World Lines

In the last lecture, we introduced the notion of four-dimensional space-time, and used it to describe Lorentz transformations as hyperbolic rotations, and also to discuss the causal structure of light cones. However, space-time diagrams are also very useful for describing the trajectories of material objects, especially when making quantitative statements about the proper time experienced by an observer.

As an example, let's again consider an inertial observer, describing the events he witnesses using the coordinates (x, y, z, t). For concreteness, let's imagine there is a material body he is observing, which follows the one-dimensional trajectory

$$x(t) = \frac{c}{\sqrt{8}}\left(2t - \lambda c t^2\right) \;\; ; \;\; 0 < t < 2/\lambda c \tag{1}$$

where c is the speed of light, and λ is some parameter with units of inverse length. A plot of the position and velocity as a function of time is indicated in Figure 1. This describes a particle which leaves the origin at time zero, travels for a ways in the positive direction, turns around, and then comes back to the origin.

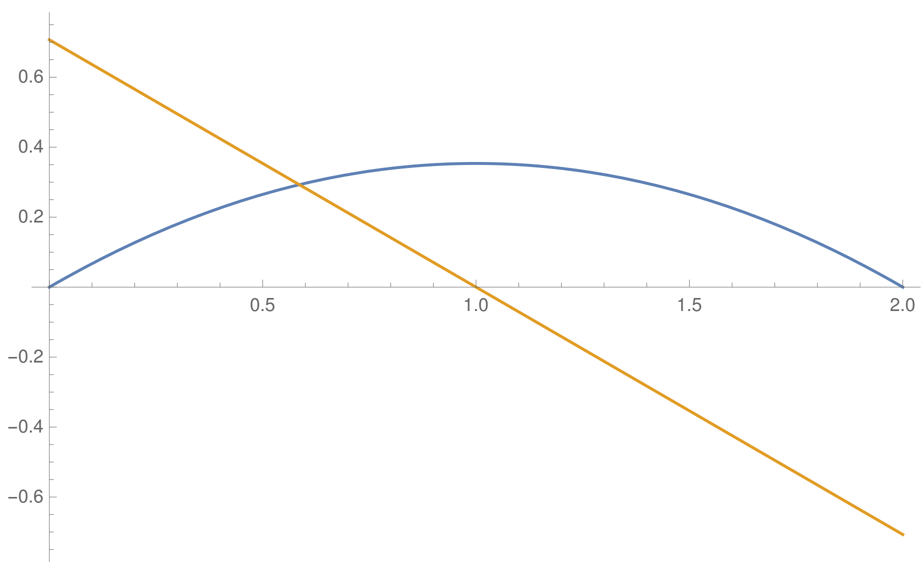

Figure 1: The position and velocity of our material object as a function of time. The blue curve is λx, while the orange curve is v/c, both functions of λct.

In addition to plotting the position and velocity of the particle as a function of time, we can also indicate the particle's trajectory on a space-time diagram, using something known as a **world-line**. The world-line of a particle is just the curve in space-time which indicates its trajectory. For our particle, this is shown in Figure 2. The world-line of the material particle is indicated in black, while we have also shown, in red, the world-line of the inertial observer. Notice that in his own set of inertial coordinates, the world-line of the observer is just a vertical line - his position for all time is zero while time ticks steadily forward.

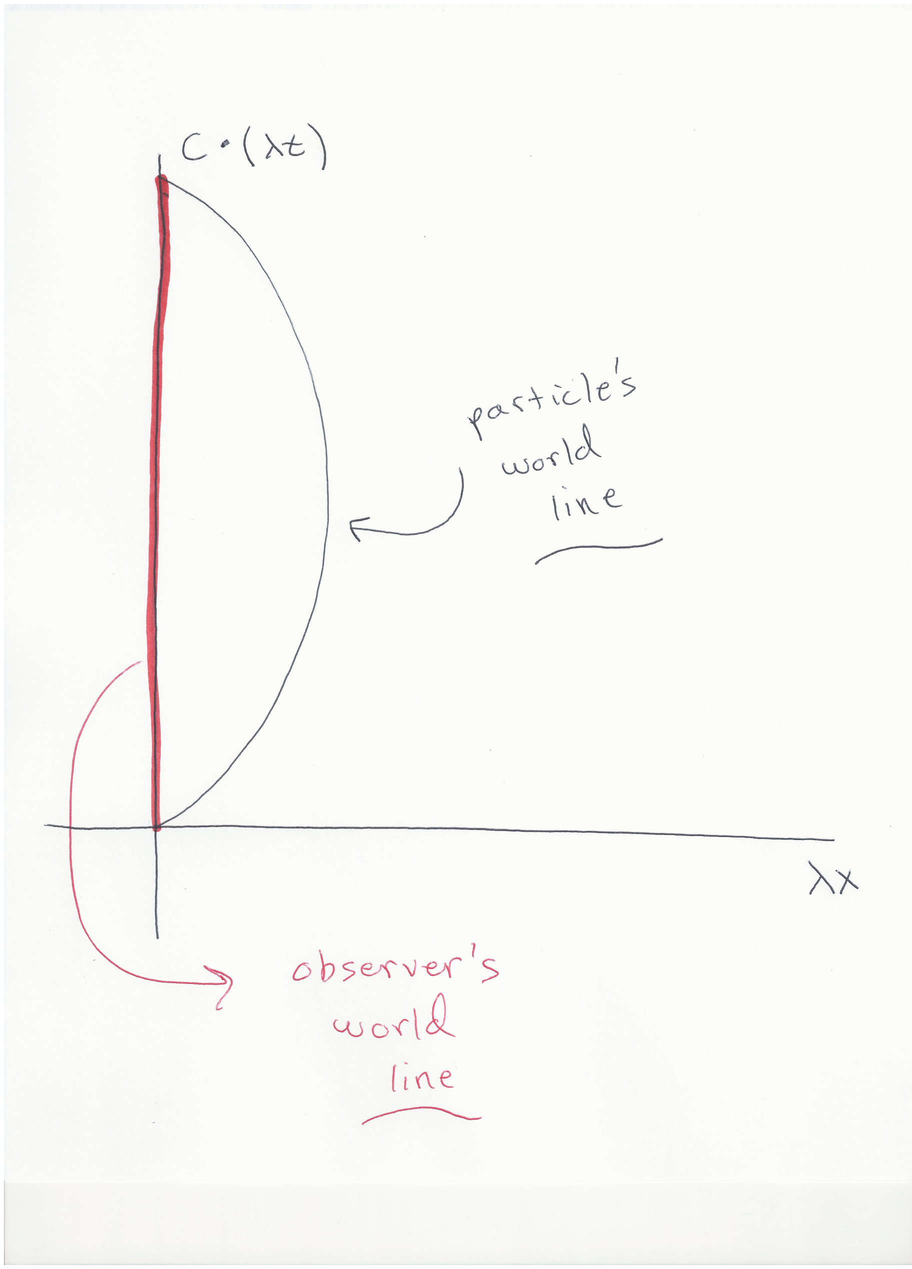

Figure 2: The world-line of our material object.

Now that we've introduced the idea of a world-line, what can we do with it? To give one answer to this question, let's recall that in the last few lectures, we have been pursuing an analogy in which we view space and time as one sort of "geometric" object, known as space-time. We have also typically been interested in objects which are invariant, or that do not depend on a particular choice of inertial coordinates. In regular three-dimensional geometry, the length of a line segment is something which is independent of coordinates. Thus, pursuing our analogy further, let's consider the space-time "length" along an infinitesimal portion of the curve, given by

$$\frac{1}{c}\sqrt{-ds^2} = \frac{1}{c}\sqrt{c^2 dt^2 - dx^2}. \tag{2}$$

Notice that in my definition, I've inserted a minus sign under the square root, since for a particle moving with a velocity less than c, ds^2 will be negative along the entire portion of the curve (remember, particles moving with velocities less than c follow time-like vectors).

What is the physical significance of this space-time length I have introduced? To answer that question, let's rewrite our quantity, as

$$\frac{1}{c}\sqrt{-ds^2} = \sqrt{dt^2 - dx^2/c^2} = \sqrt{dt^2\left(1 - dx^2/c^2 dt^2\right)} = \sqrt{1 - v\left(t\right)^2/c^2}\ dt. \tag{3}$$

But this is nothing other than the infinitesimal amount of proper time that the material object experiences!

$$dt' = \sqrt{1 - v\left(t\right)^2/c^2}\ dt = \gamma dt. \tag{4}$$

Therefore, if we integrate the space-time length of the entire world-line, we have

$$\int \frac{1}{c}\sqrt{-ds^2} = \int_0^{2/\lambda c} \sqrt{1 - v\left(t\right)^2/c^2}\ dt = \int_0^{T'} dt' = T'. \tag{5}$$

Thus, *the space-time length of a world-line is precisely equal to the amount of time the particle experiences while travelling this trajectory.* In particular, if an observer were travelling over this trajectory, T' would tell us what his proper time is, or how much time he experiences on his own which. For this reason, the space-time length of a world-line is typically referred to as the proper time of that world-line.

Notice that because the space-time length is something which is manifestly invariant (we showed in the last lecture that the scalar product was indeed Lorentz invariant), the answer we will find for this computation is independent of which set of inertial coordinates we use. We can see that this must be so, in order for the theory to be sensible at all. As the particle follows a trajectory in which it meets up with our observer, travels some distance away, and then meets up with our observer again, it will experience some amount of time. Certainly, the amount of time the particle experiences must be agreed upon by all observers. If I take a journey from the Earth to Mars, and measure some amount of time on my watch, and then record the number, anyone inquiring as to how long the journey took could simply ask me. We would indeed have a nonsensical situation if different observers disagreed on how much time I experienced.

If we want to be specific, we can compute the value for the specific trajectory given above, with

$$\frac{1}{c}v\left(t\right)=\frac{2}{\sqrt{8}}\left(1-\lambda ct\right)\ ;\ 0<t<2/\lambda c. \tag{6}$$

Performing the integration above for this velocity, we find

$$T'=\frac{\left(2+\pi\right)}{4\sqrt{2}}\frac{2}{\lambda c}\ \approx\ 0.908914\frac{2}{\lambda c}. \tag{7}$$

Notice that the time experience by the accelerated object is **less** than the time experienced by the inertial observer, as we know must be the case. But now we have a nice graphical way of helping us remember how to compute the proper time - simply choose a handy set of inertial coordinates, and integrate the space-time length over the world-line. While it is not immediately obvious, it turns out that given any two events in space-time, among all possible curves we can draw between them, the curves with the **longest** possible proper time are straight-lines - in other words, particles moving at constant velocity. This is just to say that inertial observers experience the longest possible proper time between two points in space-time. This result will be important to keep in mind later, when we give a brief introduction to General Relativity.

Four-Velocity

Continuing our geometric analogy, we know that for a particle following a curve in three-dimensional space, the velocity of the particle is

$$\mathbf{v}\left(t\right)=\frac{d\mathbf{r}}{dt}=\frac{d}{dt}\left(x,y,z\right). \tag{8}$$

Additionally, the arc length that the particle traces out as it follows the curve is

$$s\left(t\right)=\int_0^t\left|\mathbf{v}\left(t'\right)\right|dt'=\int_0^t v\left(t'\right)dt'. \tag{9}$$

The distance traced out by the particle is certainly a monotonically increasing function of time. Because of this, we can always invert $s\left(t\right)$, in order to find $t\left(s\right)$. Using this, it is always possible to instead parametrize the curve in three-dimensional space by using the arc length along the curve,

$$\mathbf{r}\left(t\right)=\left[x\left(t\right),y\left(t\right),z\left(t\right)\right]\ \rightarrow\ \mathbf{r}\left(s\right)=\left[x\left(s\right),y\left(s\right),z\left(s\right)\right]. \tag{10}$$

In other words, we can just change variables from the time elapsed, t, to the distance travelled so far, s. While it may not be obvious, it turns out to be true that if we differentiate the position with respect to s instead of t,

$$\widetilde{\mathbf{v}} = \frac{\mathbf{r}(s)}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right), \tag{11}$$

then we will always have

$$|\widetilde{\mathbf{v}}| = 1 \tag{12}$$

at all points along the curve. Notice that this is consistent with our expression for the total length, since

$$s = \int_0^s |\widetilde{\mathbf{v}}| \, ds' = \int_0^s 1 ds' = s. \tag{13}$$

While in physics we are usually interested in how the motion of a particle evolves with time, if we are simply interested in the geometry of a curve, it is often more convenient to use the arc length as the parameter along the curve. The resulting “velocity” that we find has a variety of nice properties, such as always having a norm of one (using the arc length as a parameter has a variety of other nice properties beyond the scope of our discussion, but they are properties which are incredibly convenient when studying the **differential geometry** of curves, a topic which is a very important one in mathematics). For this reason, let’s see if we can use this idea to further our geometric analogy. Let’s see if we can define a **four-velocity** for a world-line, which is of course a curve in a four-dimensional space-time. For a given world-line, the trajectory we need to differentiate is the four-vector,

$$x = (x_0, x_1, x_2, x_3). \tag{14}$$

Now, since we want to pursue the analogy of using the “arc length” as our parameter, we can define the **four-velocity** of the world line as the derivative of the particle’s four-vector, with respect to the particle’s proper time,

$$u(\tau) = \frac{dx}{d\tau} = \left(\frac{dx_0}{d\tau}, \frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}\right). \tag{15}$$

We’ve adopted the notation that τ is the space-time length along the world line, which is of course the amount of proper time which has elapsed as the particle follows the world line. Now, remember that in terms of the time coordinate, t, of the inertial observer, we found the relation

$$dt' = d\tau = \frac{dt}{\gamma}. \tag{16}$$

Therefore, using the chain rule, we can write the four-velocity as

$$u(t) = \gamma \frac{d}{dt}(ct, x_1, x_2, x_3). \tag{17}$$

Thus, in terms of the normal three-velocity we are more familiar with, we can write the four-velocity as

$$u(t) = \gamma (c, \mathbf{v}(t)). \tag{18}$$

We will typically refer to

$$u_0 = \gamma c \tag{19}$$

as the **time-component** of the four-velocity, and

$$\mathbf{u} = \gamma \mathbf{v} \tag{20}$$

as the **spatial component**, or **three-component**, of the four-velocity.

So now we have a new object we've defined called the four-velocity, and we've found how to relate it to the more familiar quantities we usually work with. So why is this useful? Well, it turns out that the four-velocity, not surprisingly, is itself a four-vector. Up until now, we've been considering four-vectors which involve the position and time coordinates of an event,

$$x = (x_0, x_1, x_2, x_3)\,. \tag{21}$$

We've found that if we want to write the four-vector in the components of another inertial observer, x', we can find these new coordinates by simply applying a Lorentz Transformation,

$$x' = \Lambda x, \tag{22}$$

where

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

However, we can check for ourselves that if we have the four-velocity in one set of inertial coordinates, u, then if we write the same quantity in another set of coordinates, u', the two will be related by

$$u' = \Lambda u. \tag{24}$$

This transformation rule is the same as for the four-vector of a space-time event. Thus, anything which satisfies this type of transformation rule, we will in general refer to as a four-vector.

The fact that the four-velocity is a four-vector is significant, because working with quantities and objects which transform in a nice way under Lorentz transformations is very handy. In the next few sections we will see some concrete examples of why this is so. Before moving on to consider those examples, however, we want to notice something important about the four-velocity, which is that its inner product is invariant. Remember that under an arbitrary Lorentz transformation, the scalar product between any two space-time four-vectors is invariant,

$$x' \cdot y' = (\Lambda x) \cdot (\Lambda y) = x \cdot y. \tag{25}$$

A transformation is a valid Lorentz transformation only if it satisfies the above condition, for **all** space-time four-vectors. Because the four-velocity also transforms in this way, however, we can show that for any two arbitrary four-velocities,

$$u' \cdot w' = (\Lambda u) \cdot (\Lambda w) = u \cdot w = -u_0 w_0 + \mathbf{u} \cdot \mathbf{w}. \tag{26}$$

We could prove this for ourselves by going back to the original definition of the four-velocity, or we could simply argue that since the four-velocity "rotates" according to Λ, its space-time inner product must also be invariant. In particular, for the inner product of the four-velocity of a particle with itself, we find

$$u \cdot u = -u_0^2 + \mathbf{u}^2 = -\gamma^2 c^2 + \gamma^2 \mathbf{v}^2 = -c^2, \tag{27}$$

which is about as invariant as they come!

Four-Momentum

Now that we have spent quite a bit of time discussing what could be refereed to as "relativistic kinematics," we want to finally turn to the discussion of dynamics. One of the most basic dynamical laws we know from regular Newtonian mechanics is the conservation of momentum of an isolated system. We typically say that any theory of physics which does not make reference to any "special location" in space possesses something known as **translational invariance**. For reasons which you will come to understand in Physics 104, any theory which possesses this property will have a corresponding vector quantity, which we refer to as momentum, which will be conserved for an isolated system. Special relativity, as we have formulated it, is certainly such a theory, and so we expect that there should be some vector quantity which we can associate as momentum, which will be conserved. However, it is entirely possible that our expression for what the momentum looks like could be different in special relativity, as so many other expressions have been.

Without all of the proper tools of Lagrangian mechanics, however, at best we can only make educated guesses as to what the "correct" definition of momentum should be. Of course, there really is no such thing as a correct definition, but at a bare minimum, we want our quantity which we call momentum to possess the property that it is a three-vector, and that **according to any inertial observer**, it is conserved for any isolated system. As a fist guess, we might simply guess that the momentum of a particle is just the same quantity as in Newtonian mechanics,

$$\mathbf{p} \stackrel{?}{=} m\mathbf{v}. \tag{28}$$

In this expression, $\mathbf{v}$ is the usual three-velocity of a particle, while the quantity m is the **rest mass**. The rest mass of a particle is just that: it is the mass that someone would measure for that particle, when it is sitting at rest next to them. While other definitions of mass in special relativity are sometimes mentioned, the rest mass is the only definition of mass that an actual practicing physicist, at least in this day and age, would take seriously. However, Taylor has a brief discussion of another possible definition for mass, known as the variable mass. I will not discuss this quantity here, however.

It turns out, however, that the above quantity is not a very good definition for momentum in special relativity. In section 15.12, Taylor constructs a scenario involving colliding billiard balls, in which if this quantity is conserved in one inertial frame of reference, it is **not** conserved in another inertial frame of reference. Since this quantity is not conserved for an isolated system according to all inertial observers, it fails one of our basic criteria for any valid momentum candidate. However, since we have now learned how to generalize the notion of velocity to a general space-time context, we wonder if perhaps our new notion of four-velocity will help us with our task. To this end, we will define the **four-momentum** according to

$$p = mu = (\gamma mc, \gamma m\mathbf{v})\,. \tag{29}$$

The quantity m is again the rest mass. Notice that m must be a Lorentz invariant, since its definition is something all observers should agree on - it is the mass that someone would measure if they took the object and weighed it next to them on a scale. Since the four-momentum is the result of multiplying a Lorentz invariant number with a four-vector, it is easy to show that the four-momentum is also a four-vector itself. In particular, the norm of the four-momentum is given by

$$p^2 = m^2 u \cdot u = -m^2c^2, \tag{30}$$

which is most certainly an invariant quantity.

One of the most important things to notice about the four-momentum is that if it is conserved in one frame, it must be conserved in another frame, simply by virtue of being a four-vector. To see why, let's imagine that in a given frame of reference, the conservation of four-momentum takes the form

$$p_{\text{final}} = p_{\text{initial}}. \tag{31}$$

Now, to find the final and initial four-momentum in another frame, we have

$$p'_{\text{final}} = \Lambda p_{\text{final}} \quad ; \quad p'_{\text{initial}} = \Lambda p_{\text{initial}}, \tag{32}$$

since finding the expression for a four-vector in another frame is simply a matter of performing a Lorentz transformation. However, if we take both sides of our momentum conservation equation, and apply a Lorentz transformation to each side, we have

$$p_{\text{final}} = p_{\text{initial}} \;\Rightarrow\; \Lambda p_{\text{final}} = \Lambda p_{\text{initial}} \;\Rightarrow\; p'_{\text{final}} = p'_{\text{initial}} \tag{33}$$

Thus, if four-momentum is conserved in one frame, it is conserved in **every** frame, which is the sort of property we want for a conserved quantity that agrees with the postulates of relativity. Notice that this statement is in fact **four** conservation equations, one for each component of the four-velocity. What we have found here is a special case of a much more general, very important result: *if a statement can be written as an equality involving four-vectors, then if it is true in one frame, it is true in all frames.*

Now, whether or not four-momentum actually is conserved, of course, is ultimately an experimental question. Focusing for the moment on the three-component, we will take

$$\mathbf{p} = \gamma m\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \tag{34}$$

to be the vector quantity we have been looking for, and refer to it as the spatial momentum. Notice that when the velocity of the object is non-relativistic, $\gamma \approx 1$, and our definition reduces to the usual non-relativistic expression for momentum, which is certainly an important check on whether our result is sensible. It turns out that indeed, this is the vector quantity which is conserved for an isolated system in special relativity. Providing a more convincing theoretical argument is possible once we have learned the techniques of Lagrangian mechanics, but ultimately it is the long list of experimental results which convince us that this quantity is indeed a conserved one.

Relativistic Energy

Now that we have found the conserved momentum that we were looking for, we may naturally ask about the interpretation of the time component of the four-momentum,

$$p_0 = \gamma mc. \tag{35}$$

As is always the case when confronted with a new object in an unfamiliar realm of physics, we should attempt to gain an intuitive understanding of this object by examining its behaviour in a more familiar limit - in this case, the limit of small velocities. To this end, let us perform a Taylor series expansion of the Lorentz factor, in order to find

$$p_0 = \frac{mc}{\sqrt{1 - v^2/c^2}} \approx mc\left(1 + \frac{1}{2}\left(v/c\right)^2\right) = mc + \frac{1}{2}\frac{m}{c}v^2. \tag{36}$$

Rewriting this slightly, we find

$$cp_0 \approx mc^2 + \frac{1}{2}mv^2 + ... \tag{37}$$

What we have found here is what appears to be a "constant," followed by an expression we would typically refer to as the kinetic energy of the particle. It is therefore tempting to refer to this quantity as the **energy** of the particle, so that

$$p_0 = E/c. \tag{38}$$

Keep in mind that however we refer to this quantity, its conservation has already been argued above - it is one of the components of the four-velocity, all of which, as we have previously argued, should be conserved. Furthermore, it is again an experimental fact that this quantity above is a constant for an isolated particle, in any reference frame we so choose to work in. Because it reduces, in the non-relativistic limit, to our usual expression for the kinetic energy, it seems clear that this should be our relativistic generalization of energy.

We should pause here and notice an interesting conclusion we have come to - in relativity, the energy and momentum of a particle naturally come packaged together as a four-vector quantity. While each component of the four-velocity is conserved in any particular frame, the values of the momentum and energy can change if we change to a new reference frame, just as in Galilean Relativity. However, the new phenomenon which occurs in special relativity is that the energy and momentum can be "rotated" into each other under a Lorentz transformation,

$$p' = \Lambda p. \tag{39}$$

While an ordinary rotation in three-dimensional space rotates the three spatial components of the momentum together, a more general Lorentz transformation can mix all four components of the four-momentum together.

The conservation of our new energy expression has some quite profound implications in special relativity. Notice that if we rearrange our expression slightly for the energy of a particle, we can write

$$E = p_0 c = \gamma m c^2 = mc^2 + (\gamma - 1)\, mc^2 = mc^2 + K, \tag{40}$$

where we have defined the (velocity-dependent) Kinetic energy of a particle as

$$K = (\gamma - 1)\, mc^2. \tag{41}$$

Notice that

$$v = 0 \;\Rightarrow\; \gamma = 1 \;\Rightarrow\; K = 0, \tag{42}$$

which is certainly a good property for a kinetic energy to have. Also,

$$\gamma \geq 1 \;\Rightarrow\; K \geq 0. \tag{43}$$

With these general observations out of the way, let's now imagine a **sticky collision** between two particles - a collision in which two particles approach each other with equal and opposite momentum, and then stick together as one mass (coming to a stop, by virtue of conservation of momentum). This is illustrated in Figure 3. We of course know from freshman mechanics that such a thing is possible - simply throw two lumps of clay against each other so that they stick, thus forming one larger lump. The initial energies of the two particles are, according to our new expression

$$E_1 = m_1 c^2 + K_1 \;;\; E_2 = m_2 c^2 + K_2. \tag{44}$$

After the two particles stick and come to rest, with no kinetic energy, their final energy is

$$E = Mc^2. \tag{45}$$

By conservation of energy, we find

$$Mc^2 = (m_1 + m_2)\, c^2 + K_1 + K_2, \tag{46}$$

or,

$$M = (m_1 + m_2) + \frac{1}{c^2}\,(K_1 + K_2)\,. \tag{47}$$

Because the kinetic energy of the initial particles is always positive, our result we have found is that the total mass of the final lump of clay is **more** than the sum of the initial masses!

In freshman mechanics, we typically say that for sticky collisions, the kinetic energy of the initial particles is lost to various forms of **internal energy** - heat, electrostatic binding, etc. But here, in our relativistic collision, we have seen that all of these internal sources of energy are in fact reflected in an **increase in mass** of our final object, given by

$$\Delta M = \frac{1}{c^2}\left(K_1 + K_2\right). \tag{48}$$

Thus, we find the profound result that **the mass of an object is a measure of its internal energy**. Even when a particle is not moving, it has an intrinsic **rest energy**, given by

$$E = mc^2, \tag{49}$$

perhaps the most famous equation in all of physics.

In everyday life, we don't typically notice this type of effect. For two particles each with a kinetic energy of one Joule, the resulting mass difference would be

$$\Delta M \approx 2.23 \times 10^{-17} \text{ kg}, \tag{50}$$

which is certainly an incredibly tiny number. After Einstein published his famous paper, "Does the inertia of a body depend upon its energy content?" in 1905, in which he first proposed the above relation, he suspected that it would not be experimentally verified for hundreds of years. However, it was in fact verified only a few years later in 1938 by Lise Meitner, while she was in exile in the Netherlands, fleeing Nazi Germany. Only a few years after that, the effect found applications in both energy production and warfare, as we all know.

Thus, by considering the geometry of space-time, we've been led to perhaps the most famous equation in all of physics, an equation whose technological applications (for better or worse) defined the 20$^{\text{th}}$ century as we know it. Not bad for 10 pages...

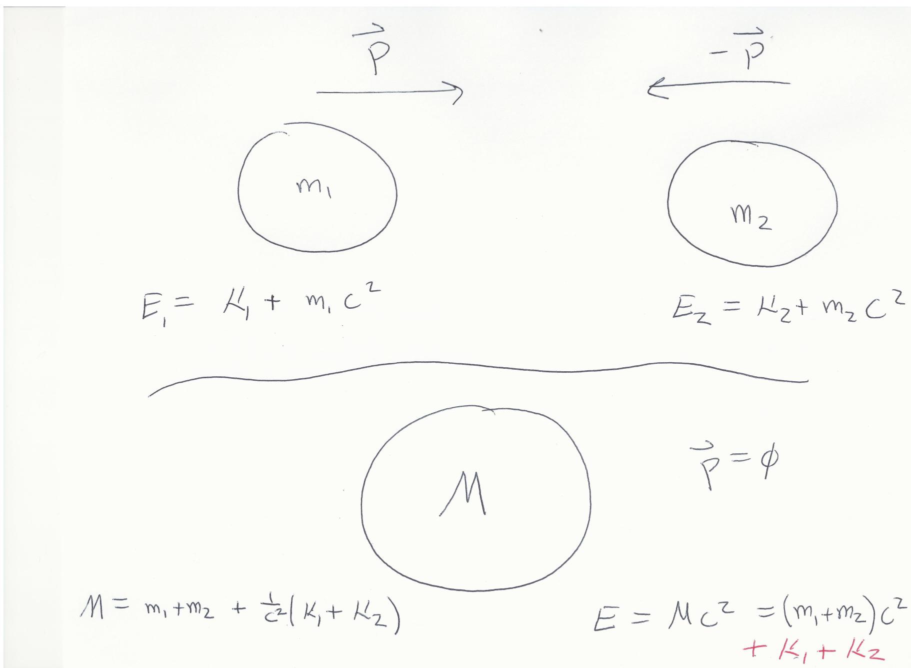

Figure 3: A sticky, relativistic collision between two particles, whose kinetic energies end up contributing to the total mass of the final combined object.