Work

Let's imagine I have a block which I'm pushing across the floor, shown in Figure 1. If I'm moving the block at constant velocity, then I know that I have to apply a force to compensate the effects of kinetic friction,

$$\vec{F} = -\vec{f}_k = \mu_k N\hat{x} = \mu_k mg\hat{x}, \tag{1}$$

where I'm assuming I'm moving the block in the positive x direction. N is the magnitude of the normal force, m is the mass of the block, and the coefficient of kinetic friction is μ_k.

I know that in order to perform this task, there is some sense in which I need to put some "effort" into compensating the effects of friction. In order to quantify this statement, let's assume I move the block a total distance d. Then, if I'm applying a force along the direction of the motion, a simple definition to quantify this amount of effort might be

$$W = Fd. \tag{2}$$

From our freshman mechanics course, we remember that this is defined as the amount of *work* that I have done on the block. This definition takes into account how much force I am applying, and for how long of a distance I do that. Intuitively, I would expect that increasing both of these things would correspond to me putting more overall effort into moving the block.

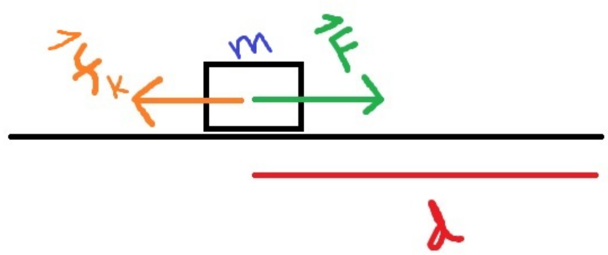

Figure 1: Doing work on a block by pushing it across the floor.

Now, we can also say that because the floor exerts a frictional force on the block, it also does work on the block. In this case, however, the frictional force opposes the motion of the block - its the reason we need to do anything to make the block move - so perhaps we want to take this into account as well. In this case, we say that the floor does *negative work* on the block, and we write

$$W_f = -f_k d = -Fd, \tag{3}$$

where I've used the fact that the magnitudes of the two forces are the same, and I've also included a subscript to clarify that this is the work the floor is doing.

If I add these two together, I find that I get zero, since

$$W_t = W_m + W_f = Fd - Fd = 0, \tag{4}$$

where the subscripts stand for "total," "me," and "floor." *Notice that zero net force along the direction of motion implies zero net work.*

I can generalize this definition to include forces pointing in arbitrary directions, which I've shown in Figure 2. Intuitively, it seems like this should involve the component of the force along the direction of the block. If the displacement vector for the block's net motion is

$$\vec{d} = d\hat{x}, \tag{5}$$

then I define the work to be

$$W_m = \vec{F} \cdot \vec{d} = Fd\cos\theta = F_x d. \tag{6}$$

While it may not be obvious that this is the best possible definition, it will become clear in a moment why it is useful.

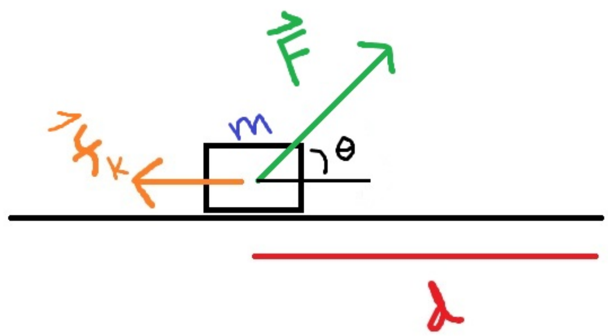

Figure 2: Work done by a force which is not along the direction of motion.

More generally, we can imagine that the object moves along an arbitrary path, feeling a force which depends on where it is located. This is shown in Figure 3. The notation on the forces emphasizes that the force can depend on where the object is, and the notation on the position of the object emphasizes that this changes with time. Now, over short enough distances, we know an arbitrary curve will look approximately straight. We can imagine that at time t_1, the object moves a tiny, infinitesimal distance which we call $d\vec{r}$. Over this small section of path, the path is roughly straight, and so we define the infinitesimal work to be

$$dW = \vec{F}\left(\vec{r}\left(t_1\right)\right) \cdot d\vec{r}. \tag{7}$$

We write the total work over the path from $\vec{r}\left(t_1\right)$ to $\vec{r}\left(t_2\right)$ as

$$W = \int_{\vec{r}(t_1)}^{\vec{r}(t_2)} \vec{F}\left(\vec{r}\left(t\right)\right) \cdot d\vec{r}\left(t\right). \tag{8}$$

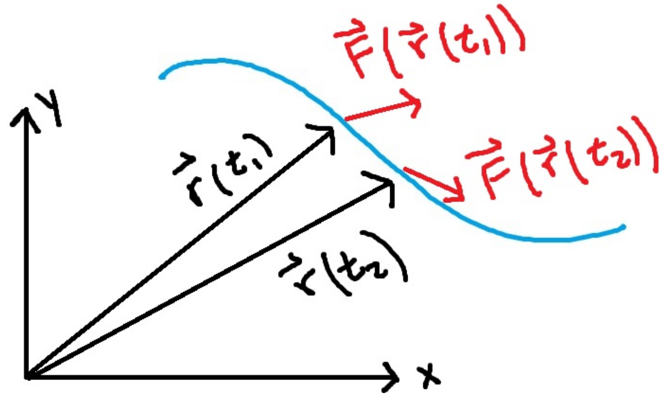

Figure 3: The work done along an arbitrary path.

The above object is called a line integral, which you've probably seen before in your multivariable calculus course. It tells us to sum up all of the infinitesimal contributions to the work along the path the object takes. As a practical matter, the simplest way to compute such a quantity is to remember that the infinitesimal displacement at a given time is

$$d\vec{r}\,(t) = \vec{v}\,(t)\,dt, \tag{9}$$

or the velocity at that instant times the time difference. Thus, we can write our integral as

$$W = \int_{t_1}^{t_2} \left[\vec{F}\,(\vec{r}\,(t)) \cdot \vec{v}\,(t)\right] dt, \tag{10}$$

which is now just a regular time integral which I know how to compute, assuming I know the path as a function of time, and also what the force is at each point.

While it certainly doesn't look like it from the above expression, we remember from our multivariable course that the amount of work done over a path is actually independent of how quickly the object moves, so long as the forces only depend on where the particle is in space. This useful fact simplifies most work calculations, since it implies that I don't actually need to know the position of an object as a function of time. For example, if I want to compute the work done while going around in a circle, then so long as the forces doing work on the object only depend on where the particle is located on the circle, I can assume the particle moves in uniform circular motion when computing the above quantity, even if this does not accurately describe the true motion of the object.

Kinetic Energy and the Work-Energy Principle

Now that we've carefully defined what it means to do work on an object, let's put this definition to use. From Newton's second law, we know that we can write the total force on the particle as

$$\vec{F}(\vec{r}(t)) = m\vec{a}(t), \tag{11}$$

where I've assumed that the force, and thus acceleration, may change in time. Using this, I can write the formula for the work as

$$W = m\int_{t_1}^{t_2} \vec{a}(t)\cdot\vec{v}(t)\,dt. \tag{12}$$

Now, on your homework, you'll derive the identity

$$\frac{d}{dt}(\vec{p}(t)\cdot\vec{q}(t)) = \frac{d\vec{p}}{dt}\cdot\vec{q} + \frac{d\vec{q}}{dt}\cdot\vec{p}, \tag{13}$$

valid for any two vectors. With this formula, notice that I can write

$$\frac{d}{dt}\left(v^2(t)\right) = \frac{d}{dt}(\vec{v}(t)\cdot\vec{v}(t)) = \frac{d\vec{v}}{dt}\cdot\vec{v} + \frac{d\vec{v}}{dt}\cdot\vec{v} = 2\frac{d\vec{v}}{dt}\cdot\vec{v} = 2\vec{a}\cdot\vec{v}. \tag{14}$$

Using this, I can write my expression for the work as

$$W = \frac{1}{2}m\int_{t_1}^{t_2} \frac{d}{dt}\left(v^2(t)\right)dt. \tag{15}$$

Since this is just the integral of a derivative, we finally see that

$$W = \frac{1}{2}mv^2(t_2) - \frac{1}{2}mv^2(t_1). \tag{16}$$

The quantity $K = \frac{1}{2}mv^2$, which depends on the speed of the body, is familiar to us as the *kinetic energy* of the body. The above equation tells us that the amount of work done on an object as it travels between two points is the same as the difference between the kinetic energies that that body has at those two points. This is known as the work-energy principle, which you've used many times before in your freshman mechanics course.

As a reminder of how this quantity is helpful, let's revisit our block sliding on a table with friction. I'll give the block an initial shove so that I start it off with some initial velocity $\vec{v}_0$, which we'll take to be entirely along the surface of the table. Additionally, let's say I've taken the table and sanded it down in some strange way, so that the coefficient of friction is not uniform over the table. As a specific example, let's say that

$$\mu_k = Ax^2, \tag{17}$$

where A is some constant number. As the block moves, the frictional force will reduce the block's velocity, until it eventually comes to a halt. How far does the block travel before this happens?

Well, if my block travels a distance d, we can compute the work done by friction as

$$W = \int \vec{f}_k\cdot d\vec{x} = -\int_0^d Amgx^2\;dx = -\frac{1}{3}Amgd^3. \tag{18}$$

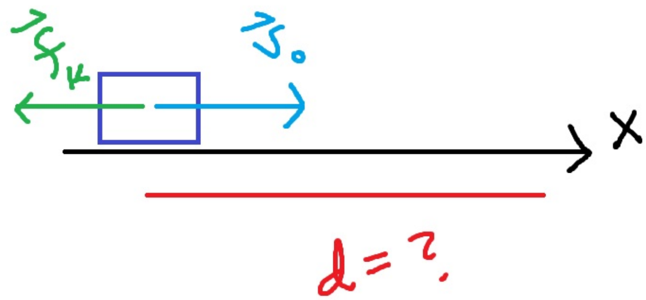

Figure 4: A block being brought to a stop by friction.

If I now equate the work done with the change in kinetic energy, I find that

$$-\frac{1}{3}Amgd^3 = -\frac{1}{2}mv_0^2 \Rightarrow d = \left[\frac{3}{2}\frac{v_0^2}{Ag}\right]^{1/3}. \tag{19}$$

Needless to say, this is much, much easier than using kinematics! No information about the motion of the block as a function of time was needed. While the differential equation describing the motion of the block in this case is not too terribly complicated, this example still demonstrates the fact that energy considerations often allow us to extract certain information about a system, without having to completely solve for its kinematics.

Power

In many applications, the amount of *power* supplied to an object is more interesting than the total work done on it. Power is defined to be the rate at which work is done,

$$P = \frac{dW}{dt}. \tag{20}$$

Using our general expression for work, we can see that

$$P = \frac{d}{dt}\int \vec{F}\cdot\vec{v}\,dt = \vec{F}\cdot\vec{v}. \tag{21}$$

Because the force acting on an object, along with its velocity, can be functions of time, the power, in general, will also be some function of time. It is equal to the rate of change of the kinetic energy, since we can easily calculate

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = m\vec{a}\cdot\vec{v} = \vec{F}\cdot\vec{v} = P. \tag{22}$$

Notice that while in many cases, the net force on an object may be zero, it is often still true that some agent is providing power. In the case of the sliding block, if I am pushing the block along to compensate friction, I am doing work on the block, and so I am expending some effort to do this. The rate at which I do this is the power I am providing.

Actually, in an indirect way, I am actually doing work on the floor. Because the floor does negative work on the block, the net work done on the block from the floor and me is zero, and so is the net power supplied to the block. So where does the result of my work go? Well, we know that when I rub two surfaces together, they get hot. So really I am heating the floor! The power I am providing is going into the thermal energy of the floor, although that is a subject more appropriate for a class on thermodynamics, not classical mechanics.

In many situations, power is a much more important quantity than total work. Walking five miles over the course of a day is pretty easy, but running five miles in an hour is a lot harder! Runners have to train their bodies to be able to output power at a much higher rate than the average person.

Potential Energy

Let's consider another type of system which is an old favourite of ours from freshman mechanics. Let's imagine I have a block sitting on the floor, but it's attached to a nearby wall with a spring. This is shown in Figure 5. I'll imagine that the floor has a negligible amount of friction, so that I can effectively ignore it (maybe the floor is actually an air hockey table or something). Initially, I've placed the block so that the spring is not stretched or compressed at all, and is not exerting any forces on the block. I've labeled this position as x_0. If I've taken the location of the wall to be $x = 0$, then we say that the spring has an *equilibrium length* of x_0.

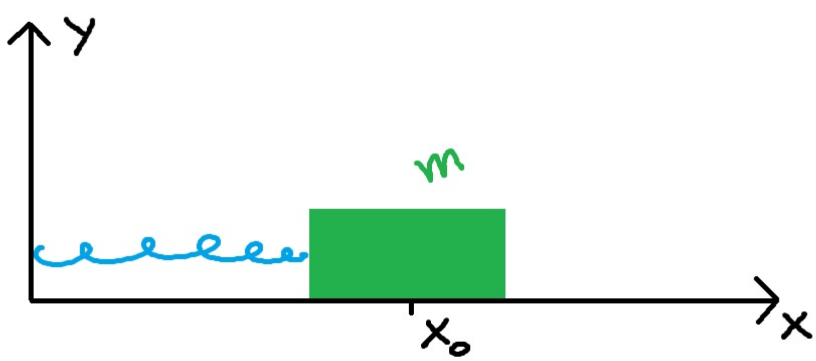

Figure 5: A block attached to a spring, sitting at rest.

Now, Hooke's law tells us that as we start to move the block a little bit, the force exerted by the spring on the block is given by

$$F = -k\,(x_1 - x_0)\,, \tag{23}$$

where x_1 is the new location of the block, and k is some positive constant which characterizes the spring. This is shown in Figure 6. Notice that when x_1 is to the left of the equilibrium position, the sign of the force is positive, as it should be.

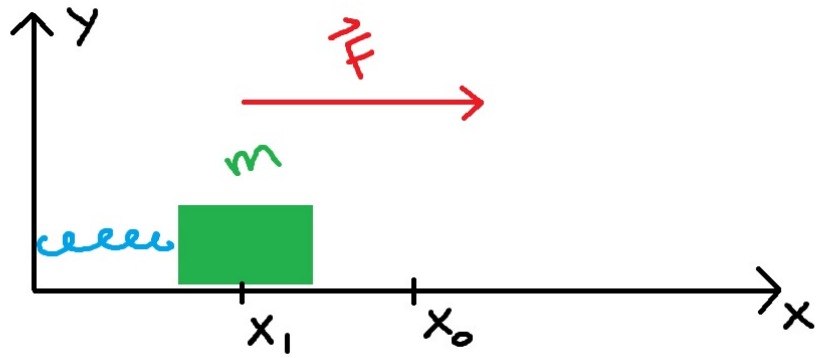

Figure 6: Doing work on a block by moving it against the action of a spring.

Now, if I compress the spring by some amount

$$\Delta x = x_1 - x_0. \tag{24}$$

I might be interested to know how much work I've done on the block-spring system. Well, I know that the force I exert will be opposite to what the spring is exerting on the block,

$$F_m = +k\,(x_1 - x_0)\,. \tag{25}$$

If I now compute the amount of work that I've done, the result is

$$W_m = \int_{x_0}^{x_1} \vec{F}_m \cdot d\vec{x} = +k \int_{x_0}^{x_1} (x - x_0)\, dx = \frac{1}{2} k\,(x_1 - x_0)^2 = \frac{1}{2} k\,(\Delta x)^2\,. \tag{26}$$

Notice that my final result doesn't depend on the sign of Δx, and is always positive - this agrees with our intuition that compressing and stretching a string both require some positive effort on my part.

In some sense, while I'm applying a force to the block directly, it might be more natural for me to think of doing work on the spring. The reason I say this is because after the compression has finished, the block is more or less in the same state it was in before I started, and has had zero net work done on it, since it is sitting still before and after the compression, so it's kinetic energy has not changed. The spring, however, has changed noticeably - it is physically smaller as a result of being compressed. For this reason, we typically say that the quantity $\frac{1}{2}k\left(\Delta x\right)^2$ is the *potential energy stored in the spring.*

In particular, why do we say that the potential energy is "stored" in the spring? Let's remember what happens when I let go of the block. Because the spring is compressed, there will be a force exerted on the block, and once I let go, there is nothing to compensate that force. So the block will start to move back to its equilibrium position. As it does so, the spring will be doing work on the block. The amount of work done will be

$$W_s = \int_{x_1}^{x_0} \vec{F}_s \cdot d\vec{x} = -k \int_{x_1}^{x_0} \left(x - x_0\right) dx = \frac{1}{2}k\left(x_1 - x_0\right)^2 = \frac{1}{2}k\left(\Delta x\right)^2 . \quad (27)$$

This is the *same* amount of work I did on the spring-block system when I compressed the spring. So I did work on a system, did something to the spring to change its physical nature, and then, when I let go, the spring restored itself to its original state, while transferring the same amount of work to the block. Furthermore, because the work done on the block is the change in its kinetic energy, I know that after the spring has restored itself to its initial state, the block has acquired a nonzero velocity, given by

$$\frac{1}{2}mv^2 = \frac{1}{2}k\left(\Delta x\right)^2 . \quad (28)$$

All of this gives me the idea that maybe there is some sort of "stuff" which is being transferred around from place to place in my system, whose amount seems to stay the same. Of course, we remember that this "stuff" is what we call energy, and that the principle I've discovered is the **conservation of energy**. In order to quantify this idea in general, let me take the expression for the work done during the compression and use it to define a new quantity,

$$U\left(x\right) = -\int_{x_0}^{x} F_s\left(x'\right) dx' = k \int_{x_0}^{x} \left(x' - x_0\right) dx' = \frac{1}{2}k\left(x - x_0\right)^2 , \quad (29)$$

which I will call the *potential energy* stored in the spring. While I originally considered compression from x_0 to x_1, I'm now considering compression to some arbitrary point x. Notice that it is a function only of the material properties of the spring, described by k, and the location of the end of the spring (where the block is). It is equal to the amount of work I did on the system when I compressed the spring.

Also, notice that the only reason I could define this function in this way is because it was possible for me to write down the force as a function of position. I could not do the same thing for friction, because the frictional force acting on a block is not just something I can write down as a function of position. The force of kinetic friction depends on whether or not the block is moving (it is some constant if it is moving, and zero otherwise), and so it depends on the velocity, not just the location. I describe this by saying that the force from the spring is a *conservative* force, whereas the force from friction is a *nonconservative force.*

In one dimension, for an arbitrary force which can be written as a function of position, we define the potential energy as

$$U(x) = -\int_{x_0}^{x} F(x')\, dx'. \tag{30}$$

A little bit later we'll generalize this definition to more than one dimension. Again, the choice of x_0 is up to me, and does not affect any of the physics. You'll explore exactly why this is the case in the homework. I can also use the fundamental theorem of calculus to invert the above relationship, and write the force as

$$F(x) = -\frac{dU}{dx}. \tag{31}$$

So alternatively, in a system where I can define a potential energy, I can either specify the force as a function of position, or the potential energy as a function of position, and either one will tell me what the physics is. To see this explicitly, we can take Newton's second law for the spring acting on the block,

$$F_s = ma, \tag{32}$$

and use my new definition to write

$$m\ddot{x} = -\frac{dU}{dx}. \tag{33}$$

So I've rewritten Newton's laws in terms of this potential energy function.

Conservation of Energy

For a general system with a conservative force, let's define the total energy as

$$E = \frac{1}{2}mv^2 + U(x), \tag{34}$$

Let's check explicitly to make sure that this quantity is conserved. Taking a time derivative, we have

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}mv^2 + U(x)\right] = mv\frac{dv}{dt} + \frac{dU}{dx}\frac{dx}{dt}, \tag{35}$$

where I've used the chain rule in a few places in order to take time derivatives. If I use the definitions of the time derivatives in terms of velocity and acceleration, and the definition of force in terms of potential energy, I have

$$\frac{dE}{dt} = mva - F(x)\, v = [ma - F(x)]\, v. \tag{36}$$

However, Newton's second law tells me that

$$F(x) = ma, \tag{37}$$

and so

$$\frac{dE}{dt} = 0, \tag{38}$$

just as we suspected. So indeed, this total energy function is *constant* in time - it never changes throughout the motion. Be careful to notice in particular that I have not introduced any new physics here - at the end of the day, my derivation crucially relied on Newton's laws, which means that conservation of energy is simply a consequence of the underlying Newtonian laws that describe my system.

Of course, energy conservation is an incredibly useful tool. As an example, imagine a situation where instead of slowly compressing the spring, I give it an initial shove, so that it possesses an initial velocity, shown in Figure 7. Because the spring is not initially compressed, the initial energy is solely a result of the block's kinetic energy, and so we have

$$E = \frac{1}{2}mv_0^2 \tag{39}$$

at the beginning of the motion. Of course, because the total energy is conserved, this will always be the total energy, so that for a general position and general velocity,

$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}k\left(x - x_0\right)^2. \tag{40}$$

I can use this information to easily answer a lot of questions. For example, if I want to know how much the spring will compress, this corresponds to the block transferring all of its kinetic energy to the spring. This occurs when the block has zero velocity, and so that maximum compression is given by

$$\frac{1}{2}mv_0^2 = \frac{1}{2}k\left(x - x_0\right)^2. \tag{41}$$

Rearranging and taking a square root, I find

$$\pm v_0\sqrt{\frac{m}{k}} = x - x_0, \tag{42}$$

or,

$$x = x_0 \pm v_0\sqrt{\frac{m}{k}}. \tag{43}$$

So I see that there are two different solutions for the location of the block. This of course makes sense - I know that initially the block will compress, come to a stop, and then start uncompressing. But of course, the block is now traveling in the opposite direction and has some kinetic energy when it comes back to its starting point. Thus, it will keep going, and now it will stretch out the spring, until the spring is extended to some final stopping point. These two points where the block comes to rest are the two points given above.

Keep in mind that if I wanted to find the motion of the block as a function of time, I would need to solve the differential equation

$$m\ddot{x} = -k\left(x - x_0\right). \tag{44}$$

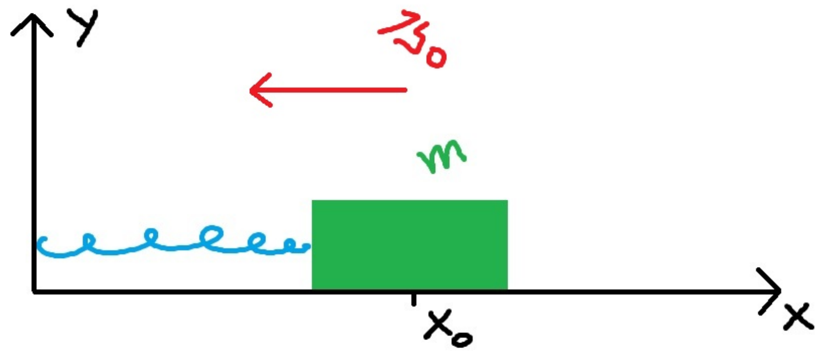

Figure 7: Giving a spring an initial velocity.

While this differential equation is easy to solve (and we will study its solutions soon), we will later encounter systems whose differential equations are hard to solve, and the type of reasoning we just used will be absolutely crucial to understanding something about the system.

Potential Energy Diagrams

There's a useful fact about the way that kinetic and potential energies behave, which helps us easily visualize what's going on in our system. Notice that by definition, the kinetic energy is always positive,

$$K = \frac{1}{2}mv^2. \tag{45}$$

As a result, we have

$$K \geq 0, \tag{46}$$

since the square of the speed can never be negative. This is the reason that the spring compressed to some distance, and then stopped compressing. In order to compress further, it would need to have more potential energy put into it. However, the potential energy must come at the expense of the kinetic energy, since their sum is conserved. Thus, because the potential energy increases as the kinetic energy decreases, and the kinetic energy cannot be less than zero, there is a maximum amount to which the potential energy can increase. This is, of course, the total energy E itself. If we rewrite the above equation in terms of total and potential energy, we have

$$E - U(x) \geq 0 \Rightarrow U(x) \leq E. \tag{47}$$

Now, if I think about it, this statement actually tells me something about the motion of the particle. The above statement says that the potential energy can never exceed the total energy. Since the potential energy is a function of position, this means that some regions of space are forbidden to the particle - these are regions in which the potential energy function is more than E. If the particle were to travel to these regions, its potential energy would exceed the total, which is not allowed.

To see how we can visualize these ideas, let's draw a plot of the potential energy function of the spring system, shown in Figure 8. I've plotted the potential energy function, along with a horizontal line, equal to some value E. I'll assume that this is the total energy of my system. The point x_0 represents the equilibrium point, and the points x_i and x_f are the points which satisfy

$$U(x_i) = U(x_f) = E. \tag{48}$$

Specifically, this means that, taking x_i to be the smaller value, we have

$$x_i = x_0 - \sqrt{\frac{2E}{k}}, \tag{49}$$

as well as

$$x_f = x_0 - \sqrt{\frac{2E}{k}}. \tag{50}$$

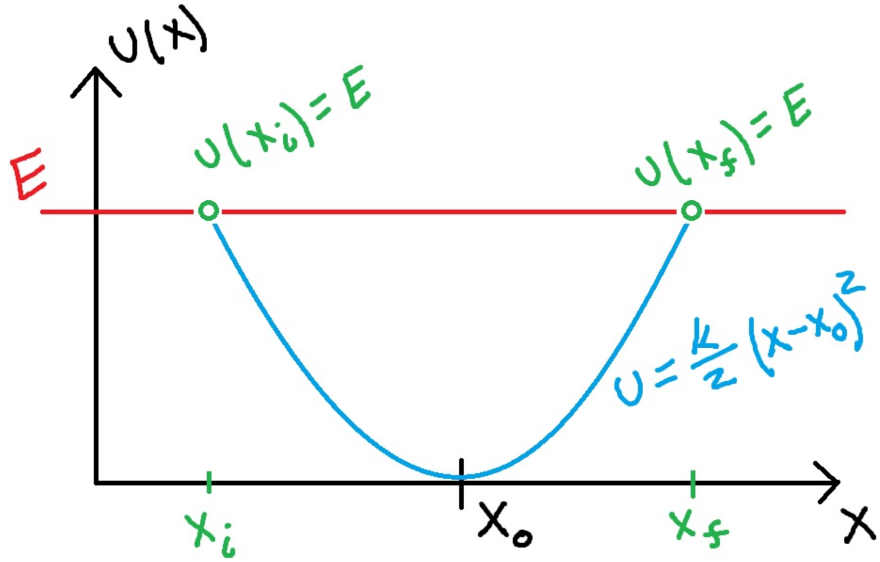

Figure 8: Using a potential energy diagram to understand the motion of the block under the influence of the spring force.

From the above considerations, I know the block cannot venture outside of these two points. This is because the potential energy is quadratic, and increases outside of this region. So the block cannot move beyond the region indicated, since otherwise its potential energy would exceed the total energy.

Let's see what else I can say about the motion of the block, just by looking at this potential energy plot. At the bottom of the plot, we have a minimum of the potential energy, and so the first derivative must be zero,

$$\frac{dU}{dx} = 0. \tag{51}$$

However, we know this is equal to the force, and so

$$F = 0 \tag{52}$$

at a minimum of the potential energy. If I take the block and place it at this point, and then let go, it will sit there, since there is no force acting on it. Thus, we see that places where the potential energy has a minimum correspond to equilibrium points of the system.

Let's also see what happens when I move the block to either side of the minimum. If I place the block slightly to the left, then the force is

$$F = -\frac{dU}{dx} > 0, \tag{53}$$

which we can say because the derivative of the plot in this region is negative, since the function is decreasing. This means that the force points in the positive direction, bringing it back to the equilibrium point. If we place the block to the right, then we have

$$F = -\frac{dU}{dx} < 0, \tag{54}$$

and the force points to the left, again tending to push it back to the minimum. These ideas help us to develop some intuition about what our potential energy diagram is telling us. Placing the block at the minimum will result in no motion, while displacing it slightly will tend to bring it back towards the equilibrium point.

In general, if I displace my block slightly, then it will continue to experience a force pushing it back to the equilibrium point. Once it reaches this point, the force will begin to oppose its motion. This will continue to occur until the block loses all of its kinetic energy, and the potential energy is again equal to E. The motion then reverses. So we see in general, the block will oscillate back and forth between the two end points of the motion, x_i and x_f.

Because of these behaviors, it is common to make an analogy where we liken this to the rolling of a ball down a "hill." If I were to imagine taking a parabolic surface, shaped like my potential energy plot, and placing a ball on it, when I let go, it would start rolling down towards the bottom of the well, and then roll up the other side, eventually oscillating back and forth. This is a handy tool for thinking about what happens in a potential energy function, but remember that there is a crucial distinction between these two cases. In the case of the ball, I am considering motion throughout two dimensional space, and my axes are x

and y, two directions of space. In the case of the potential energy function I am considering, it is really just a one dimensional problem - the block only moves in one dimension, oscillating back and forth between x_i and x_f. This point can cause lots of confusion, if it is not fully appreciated.

We can actually say something even more specific than this. Let's imagine that I take my block and move it to x_i, where it sits still. Once I let go, the block will move to the right. Using the relation between kinetic and potential energy, I can write

$$\frac{1}{2}mv^2 = E - U(x), \tag{55}$$

or

$$v = \pm\sqrt{\frac{2}{m}(E - U(x))}. \tag{56}$$

Let me consider first the motion from x_i to x_f. Because the block is moving to the right, the velocity is positive, and I have

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}(E - U(x))}. \tag{57}$$

If I consider this to be a differential equation involving the position, then after using the method of separation, I find

$$\int_0^{T_1} dt = \int_{x_i}^{x_f} \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U(x)}}, \tag{58}$$

where T_1 is the time it takes to travel from x_i to x_f. Thus,

$$T_1 = \sqrt{\frac{m}{2}} \int_{x_i}^{x_f} \frac{dx}{\sqrt{E - U(x)}}. \tag{59}$$

After the block makes it to the right, it will move back to the left, and since its velocity is now negative, we have

$$T_2 = -\sqrt{\frac{m}{2}} \int_{x_f}^{x_i} \frac{dx}{\sqrt{E - U(x)}} \tag{60}$$

as the time it takes to move from right to left. However, I can get rid of the minus sign by flipping the order of integration, and I get

$$T_2 = \sqrt{\frac{m}{2}} \int_{x_i}^{x_f} \frac{dx}{\sqrt{E - U(x)}} = T_1. \tag{61}$$

So already, I have another useful conclusion - the time it takes to move from left to right is the same as the time to move from right to left. Notice that this doesn't make any assumption about the shape of the potential. But I can actually do better than this. The full period of oscillation between the two points is

$$T = T_1 + T_2 = \sqrt{2m} \int_{x_i}^{x_f} \frac{dx}{\sqrt{E - U(x)}}. \tag{62}$$

The important conclusion here is that I can find the period of oscillation just by integrating a function that depends only on the *shape* of the potential energy function, in between the two end points. I don't need to know anything about the intermediate kinematics of the motion.

As an application of this, let's find the period of an oscillating spring. For a given energy, we know the form of x_i and x_f in terms of the energy E, and so we have

$$T = \sqrt{2m} \int_{x_0 - \sqrt{2E/k}}^{x_0 + \sqrt{2E/k}} \frac{dx}{\sqrt{E - (k/2)\,(x - x_0)^2}}. \tag{63}$$

After doing a little rearranging on this expression, and making some changes of variables in the integral, I can get this expression into the form

$$T = 2\sqrt{\frac{m}{k}} \int_{-1}^{1} \frac{du}{\sqrt{1 - u^2}}. \tag{64}$$

Now, the value of this integral turns out to be

$$\int_{-1}^{1} \frac{du}{\sqrt{1 - u^2}} = \pi, \tag{65}$$

and thus

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{66}$$

Notice that the above expression for the period has no dependence on the total energy. This is a unique feature of the harmonic oscillator potential, and is not true for a generic potential.

This example clearly demonstrates that we've just developed an enormously powerful set of tools. I have qualitatively analyzed the motion of the block over its entire range of motion, and even said something exact about its period of oscillation. And I never even had to actually solve for the motion! In addition to helping us gain an intuitive understanding of the motion of a system, these tools will be crucial in a few days when we analyze systems whose motion cannot be solved exactly.