Problem 1

In the following, \( U(t) = e^{-iHt} \) denotes the time evolution operator, and \( U(x,t;x') = \langle x|U(t)|x' \rangle \) is the propagator. First let’s note that since \( \hat{x}(t) \) is a Heisenberg operator, so must be \( \langle \hat{x}(t) \rangle \). Then \( \langle \hat{x};t \rangle \langle x; t \rangle \) must also be a Heisenberg operator. This precludes interpreting \( \langle \hat{x};t \rangle \langle x; t \rangle \) as \( \langle U(t)|x \rangle \langle x|U(t)^\dagger \rangle \), since this has the opposite time evolution of a Heisenberg operator. Instead, \( \langle \hat{x};t \rangle \langle x; t \rangle = U(x)^\dagger(t) \langle \hat{x};t \rangle \langle x; t \rangle U(t) \), the Heisenberg picture version of \( \langle \hat{x};t \rangle \langle x; t \rangle \).

(a) Let’s act on an arbitrary state \( |\psi \rangle = \int dy \psi(y)|y \rangle \) with each operator. First we have

\[
\delta(\hat{x}(t) - x)|\psi \rangle = \int \frac{d\alpha}{2\pi} dy \psi(y)e^{i\alpha(\hat{x}(t) - x)}|y \rangle
\]

\[
= \int \frac{d\alpha}{2\pi} dy \psi(y)U(t)^\dagger e^{i\alpha(\hat{x} - x)}U(t)|y \rangle
\]

\[
= \int \frac{d\alpha}{2\pi} dy dz \psi(y)U(t)^\dagger e^{i\alpha(\hat{x} - x)}z|z \rangle U(t)|y \rangle
\]

\[
= \int dy dz \psi(y)U(t)^\dagger \delta(z - x)|z \rangle U(z,t;y)
\]

\[
= \int dy \psi(y)U(x,t;y)U(t)^\dagger|x \rangle
\]

On the other side we have

\[
|x, t \rangle \langle x, t | = \int dy \psi(y)U(t)^\dagger|x \rangle \langle x|U(t)|y \rangle
\]

\[
= \int dy \psi(y)U(t)^\dagger|x \rangle U(x,t;y)
\]

\[
= \psi(x,t)U(t)^\dagger|x \rangle
\]

Since \( |\psi \rangle \) is arbitrary, we find the desired equality.
Using the result from part (a), we have

\[ \langle \psi_2 [\delta(\hat{x}(t) - x_2), \delta(\hat{x}(0) - x_1)] | \psi_1 \rangle = \langle \psi_2 | x_2, t \rangle \langle x_2, t | x_1, t \rangle \langle x_1 | \psi_1 \rangle - \langle \psi_2 | x_1, t \rangle \langle x_1 | x_2, t \rangle \langle x_2, t | \psi_1 \rangle \]

If we choose \(|\psi_2\rangle = |x_2\rangle\), \(|\psi_1\rangle = |x_1, t\rangle\), then we have

\[ \langle \psi_2 [\delta(\hat{x}(t) - x_2), \delta(\hat{x}(0) - x_1)] | \psi_1 \rangle = \langle x_2 | x_2, t \rangle \langle x_2, t | x_1, t \rangle \langle x_1 | x_1, t \rangle \]

\[ = \langle x_2 | x_2, t \rangle \langle x_2, t | x_1, t \rangle \langle x_1 | x_1, t \rangle \neq 0 \]

since \langle x_2 | x_1 \rangle = 0 for spacelike separation, and the propagators in the last line do not vanish.

**Problem 2**

This problem deals with the *Klein Paradox*; it is closely related to properties of electron transport in graphene. It is also related to the physics of Hawking radiation.

a) We begin by matching \( \phi \) and its first space derivative, which gives the equations

\[ 1 + A = B \quad , \quad k(1 - A) = qB \]

These have solution

\[ A = \frac{k - q}{k + q} \quad , \quad B = \frac{2k}{k + q} \]

The numerator of \(|A|^2 + q|B|^2|\) is then \(k(k - q)^2 + 4qk^2 = k(k + q)^2\), from which \(|A|^2 + q|B|^2 = k\) immediately follows.

b) To clarify: \( E = \omega \). Let us look at \( q \): the quantity inside the square root is negative for \(|E - V_0| < m\), or \( E + m > V_0 > E - m\). This means that the \( x > 0 \) solution has exponential falloff, when the correct sign for \( q \) is chosen, so no transmitted flux to the right. But, for \( V_0 > E + m - a \text{ large} \) barrier – transmission appears to begin again! This is very strange, since you might expect a very large barrier to be even more effective in blocking transmission.

c) The differential of \((E - V_0)^2 = q^2 + m^2\) yields group velocity

\[ v_g = \frac{dE}{dq} = \frac{q}{E - V_0} \]

If a disturbance (wave packet superposition of plane waves) is incident from the left, causal behavior means it should be right moving for \( x > 0 \) as well (i.e. it doesn’t come from the right), so \( v_g > 0 \). This means that for \( V_0 > E \), \( q < 0 \) for the correct group velocity – i.e. we must choose the lower sign on the square root.

But this looks even weirder! Now \( A > 1 \) – the reflected wave is larger than the incident wave! The “probability current” appears to be conserved – since \( B < 0 \), and the relation in part a) still holds. But it’s hard to interpret it as probability current if it is negative for \( x > 0 \).

d) If \( j_x \) is instead electric current, then we don’t see any fundamental problem. Electric current is also conserved, but can be positive or negative. So, the natural interpretation is
that for $V_0 > E + m$, there is an incident electric current from the left, and a larger “reflected” current back to the left. This is conserved because the barrier interface produces a positive current to the left, and a negative current to the right, conserving charge. But, this only makes sense if there are both particles and antiparticles. If we only had particles, their charges could be taken to be positive, and we would only have positive currents. We find that for a consistent interpretation we require our theory to always have both positive and negative charges – particles and antiparticles. The sharp potential pair produces particles and antiparticles.

Problem 3

(a)
The solution can be find through employing standard separation of variables techniques. That is, we assume that $\phi(x)$ can be written as $\phi(x) = T(t)X(x)Y(y)Z(z)$. Expanding the equation, and dividing through by $\phi$, we see that each piece must individually be equal to a constant. So, we find that for each variable, we have a differential equation of the form

$$\partial^2_x X(x) = C$$

We know the solution to this equation to be exponential, as we’ve all seen it many times before. We then find

$$\phi(x) = e^{\pm i \omega_k t} e^{\pm i k \cdot x}$$

where $\omega_k = \sqrt{k^2 + \mu^2}$, which can be found by plugging our solution into the full differential equation. Really only $\pm i \omega_k t - ik \cdot x$ are distinct plane waves, and we can form arbitrary solutions by superposition of solutions with different $k$, with $a_k, a_k^\dagger$ as Fourier modes. The expression given in the problem set is manifestly real.

To invert for the modes, we want to Fourier transform convenient combinations of $\phi(x)$ and $\partial \phi / \partial t(x)$. We first note that

$$\int d^3 x e^{-i k \cdot x} \phi(x) = \int d^3 x \frac{d^3 k'}{(2\pi)^{3/2} \sqrt{2 \omega_{k'}}} e^{-i k \cdot x} \left( a_{k'} e^{i k' \cdot x} e^{-i \omega_{k'} t} + a_{k'}^\dagger e^{-i k' \cdot x} e^{i \omega_{k'} t} \right)$$

$$= \int \frac{(2\pi)^{3/2} d^3 k'}{\sqrt{2 \omega_{k'}}} \left( a_{k'} \delta^3(k' - k) e^{-i \omega_{k'} t} + a_{k'}^\dagger \delta^3(k' + k) e^{i \omega_{k'} t} \right)$$

$$= \frac{(2\pi)^{3/2}}{\sqrt{2 \omega_k}} \left( a_k e^{-i \omega_k t} + a_k^\dagger e^{i \omega_k t} \right)$$

and that

$$\int d^3 x e^{-i k \cdot x} \partial \phi / \partial t(x) = \frac{(2\pi)^{3/2}}{\sqrt{2 \omega_k}} \left( -i \omega_k a_k e^{-i \omega_k t} + i \omega_k a_k^\dagger e^{i \omega_k t} \right)$$

Multiplying each of these expressions by the appropriate powers of $\omega_k$ and $e^{i \omega_k t}$, and adding them allows us to isolate the modes. The final result is
The mode expansion gives us

\[ a_k = \frac{1}{2} \int d^3x \frac{\sqrt{2\omega_k}}{(2\pi)^{3/2}} \left( \phi(x) + \frac{i}{\omega_k} \frac{\partial \phi}{\partial t}(x) \right) e^{-ikx} \]

\[ a_k^\dagger = \frac{1}{2} \int d^3x \frac{\sqrt{2\omega_k}}{(2\pi)^{3/2}} \left( \phi(x) - \frac{i}{\omega_k} \frac{\partial \phi}{\partial t}(x) \right) e^{ikx} \]

(b)

The mode expansion gives us

\[
[\phi(x, t), \phi(y, t)] = \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \frac{d^3k'}{(2\pi)^{3/2} \sqrt{2\omega_{k'}}} \times \left( [a_k, a_{k'}^\dagger] e^{ikx} e^{ik'y} + [a_k^\dagger, a_{k'}] e^{-ikx} e^{-ik'y} + [a_k^\dagger, a_{k'}^\dagger] e^{-ikx} e^{-ik'y} \right)
\]

\[
[\partial_0 \phi(x, t), \phi(y, t)] = \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \frac{d^3k'}{(2\pi)^{3/2} \sqrt{2\omega_{k'}}} \times \left( -i\omega_k [a_k, a_{k'}] e^{ikx} e^{ik'y} - i\omega_k [a_k^\dagger, a_{k'}^\dagger] e^{-ikx} e^{-ik'y} + i\omega_k [a_k^\dagger, a_{k'}^\dagger] e^{-ikx} e^{-ik'y} \right)
\]

If \([a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0\) and \([a_k, a_{k'}^\dagger] = \delta^3(k' - k)\), we have at equal times

\[
\phi(x, t), \phi(y, t)] = \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \frac{d^3k'}{(2\pi)^{3/2} \sqrt{2\omega_{k'}}} \delta^3(k' - k) \left( e^{-i(\omega_k - \omega_{k'})t} e^{ikx} e^{-ik'y} - e^{i(\omega_k - \omega_{k'})t} e^{-ikx} e^{ik'y} \right) = 0
\]

\[
[\partial_0 \phi(x, t), \phi(y, t)] = \int \frac{-i\omega_k d^3k}{(2\pi)^{3/2} \sqrt{2\omega_k}} \frac{d^3k'}{(2\pi)^{3/2} \sqrt{2\omega_{k'}}} \delta^3(k' - k) \left( e^{-i(\omega_k - \omega_{k'})t} e^{ikx} e^{-ik'y} + e^{i(\omega_k - \omega_{k'})t} e^{-ikx} e^{-ik'y} \right) = -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} 2\omega_k e^{ik(x-y)} = -i \delta^3(x - y)
\]

To prove the converse, we use the expressions for the modes we derived in 2(a), and follow the same procedure.

\[
[a_k, a_{k'}] = \int d^3x d^3y \frac{\sqrt{2\omega_k}}{(2\pi)^{3/2}} \frac{\sqrt{2\omega_{k'}}}{(2\pi)^{3/2}} \left( [\phi(x, t), \phi(y, t)] + i \frac{\omega_{k'}}{\omega_k} [\phi(x, t), \frac{\partial \phi}{\partial t}(y, t)] \right) + i \frac{\omega_k}{\omega_{k'}} \left( \frac{\partial \phi}{\partial t}(x, t), \frac{\partial \phi}{\partial t}(y, t) \right) e^{-ikx} e^{-ik'y}
\]
\[ [a_k, a_{k'}^\dagger] = \int d^3x \int d^3y \frac{\sqrt{2\omega_k}}{(2\pi)^{3/2}} \frac{\sqrt{2\omega_{k'}}}{(2\pi)^{3/2}} \left( [\phi(x, t), \phi(y, t)] - \frac{i}{\omega_k} [\phi(x, t), \frac{\partial \phi}{\partial t}(y, t)] + \frac{i}{\omega_{k'}} [\phi(x, t), \frac{\partial \phi}{\partial t}(y, t)] \right) e^{-ikx} e^{ik'y} \]

If \([\phi(x, t), \phi(y, t)] = \frac{i}{\omega_k} [\phi(x, t), \frac{\partial \phi}{\partial t}(y, t)] = 0\) and \([\frac{\partial \phi}{\partial t}(x, t), \phi(y, t)] = -i\delta^3(x - y)\), then we have at equal times

\[ [a_k, a_{k'}] = \int d^3x \int d^3y \frac{\sqrt{2\omega_k}}{(2\pi)^{3/2}} \frac{\sqrt{2\omega_{k'}}}{(2\pi)^{3/2}} \left( -\frac{1}{\omega_{k'}} \delta^3(x - y) + \frac{1}{\omega_k} \delta^3(x - y) \right) e^{-ikx} e^{-ik'y} \]

\[ = \int d^3x \frac{\sqrt{2\omega_k} \sqrt{2\omega_{k'}}}{(2\pi)^{3/2}} \left( -\frac{1}{\omega_{k'}} + \frac{1}{\omega_k} \right) e^{-ix(k' + k)} e^{it(\omega_k + \omega_{k'})} \]

\[ = \sqrt{2\omega_k} \sqrt{2\omega_{k'}} \left( -\frac{1}{\omega_{k'}} + \frac{1}{\omega_k} \right) e^{it(\omega_k + \omega_{k'})} \delta^3(k' + k) \]

\[ = 0 \]

\[ [a_k, a_{k'}^\dagger] = \int d^3x \int d^3y \frac{\sqrt{2\omega_k}}{(2\pi)^{3/2}} \frac{\sqrt{2\omega_{k'}}}{(2\pi)^{3/2}} \left( \frac{1}{\omega_{k'}} \delta^3(x - y) + \frac{1}{\omega_k} \delta^3(x - y) \right) e^{-ikx} e^{ik'y} \]

\[ = \int d^3x \frac{\sqrt{2\omega_k} \sqrt{2\omega_{k'}}}{(2\pi)^{3/2}} \left( \frac{1}{\omega_{k'}} + \frac{1}{\omega_k} \right) e^{ix(k' - k)} e^{it(\omega_k - \omega_{k'})} \]

\[ = \sqrt{2\omega_k} \sqrt{2\omega_{k'}} \left( \frac{1}{\omega_{k'}} + \frac{1}{\omega_k} \right) e^{it(\omega_k - \omega_{k'})} \delta^3(k' - k) \]

\[ = \delta^3(k' - k) \]

The last equality in the first commutation relation follows because the prefactor of the delta function vanishes for \(k' = -k\), and the delta function vanishes for all \(k' \neq -k\), so it vanishes for all \(k\). The last equality in the second commutation relation follows because the prefactor of the delta function is 1 for \(k' = k\), and is irrelevant for all \(k' \neq k\) because the delta function vanishes. Of course, \([a_k^\dagger, a_{k'}^\dagger] = 0\) follows by hermitian conjugation.

(c)

On one hand,

\[ U^\dagger(\Lambda)\varphi(x)U(\Lambda) = \int \frac{d^3k}{(2\pi)^{3/2}2\omega_k} \left( U^\dagger(\Lambda)(\sqrt{2\omega_k}a_k)U(\Lambda) e^{ikx} + U^\dagger(\Lambda)(\sqrt{2\omega_k}a_{k'}^\dagger)U(\Lambda) e^{-ikx} \right) \]

since the integration measure \(\frac{d^3k}{(2\pi)^{3/2}2\omega_k}\) and inner product \(kx\) are Lorentz invariant.

On the other hand,

\[ U^\dagger(\Lambda)\varphi(x)U(\Lambda) = \varphi(\Lambda x) = \int \frac{d^3k}{(2\pi)^{3/2}2\omega_k} \left( (\sqrt{2\omega_k}a_k) e^{ik\Lambda x} + (\sqrt{2\omega_k}a_{k'}^\dagger) e^{-ik\Lambda x} \right) \]
so that

\[
\int \frac{d^3k}{(2\pi)^{3/2}2\omega_k} \left( U^\dagger(\sqrt{2\omega_k}a_k)U(\Lambda) e^{ikx} + U^\dagger(\Lambda) (\sqrt{2\omega_k}a_k^\dagger)U(\Lambda) e^{-ikx} \right)
\]

\[
= \int \frac{d^3k}{(2\pi)^{3/2}2\omega_k} \left( e^{ik(\Lambda x)} + (\sqrt{2\omega_k}a_k^\dagger) e^{-ik(\Lambda x)} \right)
\]

\[
= \int \frac{d^3k}{(2\pi)^{3/2}2\omega_k} \left( e^{(\Lambda^{-1}k)x} + (\sqrt{2\omega_k}a_k^\dagger) e^{-i(\Lambda^{-1}k)x} \right)
\]

\[
= \int \frac{d^3k}{(2\pi)^{3/2}2\omega_k} \left( e^{ikx} + (\sqrt{2\omega_k}a_k^\dagger) e^{-ikx} \right)
\]

Since Fourier expansions (or this one with oddly-normalized modes) are unique, we conclude that

\[
U^\dagger(\Lambda)(\sqrt{2\omega_k}a_k)U(\Lambda) = \sqrt{2\omega_k}a_k
\]

**Problem 4**

\[
\int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx}}{\omega_k} = 2\pi \int_{-1}^1 d(\cos \theta) \int_0^\infty \frac{dk k^2}{(2\pi)^3} \frac{e^{ikr \cos \theta}}{\omega_k} e^{-i\omega_k t}
\]

\[
= \frac{1}{(2\pi)^2} \frac{1}{ir} \int_0^\infty \frac{dk}{\omega_k} k e^{-i\omega_k t} \left( e^{ikr} - e^{-ikr} \right)
\]

\[
= \frac{1}{(2\pi)^2} \frac{1}{ir} \int_{-\infty}^\infty \frac{dk}{\omega_k} k e^{-i\omega_k t+ikr}
\]

This can be evaluated with a contour integral in the complex $k$ plane on the contour pictured below. The square root in $\omega_k$ introduces branch points at $\pm i\mu$; we connect them with a branch cut along the imaginary axis through complex infinity, out of the way of our desired integration path.

The integrand has no poles inside this contour, so the contour integral vanishes.
\[ 0 = \int dk \frac{k}{\sqrt{\mu^2 + k^2}} e^{-i\sqrt{\mu^2 + k^2}t + ikr} = \int_{-\infty}^{\infty} dk \frac{k}{\sqrt{\mu^2 + k^2}} e^{-i\sqrt{\mu^2 + k^2}t + ikr} + \int_{i\infty+\varepsilon}^{i\infty-\varepsilon} idk \frac{ik}{\sqrt{\mu^2 + (ik)^2}} e^{-i\sqrt{\mu^2 + (ik)^2}t + i(ik)r} + \int_{i\mu-\varepsilon}^{i\mu+\varepsilon} idk \frac{ik}{\sqrt{\mu^2 + (ik)^2}} e^{-i\sqrt{\mu^2 + (ik)^2}t + i(ik)r} + \int_{0}^{\pi/2-\varepsilon} d\varphi \frac{k^2 e^{i2\varphi}}{\sqrt{\mu^2 + k^2 e^{i2\varphi}}} e^{-i\sqrt{\mu^2 + k^2 e^{i2\varphi} t + ik(e^{i\varphi})r}} + \int_{\pi/2+\varepsilon}^{\pi} d\varphi \frac{k^2 e^{i2\varphi}}{\sqrt{\mu^2 + k^2 e^{i2\varphi}}} e^{-i\sqrt{\mu^2 + k^2 e^{i2\varphi} t + ik(e^{i\varphi})r}} \]

There really should be one more integral for the small arc around the branch point at \( k = i\mu \), but it is easily seen to vanish in the \( \varepsilon \to 0 \) limit. The minus sign in the denominator of the third line comes from the discontinuity of \( \omega_k \) across the branch cut. We have abused notation and set \( k = |k| \) on the RHS. The contributions from the big arcs vanish in the \( k \to \infty \) limit:

\[
\frac{k^2 e^{i2\varphi}}{\sqrt{\mu^2 + k^2 e^{i2\varphi}}} e^{-i\sqrt{\mu^2 + k^2 e^{i2\varphi} t + ik(e^{i\varphi})r}} \xrightarrow{\mu \gg e^{\varphi}} \left( ke^{i\varphi} e^{ik(r-t) \cos \varphi} \right) e^{-k(r-t) \sin \varphi} \xrightarrow{k \to \infty} 0
\]

since \( r - t > 0 \) (because of spacelike separation), and since \( \sin \varphi > 0 \) (because \( \varphi \in [0..\pi] \) for the big arcs). The contributions from the paths along the branch cut add to give us

\[
\int_{-\infty}^{\infty} dk \frac{k}{\sqrt{\mu^2 + k^2}} e^{-i\sqrt{\mu^2 + k^2}t + ikr} = 2i \int_{\mu}^{\infty} dk \frac{k}{\sqrt{k^2 - \mu^2}} e^{-kr + t\sqrt{k^2 - \mu^2}}
\]

This integral is tractable. The 'cheap' way to proceed is to realize that we could have begun by using the Lorentz invariance of the integral to our advantage, and boosted to a frame where \( t = 0 \). This doesn’t affect any of the manipulations in the contour integration, so we can simply set \( t = 0 \) on the RHS of (1), which looks almost (but not quite) like an integral identity for a modified Bessel function of the second kind. Now we have

\[
\int \frac{d^3k}{(2\pi)^3} \frac{e^{ikx}}{\omega_k} = \frac{1}{2\pi^2} \int_{\mu}^{\infty} dk \frac{ke^{-kr}}{\sqrt{k^2 - \mu^2}} = \int_{\mu}^{\infty} dk \frac{\partial}{\partial r} \frac{e^{-k(\mu r)}}{\sqrt{k^2 - 1}} = -\int_{1}^{\infty} dk \frac{\partial}{\partial r} K_0(\mu r) = \frac{\mu}{2\pi^2} K_1(\mu r) = \frac{\mu K_1(\mu \sqrt{x^2 - t^2})}{2\pi^2 \sqrt{x^2 - t^2}}
\]

The third and fourth lines of (2) follow from standard Bessel function identities. In the fifth line we boost back to an arbitrary frame so that the time coordinate is explicit.