance between the origins O and O' is vt, while the distance to the event is simply x. This is illustrated in Figure 1. Therefore, we have two competing expressions, according to the two different observers, for the physical distance between O' and the spatial location of the event. Since we know that physical distances will contract according to the length contraction formula, we find

$$x - vt = x'/\gamma. \tag{2}$$

Notice that this formula tells us that the ground-based observer measures a shorter length, as he should. Rearranging this expression, we find

$$x' = \gamma\left(x - vt\right). \tag{3}$$

Thus, if we know the time and location of an event in the coordinates of the ground-based observer, we know the location of the event according to the train-based observer.

As for the time coordinate, we can employ a clever trick based on the above result. Since both observers are inertial, and the situation between them is entirely symmetric, we should be able to consider the inverse transformation for the position coordinate. Since this simply amounts to swapping the roles of the primed and unprimed coordinates, along with sending $v \to -v$ (since the train-based observer witnesses the ground moving in the opposite direction), similar considerations lead to the result

$$x = \gamma\left(x' + vt'\right). \tag{4}$$

If we now substitute in our result for x', and solve for t', the result we find is

$$t' = \gamma\left(t - vx/c^2\right). \tag{5}$$

Therefore, if we know the coordinates of an event as described by the ground-based observer, we know that we can find the coordinates of the event as described by the train-based observer, according to the formulas

$$\begin{aligned} x' &= \gamma\left(x - vt\right) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - vx/c^2\right) \end{aligned} \tag{6}$$

These expressions together are known as a **Lorentz transformation**. While we have derived them for a specific orientation of the two coordinate systems, deriving them in the more general case is straight-forward (although unnecessary for our purposes). Again, we can find the inverse transformation simply by swapping the roles of the coordinates, and sending $v \to -v$,

$$\left.\begin{aligned} x &= \gamma\left(x' + vt'\right) \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + vx'/c^2\right) \end{aligned}\right\} \tag{7}$$

From these equations, we can derive all of the previous results regarding time dilation and length contraction, along with some new effects which we will discuss shortly.

Notice that these equations also allow us to find time and space **differences**, since we can always write, for example,

$$\Delta t' = t'_2 - t'_1 = \gamma\left(t_2 - vx_2/c^2\right) - \gamma\left(t_1 - vx_1/c^2\right) = \gamma\left(\Delta t - v\Delta x/c^2\right). \tag{8}$$

The transformation for time and space durations is the same as the transformation for time and space coordinates, since the transformation is a linear one. The expression we've found above is actually a more general result for time dilation than we found previously. In the last lecture, we often considered events which occurred in the same physical location in one of the two frames. For example, if two events occur in the same location in the unprimed coordinate system, we would have

$$\Delta x = 0 \;\Rightarrow\; \Delta t' = \gamma \Delta t, \tag{9}$$

which is the result we found last time. More generally, however, we see that time durations in one frame depend on a combination of time and space differences in the other frame.

Notice that when $v \ll c$, we have

$$v \ll c \;\Rightarrow\; \gamma \approx 1\ ;\ v/c \ll 1 \tag{10}$$

Thus, assuming that $\Delta x/c$ is not too large, our transformation in this case reduces to

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{11}$$

Thus, the small-velocity limit of the Lorentz transformation is the Galilean transformation, which of course it must be. For hundreds of years, it was widely believed that the Galilean transformation was correct, because according to every experiment ever conducted, it was correct. If Special relativity is to be a correct theory of nature, it must explain the outcomes of all experiments, including these ones. The fact that the Lorentz transformation reduces to the Galilean one in this limit is proof that Special Relativity can account for those experiments, ones which were of course conducted long before any physicists knew anything about the postulates of special relativity.

Relativistic Velocity Addition

As we did in the Galilean case, we may ask how this change of coordinates affects the velocities that observers would measure for material objects. Again, we should expect that the resulting velocity addition formula is no longer the same as it was for the Galilean case. With the Lorentz transformations we have found, finding this new addition formula is a relatively straight-forward task.

Let's imagine that with respect to the ground observer, the velocity of some object is

$$\mathbf{w} = \frac{d\mathbf{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right), \tag{12}$$

where I've used the letter w to refer to the velocity of this third object, as to not confuse it with the relative velocity between the two observers. Focusing first on the x-component of the velocity, the train observer would find, in the primed set of coordinates,

$$w'_x = \frac{dx'}{dt'}. \tag{13}$$

Now, because we have a transformation which relates the primed to unprimed coordinates, we could use the chain rule to write the above derivative in terms of the unprimed coordinates. However, the easier way to proceed is to use the Lorentz transformation for an infinitesimal time step, so that

$$dx' = \gamma\left(dx - vdt\right) \; ; \; dt' = \gamma\left(dt - vdx/c^2\right). \tag{14}$$

Using these two expressions, we find

$$w'_x = \frac{\gamma\left(dx - vdt\right)}{\gamma\left(dt - vdx/c^2\right)}. \tag{15}$$

Cancelling the factors of γ and dividing top and bottom by dt, we find

$$w'_x = \frac{\left(dx/dt - v\right)}{\left(1 - v\left(dx/dt\right)/c^2\right)}, \tag{16}$$

or,

$$w'_x = \frac{\left(w_x - v\right)}{\left(1 - vw_x/c^2\right)}. \tag{17}$$

This is the relativistic velocity addition formula for the x-component.

Notice that the numerator of the velocity addition formula is the same as the Galilean expression, but now there is a term in the denominator which depends in a non-trivial way on the two velocities, as well as the speed of light c. While this term approaches one for small enough velocities, the fact remains that our expression is fundamentally different from the Galilean case. This result most certainly contradicts our usual intuition about the world, and at first seems impossible. If a stream is flowing past me at ten meters per second, and a boat is moving down stream with the water, moving with respect to the water at ten meters per second, how is it possible that I do not see the boat moving at twenty meters per second? In one second, the distance the boat moves through the water is ten meters, while the water around the boat is carried another ten meters down the shore, so how is the total distance travelled down stream not equal to twenty meters? The fundamental flaw in our reasoning is that we are assuming that 10 meters per second on the ground means the same thing as ten meters per second in the water. But this is not so. When the observer on the boat observes himself to be moving through the water at ten meters per second, *his notion of one second is* ***not*** *the same as our ground-based notion of one second.* At its core, the relativistic velocity addition formula reflects the fact that there is no universal notion of time in the universe.

As a check on our result, let's make sure it agrees with our basic postulate of special relativity, which is that all observers agree on the speed of a light beam, which must be c. If our ground-based observer witnesses a light beam with velocity

$$w_x = c, \tag{18}$$

then our velocity addition formula tells us that

$$w_x' = \frac{(c-v)}{(1-v/c)} = \frac{c\,(c-v)}{(c-v)} = c, \tag{19}$$

where in the second equality we've multiplied top and bottom by c. Thus, our velocity addition formula is consistent with the invariance of the speed of light. While the invariance of c is the most striking feature of the velocity transformation, the more general formula we have found allows us to perform velocity transformations for the motion of any arbitrary object.

Now, if this were the Galilean case, we would be content to stop here - we would have found everything we need to know about the velocity transformation, since it is "obvious" that only velocities along the x-direction should be affected by the coordinate transformation. However, in the relativistic case, this is not so. To see why, remember that while it is true that

$$dy' = dy, \tag{20}$$

it is emphatically **not** the case that

$$dt' = dt. \tag{21}$$

Therefore, w_y' is in fact **not** equal to w_y, but rather

$$w_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma\,(dt - v dx/c^2)} = \frac{w_y}{\gamma\,(1 - v w_x/c^2)}. \tag{22}$$

Notice that the term in the denominator is **not** a typo - the expression for w_y' depends on both w_y and w_x. Similarly, for the z-component,

$$w_z' = \frac{w_z}{\gamma\,(1 - v w_x/c^2)}. \tag{23}$$

Thus, we have found that in the relativistic case, not only must we revise our expression for the x-component of the velocity, but we must also consider the entirely new phenomenon in which the y and z components of the velocity are also affected, as a consequence of time dilation. Notice, however, that in the limit of small velocities, we recover the usual result that

$$w'_y \approx w_y \ ; \ w'_z \approx w_z, \tag{24}$$

since in this limit, the denominator term becomes very close to one.

As an application of these results, let's consider the case that the train-based observer is moving at a speed of $v = 0.8c$ with respect to the ground. While moving, the train-based observer fires a rocket straight forward in front of him, with a velocity he measures to be

$$w'_x = 0.6c. \tag{25}$$

What velocity does the ground-based observer measure for the rocket? As for the Lorentz transformations, we can also invert our velocity addition formula, by simply swapping the roles of the two coordinates, and also sending $v \to -v$. Thus, we find

$$w_x = \frac{(w'_x + v)}{(1 + v w'_x / c^2)} = \frac{1.4c}{1 + 0.8 \cdot 0.6} \approx 0.95c. \tag{26}$$

Despite adding two velocities which should naively add up to more than c, we find a result less than c, as we must to ensure causality.

Rotations in Space-Time

Unlike the Galilean transformation, we have found that the Lorentz transformation is an operation which mixes together space and time coordinates. In fact, since the transformation is a linear one, we can write it in the form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \tag{27}$$

where we've added a factor of c to both sides of the equation for the time coordinates. Written this way, we notice a striking similarity between the Lorentz transformation and a typical rotation, which we studied two weeks ago. In fact, pursuing this parallel reveals something quite amazing about the structure of our universe, and was one of the key insights which led Hermann-Minkowski to the idea of four-dimensional **space-time**.

Before pursuing this analogy further, however, it pays to modify our notation slightly. First, we will define

$$\beta = v/c, \tag{28}$$

which measures the velocity as a fraction of the speed of light. Furthermore, we will write

$$x_0 = ct \ ; \ x_1 = x \ ; \ x_2 = y \ ; \ x_3 = z, \tag{29}$$

and combine all four coordinates into a single four-component column-vector,

$$x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{30}$$

Lastly, we will define

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{31}$$

With this notation, our transformation becomes

$$x' = \Lambda x. \tag{32}$$

The four component object x is referred to as a **four-vector**. It was Hermann Minkowski's idea that in view of the Lorentz transformations of relativity, we should no longer think in terms of space and time as separate quantities, but that it is more natural to formulate physics in terms of a four-dimensional object known as space-time. In honor of Minkowski's revolutionary work, the four-dimensional space which includes both time and three-dimensional physical space is often referred to as **Minkowski space**. This type of thinking was crucial in helping Einstein formulate the proper ideas necessary for developing the theory of general relativity.

In order to make the analogy between Lorentz transformations and rotations even more explicit, notice that because the Lorentz factor is always greater than or equal to one, it is possible to define a "hyperbolic angle" ϕ such that

$$\gamma = \cosh\phi. \tag{33}$$

This quantity ϕ is often known as the **rapidity** of the transformation. With this definition, some simple algebra leads us to the conclusion that

$$\beta\gamma = \sinh\phi. \tag{34}$$

Thus, in terms of the rapidity,

$$\Lambda = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{35}$$

Because we can write the Lorentz transformation in this form, it is often said that Lorentz transformations are hyperbolic rotations in four-dimensional space-time.

The Invariant Scalar Product

Notice that using the notation we have set up for Lorentz transformations, we can easily write an ordinary three-dimensional rotation as

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 \quad 0 \quad 0 \\ 0 & \\ 0 & \mathbf{R} \\ 0 & \end{pmatrix}. \tag{36}$$

Thus, ordinary rotations in three-dimensional space, which are sometimes referred to as **pure rotations**, can be considered a special case of the more general Lorentz transformation. When we studied rotations two weeks ago, we discussed the fact that a rigid rotation should not affect physical lengths measured by observers. While this is indeed true for three-dimensional rotations, we have seen here that it is emphatically **not** true for Lorentz transformations which correspond to a change in velocity, which in Special Relativity are often known as **Lorentz boosts**. Because of the phenomenon of length contraction, two inertial observers may not necessarily agree on the physical length between two locations in space.

However, given the mathematical similarity between Lorentz transformations and pure rotations, we suspect that perhaps there may be some analogue to the three-dimensional dot product which we can define in the context of a more general Lorentz transformation. Since Lorentz transformations reduce to regular three-dimensional rotations when there is no change in velocity, this new scalar product should still look somewhat similar to the original dot product.

Motivated by this idea, we introduce the **invariant scalar product** between two four-vectors as

$$x \cdot y = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = -x_0 y_0 + \mathbf{x} \cdot \mathbf{y}. \tag{37}$$

Notice that it corresponds to the usual three-dimensional scalar product, with an additional correction term. To verify that our Lorentz transformation indeed leaves this quantity unchanged, we write the expression for the scalar product in another set of coordinates,

$$x' \cdot y' = -x'_0 y'_0 + x'_1 y'_1 + x'_2 y'_2 + x'_3 y'_3 = -x'_0 y'_0 + \mathbf{x}' \cdot \mathbf{y}'. \tag{38}$$

Using the Lorentz transformation, we find

$$x' \cdot y' = -\gamma^2 (x_0 - \beta x_1)(y_0 - \beta y_1) + \gamma^2 (x_1 - \beta x_0)(y_1 - \beta y_0) + x_2 y_2 + x_3 y_3. \tag{39}$$

Expanding this out and performing some algebra, we find

$$x' \cdot y' = -\gamma^2 \left(1 - \beta^2\right) x_0 y_0 + \gamma^2 \left(1 - \beta^2\right) x_1 y_1 + x_2 y_2 + x_3 y_3. \tag{40}$$

Some simple algebra reveals that

$$\gamma^2 \left(1 - \beta^2\right) = 1, \tag{41}$$

so that we have

$$x' \cdot y' = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = x \cdot y. \qquad (42)$$

Thus, we find that indeed, this "scalar product" is preserved by our Lorentz transformation.

While we have only demonstrated this fact for one particular Lorentz transformation, this invariance under Lorentz transformations holds quite generally for the invariant scalar product. In more advanced areas of physics, for example Quantum Field Theory, it is often taken as a starting assumption that the invariant scalar product should be an invariant of any physical theory. A valid Lorentz transformation is then any coordinate transformation in space-time which preserves this inner product. Thus, more generally, we say that our universe possesses **Lorentz symmetry**, and the fact that it does indeed possess this symmetry is well-established by experiment. In general, any number which is the same in all inertial frames is referred to as a **Lorentz scalar**. As we have seen, the inner-product between any two four-vectors is a Lorentz scalar.

The Light Cone

However, in striking contrast to the usual dot product, the invariant scalar product between two four-vectors need not be positive, not even when the two four-vectors are the same. This is of course a result of the minus sign in the definition. Whether or not the scalar product of a four-vector with itself is positive, negative, or zero in fact encodes important physical information about that four-vector, and understanding how this is so will allow us to say something more about the causal structure of spacetime.

To see how this is so, let's draw something called a **space-time diagram**: a digram which indicates both the space and time coordinates used by an inertial observer. Points on this diagram are now no longer points in three-dimensional space, but rather the *space-time coordinates of events.* Since of course we are not very skilled at visualizing four-dimensional objects, we will typically suppress one of the spatial coordinates, and draw one dimension of time, along with two dimensions of space. Such a diagram is indicated in Figure 2. On such a diagram, we will typically refer to the **space-time distance** as

$$s^2 = x \cdot x \equiv x^2, \qquad (43)$$

Notice that despite the common convention of naming this quantity s^2, it can of course be negative. Notice in particular that the origin O is now an **event** in space-time.

On such a space-time diagram, we can draw an object known as a **light cone**. The light cone, centred around the origin of our space-time diagram, is the "surface" in space-time which satisfies

$$s^2 = x \cdot x = -x_0^2 + x_1^2 + x_2^2 + x_3^2 = r^2 - c^2 t^2 = 0. \qquad (44)$$

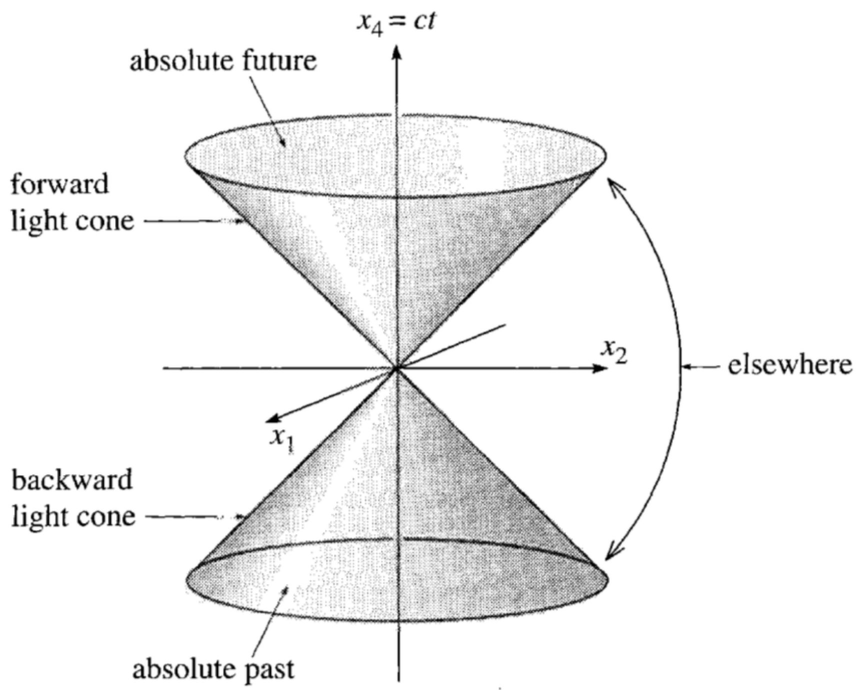

Figure 2: Taylor Figure 15.8: A light cone in space-time. Notice that Taylor uses x_4 in place of x_0.

This surface is shown in Figure 2. To understand the meaning of this object, notice that our condition is equivalent to

$$r = ct. \tag{45}$$

This is the defining equation for an expanding sphere of light - the set of all points a distance ct away from the spatial origin. Thus, if we were to imagine flashing a light bulb at the spatial origin of our coordinates, at time zero, the light cone shows the expanding sphere (or circle) of light moving away from the origin. For this reason, any vector with a scalar product of zero is referred to as a **light-like vector**. Portions of the light cone corresponding to $t > 0$ are typically referred to as the **forward light cone**, while the $t < 0$ portion is referred to as the **backward light cone**.

To be more specific, this light cone is often referred to as the light cone of O. This is because it is the light cone defined by setting the point O as the origin. Had we set the origin of our space-time coordinates at some other event P in space-time, we would be discussing the light cone of the event P.

Now, remember from our discussion of causality that in order to avoid physical paradoxes, no causal signal can propagate faster than the speed of light.

This statement can be interpreted in the context of our space-time diagram as the statement that *only events within the light cone of our space-time diagram can be in causal contact with the event at the origin.* To see why, notice that if our observer, at the place and time which corresponds to the origin of his space time diagram, wishes to send a signal out into space, the motion of this signal as a function of time certainly cannot exceed the velocity c. But given the way we have oriented our axes, with the time direction pointing vertically, velocities which are less than c correspond to lines which have a slope of **more** than 45 degrees. This corresponds to a signal which can only visit space-time events located on the interior of the future light cone. For this signal, we have

$$r < ct, \tag{46}$$

or,

$$x \cdot x < 0. \tag{47}$$

Four-vectors which satisfy the above condition are known as **time-like** vectors. The reason for this is that we can rewrite our constraint as

$$x_0^2 > x_1^2 + x_2^2 + x_3^2, \tag{48}$$

which is to say that the time portion of the four-vector is larger in magnitude than the spatial part. In fact, for any time-like vector, it is always possible to find another set of inertial coordinates such that

$$x' = \begin{pmatrix} x_0' \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{49}$$

so that the vector lies entirely in the time direction. Since all material bodies must move at a speed less than c, we say that the motion of material bodies is time-like, or that material objects move along time-like trajectories. Similarly, the set of all space-time events on the interior of the backward light cone are those events which could, in principle, have a causal effect on events which occur at the origin of our space-time diagram. Events which lie inside the forward light cone of the origin are often referred to as being part of the **absolute future** of the origin, while events which are inside the backward light cone are often referred to as being part of the **absolute past** of the origin, for reasons we will discuss momentarily.

Of course, by the same reasoning, points which are outside of the light cone are events in space-time which cannot possibly have an effect on events which occur at the origin of our space-time diagram. Any signal which propagates between the origin and a point outside of the light cone must necessarily possess a velocity which is larger than c, which we know would violate the principle of causality. For this reason, the region of space-time outside of the light cone is often referred to as **elsewhere**, since it does not have a causal influence on the origin. Vectors which lie outside of the light cone are known as **space-like** vectors, and satisfy

$$x \cdot x > 0. \tag{50}$$

Another Proof of Causality

By studying this space-time diagram, however, there is in fact another way we can come to the same conclusion about faster-than-light travel. Let's take another look at our Lorentz transformation for the time difference between two events,

$$\Delta t' = \gamma \left(\Delta t - v\Delta x / c^2 \right). \tag{51}$$

Let's assume that Δt is positive, so that our observer using the set of coordinates in our space-time diagram believes that

$$\Delta t = t_2 - t_1 > 0. \tag{52}$$

That is, event two happens **after** event one. However, due to the minus sign in the transformation, if the second term is large enough, it is entirely possible that

$$\Delta t' = t'_2 - t'_1 < 0. \tag{53}$$

That is to say, the second observer may believe that event two happens **before** event one! Thus, not only do the two observers disagree on time durations, they may also disagree on the ordering of two events.

Now, if we believe in any sort of reasonable definition of cause and effect, it certainly seems as though some event A can only have a causal influence on some other event B if event A occurs first. Otherwise, we can not make heads or tails out of which event is the cause and which event is the effect. Additionally, if we want every observer to agree on which event is the cause, and which event is the effect, it better be the case that every observer agrees on the sign of the time difference between the two events. For this reason, we are led to the conclusion that *in order for two events in space-time to be able to have a causal influence on each other, every observer must agree on the sign of the time difference between them.*

We can convert this conclusion to a more mathematical one by considering one of these two events to be the origin O of our space-time diagram. The space-time distance to any other event is

$$s^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2. \tag{54}$$

Now, if we have an inertial observer using the set of coordinates in our space-time diagram, the value x_0 corresponds to the amount of time elapsed since the event at the origin. If $x_0 > 0$, then our observer believes that this event happens after the origin of time. If $x_0 < 0$, our observer believes that the event happened before the origin of time. If $x_0 = 0$, then it occurred simultaneously with the origin of time. Whatever its value is, though, if this event is to be causally connected with the origin, *all observers must agree on the sign of* x_0. We can use this fact to put a constraint on which events can be causally connected with the origin.

Before doing so, however, we want to prove an important fact, which is that in the Lorentz transformation between any two sets of coordinates, we must always have

$$\beta < 1. \tag{55}$$

To see how, let's consider the **displacement** four-vector of a material object,

$$dx = (cdt, d\mathbf{x}) = (c, \mathbf{v})\, dt \tag{56}$$

as measured in the coordinates $\mathbf{x}$ and t of some observer. This four-vector measures the infinitesimal amount of spatial distance $d\mathbf{x}$ travelled by the object in a time dt. Now, it certainly seems as though a reasonable definition of an observer is any material object which is capable of setting up a system of coordinates in which it measures itself to be at rest. In this rest frame of the material object, we would have

$$dx = (c, 0)\, dt. \tag{57}$$

This vector is clearly time-like, satisfying,

$$dx^2 = -c^2 dt^2 < 0. \tag{58}$$

Since this fact is preserved by an arbitrary Lorentz transformation, we must have, in any other arbitrary frame

$$(dx')^2 = -c^2 (dt')^2 + (\mathbf{v}')^2 (dt')^2 = \left((\mathbf{v}')^2 - c^2\right)(dt')^2 < 0. \tag{59}$$

Thus, in any arbitrary frame, we must have

$$\left((\mathbf{v}')^2 - c^2\right) < 0 \;\Rightarrow\; \mathbf{v}' < c \;\Rightarrow\; \beta < 1. \tag{60}$$

Thus, any material object which possesses the ability to set up its own rest frame will always move at a speed less than the speed of light, according to **any** observer. Thus, any Lorentz transformation between this frame and another one will satisfy $\beta < 1$.

With this fact proven, let's start by considering events inside of the forward light cone of O. These events satisfy

$$x_0^2 > x_1^2 + x_2^2 + x_3^2\;;\; x_0 > 0. \tag{61}$$

Now, since the quantity $x \cdot x$ is a Lorentz scalar, its value in any inertial frame is the same, and thus

$$x \cdot x < 0 \;\Rightarrow\; x' \cdot x' < 0, \tag{62}$$

where x' is the four-vector in any other frame. So the defining equation which determines the forward light cone of O is invariant - every observer agrees as to what set of events corresponds to the forward light cone of O. Additionally, if we perform a Lorentz boost, the time coordinate will transform as

$$x_0' = \gamma\,(x_0 - \beta x_1)\,. \tag{63}$$

Since $\beta < 1$, and since the time-like constraint enforces $x_1 < x_0$, this implies that x_0' is positive, and thus also corresponds to an event in the future of the origin O. Therefore, if an event is in the forward light cone of the origin O in one frame, it will be in the forward light cone of O according to any other observer. Thus, the statement P is in the future of O, for any other event P, is a Lorentz invariant statement, so long as P is in the forward light cone of O. A similar conclusion can be drawn for the backward light cone. For this reason, the forward and backward light cones are often referred to as the absolute future and past of the event O. Notice that while we have shown this for only one specific example of a Lorentz boost, a more general argument shows that this holds for all possible Lorentz transformations.

Now let's ask about points outside of the light cone of O. These events satisfy

$$x \cdot x > 0. \tag{64}$$

Again, because the scalar product is invariant, this statement is true in any other inertial frame,

$$x' \cdot x' > 0. \tag{65}$$

In particular, let's assume we are considering an event which satisfies

$$x_0 > 0, \tag{66}$$

so that our observer believes that this event is in the future of O. Do all other observers agree that this point is in the future of O? For simplicity, let's assume we've oriented our axes such that

$$x = \begin{pmatrix} x_0 \\ x_1 \\ 0 \\ 0 \end{pmatrix}, \tag{67}$$

which we can always do by performing a rotation of our spatial axes. Now, if we perform a Lorentz boost, we find

$$x_0' = \gamma \left(x_0 - \beta x_1 \right). \tag{68}$$

Now, since this space-time point in question is outside of the light cone,

$$x_1 > x_0. \tag{69}$$

Therefore, we can choose

$$\beta = x_0 / x_1 < 1 \tag{70}$$

as a valid boost parameter, and find that

$$x_0' = 0. \tag{71}$$

Therefore, there is a choice of inertial reference frame in which the two events are actually simultaneous! Even worse, we could have chosen a value for β which satisfied

$$x_4 / x_1 < \beta < 1, \tag{72}$$

and in this case found

$$x'_0 < 0. \tag{73}$$

This implies that there exists an inertial observer which would claim that this space-time event actually precedes the origin. Thus, for space-time events which are outside of the origin, equally valid observers will **disagree** as to the time ordering of these events. Thus, space-time events outside of the light cone cannot be causally connected to the origin.

Thus, by arguing that any two events which are causally connected must have an absolutely unambiguous time ordering, we have found that only events within the light cone of the origin are capable of having a causal influence on it. However, by the definition of the light cone, this is just the statement that causal information cannot propagate faster than the speed of light, since any point which lies outside of the light cone of the origin would need to communicate with the origin via a signal which propagated faster than c. Thus, we again arrive at our conclusion that no causal influence can propagate faster than the speed of light.