Perturbative Structures in Gravity

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Outline

1) Significant advances in scattering amplitudes.
2) Hidden structures in gauge and gravity amplitudes.
   — a duality between color and kinematics.
   — gravity from gauge theory.
   — scattering equations.
3) Application: Demonstration and understanding of tame UV behavior in supergravity theories.
4) Links to nonperturbative physics.
Remarkable Progress

The progress is associated with some words you might have heard:

1. “On-shell Revolution”
2. “NLO QCD Revolution”
3. Remarkable structures and new insight:
   - “Twistors” (Witten; Roiban, Sparadlin, Volovich; Mason, Skinner; etc)
   - “Amplituhedron” (Arkani-Hamed, Trnka, et al)
   - “Duality between color and kinematics” (ZB, Carrasco, Johansson)
4. Most importantly: Calculations deemed impossible are now commonplace.
Duality Between Color and Kinematics

Color factors based on a Lie algebra: \([T^a, T^b] = if^{abc}T^c\)

**Jacobi Identity**

\[f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0\]

Use \(1 = s/s = t/t = u/u\) to assign 4-point diagram to others.

\[s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2 \quad u = (k_1 + k_3)^2\]

Color factors satisfy Jacobi identity:

Numerator factors satisfy similar identity:

Color and kinematics satisfy the same identity
Duality Between Color and Kinematics

Consider five-point tree amplitude:

\[ A_{5}^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i \, n_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \]

\[ c_1 \equiv f_{a_3 a_4 b} f_{b a_5 c} f_{c a_1 a_2} , \quad c_2 \equiv f_{a_3 a_4 b} f_{b a_2 c} f_{c a_1 a_5} , \quad c_3 \equiv f_{a_3 a_4 b} f_{b a_1 c} f_{c a_2 a_5} \]

\[ n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots \]

\[ c_1 - c_2 + c_3 = 0 \iff n_1 - n_2 + n_3 = 0 \]

Claim: At n-points we can always find a rearrangement where color and kinematics satisfy the same algebraic constraint equations.

Nontrivial constraints on amplitudes in field theory and string theory

BCJ, Bjerrum-Bohr, Feng, Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer; Cachazo; Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer
Gravity and Gauge Theory

**gauge theory:**
\[ \frac{1}{g^{n-2}} A_n^{\text{tree}} (1, 2, 3, \ldots, n) = \sum_i \frac{n_i c_i}{\prod_{\alpha} p_{\alpha i}^2} \]

**sum over diagrams with only 3 vertices**
\[ c_i \sim f^{a_1 a_2 b_1} f^{b_1 b_2 a_5} f^{b_2 a_4 a_5} \]

**Assume we have:**
\[ c_1 + c_2 + c_3 = 0 \iff n_1 + n_2 + n_3 = 0 \]

**Then:**
\[ c_i \Rightarrow \tilde{n}_i \quad \text{kinematic numerator of second gauge theory} \]

**Proof:** ZB, Dennen, Huang, Kiermaier

**gravity:**
\[ -i \left( \frac{2}{\kappa} \right)^{(n-2)} M_n^{\text{tree}} (1, 2, \ldots, n) = \sum_i \frac{n_i \tilde{n}_i}{\prod_{\alpha} p_{\alpha i}^2} \]

**Encodes KLT tree relations**

Gravity numerators are a double copy of gauge-theory ones.

This works for ordinary Einstein gravity and susy versions.

Cries out for a unified description of the sort given by string theory!
Gravity From Gauge Theory

$$-i \left( \frac{2}{\kappa} \right)^{n-2} \mathcal{M}^{\text{tree}}_n(1, 2, \ldots, n) = \sum_i \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2}$$

$\mathcal{N}$  $\tilde{\mathcal{N}}$

$N = 8$ sugra:  $(N = 4 \text{ sYM}) \times (N = 4 \text{ sYM})$

$N = 4$ sugra:  $(N = 4 \text{ sYM}) \times (N = 0 \text{ sYM})$

Spectrum controlled by simple tensor product of YM theories.
Recent papers show more sophisticated lower-susy cases.

Carrasco, Chiodaroli, Günaydin and Roiban (2012); Borsten, Duff, Hughes and Nagy (2013)
Loop-Level Conjecture

\[ \left( -i \right)^L \frac{1}{g^{n-2+2L}} A_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2} \]

\[ \left( -i \right)^{L+1} \frac{1}{(\kappa/2)^{n-2+2L}} M_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n_j}}{\prod_{\alpha_j} p_{\alpha_j}^2} \]

\( c_i + c_j + c_k = 0 \)

\( n_i + n_j + n_k = 0 \)

Loop-level is identical to tree-level one except for symmetry factors and loop integration. This works if numerator satisfies duality.
Gravity integrands are free!

Ideas generalize to loops:

If you have a set of duality satisfying numerators. To get:

gauge theory $\rightarrow$ gravity theory

simply take

color factor $\rightarrow$ kinematic numerator

$\text{color factor } c_k = c_i - c_j$

$\text{kinematic numerator } n_k = n_i - n_j$

Gravity loop integrands are trivial to obtain once we have gauge theory in a form where duality works.
We know that this structure has important consequences for the UV behavior of supergravity theories.

Do we have any examples where a divergence vanishes but the standard symmetries suggest valid counterterms?

Yes!

Two examples in half-maximal supergravity:

• $D = 5$ at 2 loops.
• $D = 4$ at 3 loops
One-Loop Warmup in Half-Maximal Sugra

Generic color decomposition:

\[ A_Q^{(1)} = i g^4 \left[ c_{1234} A_Q^{(1)} (1, 2, 3, 4) + c_{1342} A_Q^{(1)} (1, 3, 4, 2) + c_{1423} A_Q^{(1)} (1, 4, 2, 3) \right] \]

Q = \# supercharges

\[ Q = 0 \] is pure non-susy YM

\[ c_{1234} \] is color factor of this box diagram

\[ s = (k_1 + k_2)^2 \]

\[ t = (k_2 + k_3)^2 \]

To get \( Q + 16 \) supercharge supergravity take 2\(^{nd}\) copy \( N = 4 \) sYM

\( N = 4 \) sYM numerators very simple: independent of loop momentum

\[ n_{1234} = n_{1342} = n_{1423} = st A_{Q=16}^{\text{tree}} (1, 2, 3, 4) \]

\[ c_{1234}^{(1)} \rightarrow n_{1234} \]

\[ M_{Q+16}^{(1)} = i \left( \frac{k}{2} \right)^4 st A_{Q=16}^{\text{tree}} (1, 2, 3, 4) \left[ A_Q^{(1)} (1, 2, 3, 4) + A_Q^{(1)} (1, 3, 4, 2) + A_Q^{(1)} (1, 4, 2, 3) \right] \]
One-loop divergences in pureYM

ZB, Davies, Dennen, Huang

Go to a basis of color factors

Three independent one-loop color tensors

\[ b_1^{(0)} = \tilde{f}a_1a_2b \tilde{f}b a_3a_4 \]

\[ b_2^{(0)} = \tilde{f}a_2a_3b \tilde{f}b a_4a_1 \]

\[ b_1^{(1)} \equiv c_{1234}^{(1)} = \tilde{f}a_1b_2b_1 \tilde{f}a_2b_3b_2 \tilde{f}a_3b_4b_3 \tilde{f}a_4b_1b_4 \]

All other color factors expressible in terms of these three:

\[ A_Q^{(1)} = ig^4 \left[ b_1^{(1)} \left( A_Q^{(1)}(1, 2, 3, 4) + A_Q^{(1)}(1, 3, 4, 2) + A_Q^{(1)}(1, 4, 2, 3) \right) \right. \\
\left. - \frac{1}{2} C_A b_1^{(0)} A_Q^{(1)}(1, 3, 4, 2) - \frac{1}{2} C_A b_2^{(0)} A_Q^{(1)}(1, 4, 2, 3) \right] \]

\[ C_A = 2N_c \text{ for SU}(N_c) \]
One-loop divergences in pure YM

In a basis of color factors:

\[ \mathcal{A}_Q^{(1)} = ig^4 \left[ b_1^{(1)} \left( A_Q^{(1)}(1, 2, 3, 4) + A_Q^{(1)}(1, 3, 4, 2) + A_Q^{(1)}(1, 4, 2, 3) \right) \right. \]
\[ \left. - \frac{1}{2} C_A b_1^{(0)} A_Q^{(1)}(1, 3, 4, 2) - \frac{1}{2} C_A b_2^{(0)} A_Q^{(1)}(1, 4, 2, 3) \right] \]

\( Q \) supercharges (mainly interested in \( Q = 0 \))

- **\( D = 4 \):** \( F^2 \) is only allowed counterterm by renormalizability.
- **\( D = 6 \):** \( F^3 \) counterterm: 1-loop color tensor not allowed.

\[ F^3 = f^{abc} F_\nu^{a\mu} F_\sigma^{b\nu} F_{\mu}^{c\sigma} \]

\[ A_Q^{(1)}(1, 2, 3, 4) + A_Q^{(2)}(1, 3, 4, 2) + A_Q^{(1)}(1, 4, 2, 3) \]

\[ = 0 \quad \text{\( D=4,6 \) div.} \]

Gravity

\[ \mathcal{M}_{Q+16}^{(1)}(1, 2, 3, 4) \]

\[ = 0 \quad \text{\( D=4,6 \) div.} \]
Two Loop Half Maximal Sugra in $D = 5$

$$A_Q^{(2)} = -g^6 \left[ c_{1234}^P A_Q^P(1, 2, 3, 4) + c_{3421}^P A_Q^P(3, 4, 2, 1) 
+ c_{1234}^{NP} A_Q^{NP}(1, 2, 3, 4) + c_{3421}^{NP} A_Q^{NP}(3, 4, 2, 1) \right] + \text{cyclic}$$

$D = 5 F^3$ counterterm: 1,2-loop color tensors forbidden!

1) Go to color basis.
2) Demand no forbidden color tensors in pure YM divergence.
3) Replace color factors with kinematic numerators.

Gravity

$$M_{16+Q}^{(2)}(1, 2, 3, 4) \bigg|_{D=5 \text{ div.}} = 0$$
Half Maximal Supergravity in $D = 5$

2 loop cancellations:

\[ \mathcal{A}_{Q}^{(2)} = -g^6 \left[ c_{1234}^{P} A_{Q}^{P}(1, 2, 3, 4) + c_{3421}^{P} A_{Q}^{P}(3, 4, 2, 1) \right. \]
\[ \left. + c_{1234}^{NP} A_{Q}^{NP}(1, 2, 3, 4) + c_{3421}^{NP} A_{Q}^{NP}(3, 4, 2, 1) + \text{cyclic} \right] \]

\[ \mathcal{M}_{Q+16}^{(2)} = -i \left( \frac{\kappa}{2} \right)^6 st A_{Q=16}^{\text{tree}}(1, 2, 3, 4) \left[ s \left( A_{Q}^{P}(1, 2, 3, 4) + A_{Q}^{NP}(1, 2, 3, 4) \right. \right. \]
\[ \left. \left. + A_{Q}^{P}(3, 4, 2, 1) + A_{Q}^{NP}(3, 4, 2, 1) \right) + \text{cyclic} \right] \]

Equations that eliminate forbidden 2-loop color tensor:

\[ 0 = t(A_{Q}^{P}(1, 3, 4, 2) + A_{Q}^{P}(1, 4, 2, 3) + A_{Q}^{P}(3, 1, 4, 2) + A_{Q}^{P}(3, 2, 1, 4) \]
\[ + A_{Q}^{NP}(1, 3, 4, 2) + A_{Q}^{NP}(1, 4, 2, 3) + A_{Q}^{NP}(3, 1, 4, 2) + A_{Q}^{NP}(3, 2, 1, 4) \]
\[ + s(A_{Q}^{P}(1, 3, 4, 2) + A_{Q}^{P}(3, 1, 4, 2) + A_{Q}^{NP}(1, 3, 4, 2) + A_{Q}^{NP}(3, 1, 4, 2)) \bigg|_{D=5 \text{ div.}} \]

\[ 0 = s(A_{Q}^{P}(1, 2, 3, 4) + A_{Q}^{P}(1, 3, 4, 2) + A_{Q}^{P}(3, 1, 4, 2) + A_{Q}^{P}(3, 4, 2, 1) \]
\[ + A_{Q}^{NP}(1, 2, 3, 4) + A_{Q}^{NP}(1, 3, 4, 2) + A_{Q}^{NP}(3, 1, 4, 2) + A_{Q}^{NP}(3, 4, 2, 1) \]
\[ + t(A_{Q}^{P}(1, 3, 4, 2) + A_{Q}^{P}(3, 1, 4, 2) + A_{Q}^{NP}(1, 3, 4, 2) + A_{Q}^{NP}(3, 1, 4, 2)) \bigg|_{D=5 \text{ div.}} \]

Replace color factor with kinematic numerator:

\[ \mathcal{M}_{16+Q}^{(2)}(1, 2, 3, 4) \bigg|_{D=5 \text{ div.}} = 0 \]

Half-maximal supergravity four-point divergence vanishes because forbidden color tensor cancels in pure YM theory
At least for 2 loops in $D = 5$ we have identified the source of unexpected UV cancellations in half-maximal supergravity:

It is the same magic found by ’t Hooft and Veltman 40 years ago preventing forbidden divergences appearing in ordinary non-susy gauge theory!

Completely explains the $D = 5$ two-loop half maximal sugra case, which still remains mysterious from standard supergravity Viewpoint as investigated recently by Bossard, Howe, Stelle.

The higher-loop cases much more complicated to unravel them. Studying them now. I expect a lot of progress in the coming year because of the advent of new tools.
How can one take two copies of the gauge-theory Lagrangian and get a gravity Lagrangian? A Lagrangian formulation would give us a means from attacking nonperturbative problems.

Add zero to the YM Lagrangian in a special way:

$$L'_{YM} = -\frac{1}{2} g^3 (f^{a_1 a_2 b} f^{b a_3 c} + f^{a_2 a_3 b} f^{b a_1 c} + f^{a_3 a_1 b} f^{b a_2 c}) f^{ca_4 a_5}$$

Through five points:

- Feynman diagrams satisfy the color-kinematic duality.
- Introduce auxiliary field to convert nonlocal interactions into local three-point interactions.
- Take two copies: you get gravity!

At each order need to add more and more vanishing terms. General pattern still not understood.
Consider self dual YM. Work in space-cone gauge of Chalmers and Siegel

\[ u = t - z \]
\[ w = x + iy \]

**Generators of diffeomorphism invariance:**

\[ L_k = e^{-ik \cdot x} (-k_w \partial_u + k_u \partial_w) \]

**The Lie Algebra:**

\[ [L_{p_1}, L_{p_2}] = iX(p_1, p_2) L_{p_1 + p_2} = iF_{p_1 p_2}^k L_k \]

The \( X(p_1, p_2) \) are YM vertices, valid for self-dual configurations.

Expects why numerators satisfy Jacobi Identity

YM inherits diffeomorphism symmetry of gravity!

We need to go beyond self dual.
There has been some recent progress based on what are called the scattering equations. Exactly the saddle point equations found by Gross and Mende in tree level string theory for $s/t$ large.

Nontrivial polynomial equations with $(n-3)!$ solutions at $n$-points

\[ \sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0 \]

$\sigma_a$ is point on a sphere
$k_a$ is on-shell momentum.

On support of scattering equations a Lie algebra can be defined so that color kinematic duality holds.

\[ [\hat{V}_A^+, \hat{V}_B^+] = iX_{A,B} \hat{V}_{A+B}^+ \]

$V_A^+$ not known explicitly.
Properties imposed so that BCJ duality works correctly.

These ideas need to be pushed to higher loops.
Summary

• A new duality conjectured between color and kinematics. When manifest, it trivially gives us (super) gravity loop integrands.

• Surprisingly good UV behavior of supergravity uncovered.

• For half-maximal supergravity in $D = 5$, 2 loops we know precisely the origin of the “magical UV cancellations”: it is *standard magic* that restricts counterterms of nonsusy YM.

• We are beginning to understand the underlying symmetries but more work is needed to properly unravel this.

Key message: duality between color and kinematics is a fundamental tool for understanding perturbative gravity. Its nonperturbative implication is an important problem that needs further work.