Perturbative Structures in Gravity

March 17, 2014

UC Santa Barbara
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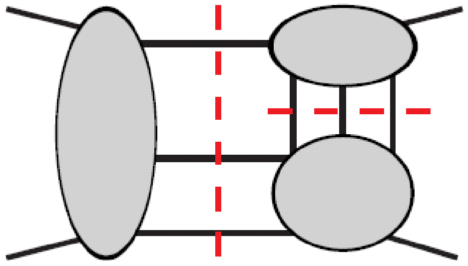


Outline

1) **Significant advances in scattering amplitudes.**

2) **Hidden structures in gauge and gravity amplitudes.**
   - **— a duality between color and kinematics.**
   - **— gravity from gauge theory.**
   - **— scattering equations.**

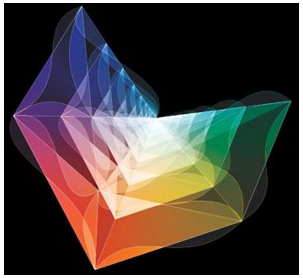
3) **Application: Demonstration and understanding of tame UV behavior in supergravity theories.**

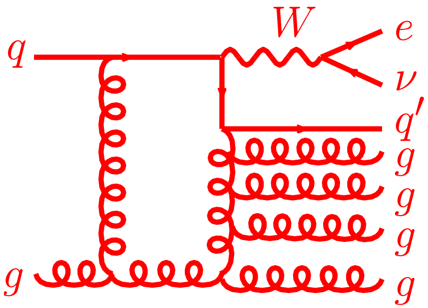
4) **Links to nonperturbative physics.**

Remarkable Progress

The progress is associated with some words you might have heard:

1. "On-shell Revolution"
2. "NLO QCD Revolution"
3. Remarkable structures and new insight:
   - "Twistors" Witten; Roiban, Sparadlin, Volovich; Mason, Skinner; etc
   - "Amplituhedron" Arkani-Hamed, Trnka, et al
   - "Duality between color and kinematics" ZB, Carrasco, Johansson
4. Most importantly: Calculations deemed impossible are now commonplace.

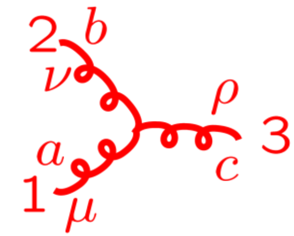




Duality Between Color and Kinematics

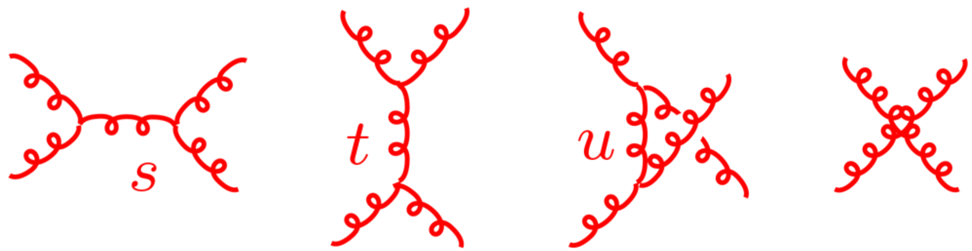
coupling constant → color factor → momentum dependent kinematic factor

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$s = (k_1 + k_2)^2$
$t = (k_1 + k_4)^2$
$u = (k_1 + k_3)^2$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Color and kinematics satisfy the same identity

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (BCJ)

Consider five-point tree amplitude:

sum is over diagrams

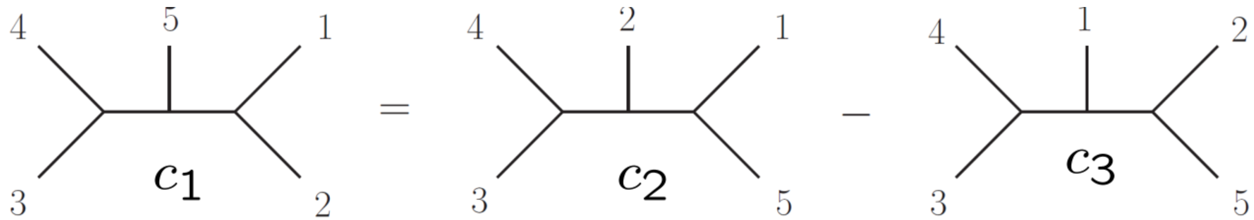
color factor

kinematic numerator factor

Feynman propagators

gauge theory

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$



$$c_1 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}, \quad c_2 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}, \quad c_3 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5\, k_2 \cdot \varepsilon_1\, \varepsilon_2 \cdot \varepsilon_3\, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$\boxed{c_1 - c_2 + c_3 = 0 \Leftrightarrow n_1 - n_2 + n_3 = 0}$$

Claim: At n-points we can always find a rearrangement where color and kinematics satisfy the same algebraic constraint equations.

Nontrivial constraints on amplitudes in field theory and string theory

BCJ, Bjerrum-Bohr, Feng, Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer; Cachazo; Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer

BCJ

Gravity and Gauge Theory

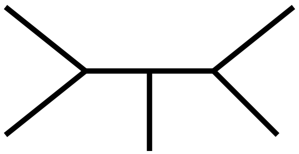
kinematic numerator　　color factor

gauge theory: $\displaystyle \frac{1}{g^{n-2}}\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$ **sum over diagrams with only 3 vertices**

$$c_i \sim f^{a_1 a_2 b_1} f^{b_1 b_2 a_5} f^{b_2 a_4 a_5}$$

Assume we have:

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$



Then: $c_i \Rightarrow \tilde{n}_i$ **kinematic numerator of second gauge theory**

Proof: ZB, Dennen, Huang, Kiermaier

gravity: $\displaystyle \boxed{-i\left(\frac{2}{\kappa}\right)^{(n-2)} \mathcal{M}_n^{\text{tree}}(1,2,\ldots,n) = \sum_i \frac{n_i\, \tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}}$ **Encodes KLT tree relations**

Gravity numerators are a double copy of gauge-theory ones.

This works for ordinary Einstein gravity and susy versions.

Cries out for a unified description of the sort given by string theory!

Gravity From Gauge Theory

BCJ

$$-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\text{tree}}(1,2,\ldots,n)=\sum_i\frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i}p^2_{\alpha_i}}$$

n $\tilde{n}$

N **= 8 sugra: (*N* = 4 sYM)× (*N* = 4 sYM)**

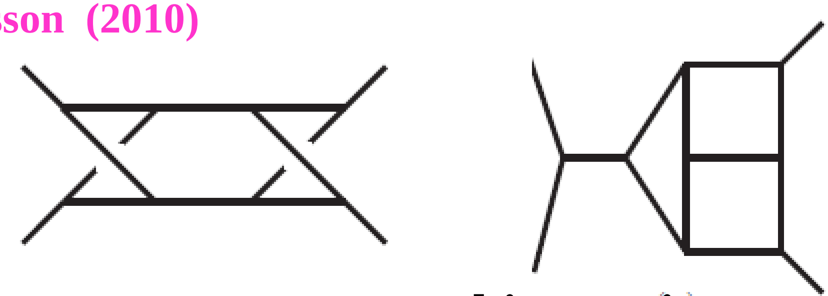
N **= 4 sugra: (*N* = 4 sYM)× (*N* = 0 sYM)**

Spectrum controlled by simple tensor product of YM theories.
Recent papers show more sophisticated lower-susy cases.

Carrasco, Chiodaroli, Günaydin and Roiban (2012); Borsten, Duff, Hughes and Nagy (2013)

Loop-Level Conjecture

ZB, Carrasco, Johansson (2010)



$$c_i + c_j + c_k = 0$$
$$n_i + n_j + n_k = 0$$

sum is over diagrams

kinematic numerator

color factor

$$\frac{(-i)^L}{g^{n-2+2L}} \mathcal{A}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j c_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gauge theory

propagators

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}} \mathcal{M}_n^{\text{loop}} = \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2}$$

gravity

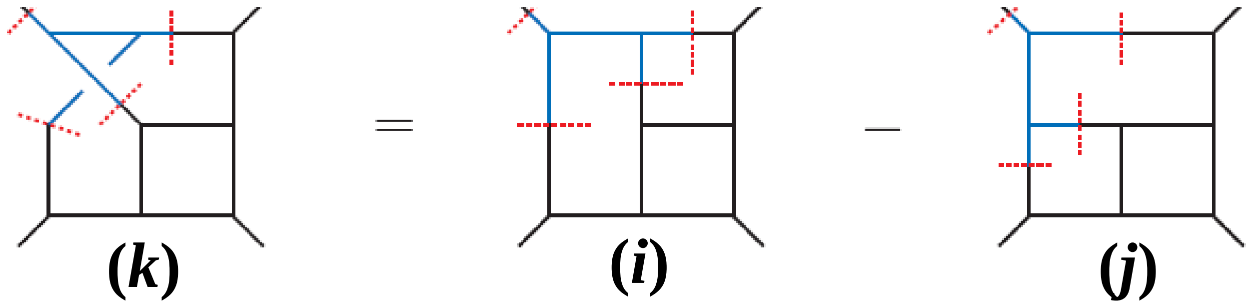
symmetry factor

Loop-level is identical to tree-level one except for symmetry factors and loop integration.
This works if numerator satisfies duality.

Gravity integrands are free!

BCJ

Ideas generalize to loops:



color factor

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

kinematic numerator

If you have a set of duality satisfying numerators.

To get:

gauge theory ⟶ gravity theory

simply take

color factor ⟶ kinematic numerator

$$c_k \longrightarrow n_k$$

Gravity loop integrands are trivial to obtain once we have gauge theory in a form where duality works.

Examples of Magical Cancellations?

We know that this structure has important consequences for the UV behavior of supergravity theories.

ZB, Carrasco, Dixon, Johansson, Roiban;
ZB, Davies, Dennen, Huang

Do we have any examples where a divergence vanishes but the standard symmetries suggest valid counterterms?

Yes!

Two examples in half-maximal supergravity :

- **$D = 5$ at 2 loops.**
- **$D = 4$ at 3 loops**

One-Loop Warmup in Half-Maximal Sugra

ZB, Boucher-Veronneau ,Johansson
ZB, Davies, Dennen, Huang

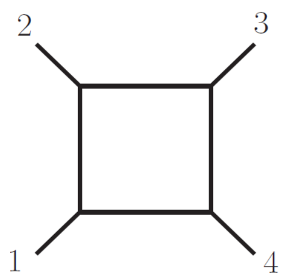
Generic color decomposition:

Dixon, Del Duca, Maltoni

color factor

$$\mathcal{A}_Q^{(1)} = ig^4\Big[c_{1234}^{(1)}A_Q^{(1)}(1,2,3,4) + c_{1342}^{(1)}A_Q^{(1)}(1,3,4,2) + c_{1423}^{(1)}A_Q^{(1)}(1,4,2,3)\Big]$$

Q = # supercharges **Q = 0 is pure non-susy YM**



$c_{1234}^{(1)}$ **is color factor of this box diagram**

$$s = (k_1 + k_2)^2$$
$$t = (k_2 + k_3)^2$$

To get Q + 16 supercharge supergravity take 2nd copy *N* = 4 sYM

***N* = 4 sYM numerators very simple: independent of loop momentum**

$$n_{1234} = n_{1342} = n_{1423} = st A_{Q=16}^{\text{tree}}(1,2,3,4) \qquad c_{1234}^{(1)} \to n_{1234}$$

$$\mathcal{M}_{Q+16}^{(1)} = i\left(\frac{\kappa}{2}\right)^4 st A_{Q=16}^{\text{tree}}(1,2,3,4)\Big[A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3)\Big]$$

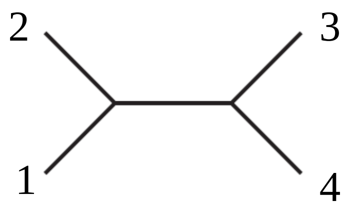
One-loop divergences in pure YM

ZB, Davies, Dennen, Huang

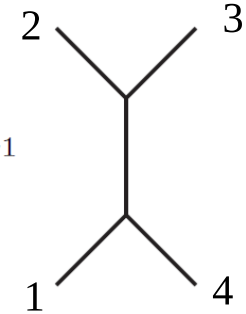
Go to a basis of color factors

Three independent one-loop color tensors

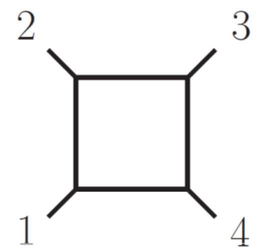
$$b_1^{(0)} = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}$$



$$b_2^{(0)} = \tilde{f}^{a_2 a_3 b} \tilde{f}^{b a_4 a_1}$$



$$b_1^{(1)} \equiv c_{1234}^{(1)} = \tilde{f}^{a_1 b_2 b_1} \tilde{f}^{a_2 b_3 b_2} \tilde{f}^{a_3 b_4 b_3} \tilde{f}^{a_4 b_1 b_4}$$



All other color factors expressible in terms of these three:

one-loop color tensor

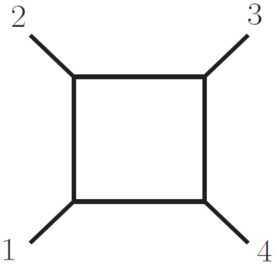
$$\mathcal{A}_Q^{(1)} = i g^4 \Big[b_1^{(1)} \Big(A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3) \Big) - \frac{1}{2} C_A b_1^{(0)} A_Q^{(1)}(1,3,4,2) - \frac{1}{2} C_A b_2^{(0)} A_Q^{(1)}(1,4,2,3) \Big]$$

tree color tensor

$C_A = 2\, N_c$ for SU(N_c)

One-loop divergences in pure YM

In a basis of color factors:



one-loop color tensor

$$\mathcal{A}_Q^{(1)} = ig^4\Big[b_1^{(1)}\Big(A_Q^{(1)}(1,2,3,4) + A_Q^{(1)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3)\Big)$$

$$-\frac{1}{2}C_A b_1^{(0)} A_Q^{(1)}(1,3,4,2) - \frac{1}{2}C_A b_2^{(0)} A_Q^{(1)}(1,4,2,3)\Big]$$

tree color tensor

Q **supercharges (mainly interested in** ***Q*** **= 0)**

D **= 4:** ***F***^{2} **is only allowed counterterm by renormalizability**

1-loop color tensor ***not*** **allowed.**

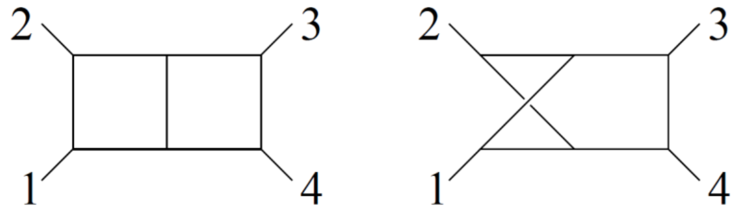
D **= 6:** ***F***^{3} **counterterm: 1-loop color tensor again** ***not*** **allowed.**

$$F^3 = f^{abc} F^{a\mu}_{\nu} F^{b\nu}_{\sigma} F^{c\sigma}_{\mu}$$

$$A_Q^{(1)}(1,2,3,4) + A_Q^{(2)}(1,3,4,2) + A_Q^{(1)}(1,4,2,3)\Big|_{D=4,6\ \mathrm{div.}} = 0$$

gravity $\boxed{\mathcal{M}_{Q+16}^{(1)}(1,2,3,4)\Big|_{D=4,6\ \mathrm{div.}} = 0}$

Two Loop Half Maximal Sugra in $D = 5$



ZB, Davies, Dennen, Huang

$$\mathcal{A}_Q^{(2)} = -g^6 \Big[c^{\rm P}_{1234} A_Q^{\rm P}(1,2,3,4) + c^{\rm P}_{3421} A_Q^{\rm P}(3,4,2,1) + c^{\rm NP}_{1234} A_Q^{\rm NP}(1,2,3,4) + c^{\rm NP}_{3421} A_Q^{\rm NP}(3,4,2,1) + \text{ cyclic} \Big]$$

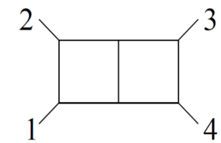
***D* = 5 *F*^{3} counterterm: 1,2-loop color tensors forbidden!**

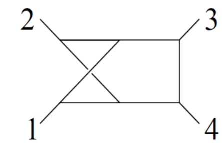
1) **Go to color basis.**
2) **Demand no forbidden color tensors in pure YM divergence.**
3) **Replace color factors with kinematic numerators.**

gravity $$\mathcal{M}^{(2)}_{16+Q}(1,2,3,4)\Big|_{D=5\,\text{div.}} = 0$$

Half Maximal Supergravity in $D = 5$

2 loop cancellations:





gauge theory

$$\mathcal{A}_Q^{(2)} = -g^6\Big[c^{\rm P}_{1234}A_Q^{\rm P}(1,2,3,4) + c^{\rm P}_{3421}A_Q^{\rm P}(3,4,2,1) + c^{\rm NP}_{1234}A_Q^{\rm NP}(1,2,3,4) + c^{\rm NP}_{3421}A_Q^{\rm NP}(3,4,2,1) + \text{ cyclic}\Big]$$

gravity

$$\mathcal{M}_{Q+16}^{(2)} = -i\left(\frac{\kappa}{2}\right)^6 st A^{\rm tree}_{Q=16}(1,2,3,4)\Big[s\Big(A_Q^{(\rm P)}(1,2,3,4) + A_Q^{(\rm NP)}(1,2,3,4) + A_Q^{(\rm P)}(3,4,2,1) + A_Q^{(\rm NP)}(3,4,2,1)\Big) + \text{cyclic}\Big]$$

Equations that eliminate forbidden 2-loop color tensor:

$$\begin{aligned} 0 = {} & t(A_Q^{\rm P}(1,3,4,2) + A_Q^{\rm P}(1,4,2,3) + A_Q^{\rm P}(3,1,4,2) + A_Q^{\rm P}(3,2,1,4) \\ & + A_Q^{\rm NP}(1,3,4,2) + A_Q^{\rm NP}(1,4,2,3) + A_Q^{\rm NP}(3,1,4,2) + A_Q^{\rm NP}(3,2,1,4) \\ & + s(A_Q^{\rm P}(1,3,4,2) + A_Q^{\rm P}(3,1,4,2) + A_Q^{\rm NP}(1,3,4,2) + A_Q^{\rm NP}(3,1,4,2))\Big|_{D=5 \text{ div.}}, \\ 0 = {} & s(A_Q^{\rm P}(1,2,3,4) + A_Q^{\rm P}(1,3,4,2) + A_Q^{\rm P}(3,1,4,2) + A_Q^{\rm P}(3,4,2,1) \\ & + A_Q^{\rm NP}(1,2,3,4) + A_Q^{\rm NP}(1,3,4,2) + A_Q^{\rm NP}(3,1,4,2) + A_Q^{\rm NP}(3,4,2,1)) \\ & + t(A_Q^{\rm P}(1,3,4,2) + A_Q^{\rm P}(3,1,4,2) + A_Q^{\rm NP}(1,3,4,2) + A_Q^{\rm NP}(3,1,4,2))\Big|_{D=5 \text{ div.}} \end{aligned}$$

Replace color factor with kinematic numerator:

$$\mathcal{M}^{(2)}_{16+Q}(1,2,3,4)\Big|_{D=5 \text{ div.}} = 0$$

gravity

Half-maximal supergravity four-point divergence vanishes because forbidden color tensor cancels in pure YM theory

Two Loop D = 5 UV Magic

ZB, Davies, Dennen, Huang

At least for 2 loops in *D* = 5 we have identified the source of unexpected UV cancellations in half-maximal supergravity:

It is the *same* magic found by ’t Hooft and Veltman 40 years ago preventing forbidden divergences appearing in ordinary non-susy gauge theory!

Completely explains the *D* = 5 two-loop half maximal sugra case, which still remains mysterious from standard supergravity Viewpoint as investigated recently by Bossard, Howe, Stelle.

The higher-loop cases much more complicated to unravel them. Studying them now. I expect a lot of progress in the coming year because of the advent of new tools.

Lagrangians

ZB, Dennen, Huang, Kiermaier; Tolloti, Weinzierl

$$L_{\mathrm{YM}} = \frac{1}{g^2} F^2 \qquad\qquad L_{\mathrm{gravity}} = \frac{2}{\kappa^2} \sqrt{-g}\, R$$

How can one take two copies of the gauge-theory Lagrangian and get a gravity Lagrangian? A Lagragian formulation would give us a means from attacking nonperturbative problems.

Add zero to the YM Lagrangian in a special way:

$$\mathcal{L}'_5 = -\tfrac{1}{2} g^3 \big(f^{a_1 a_2 b} f^{b a_3 c} + f^{a_2 a_3 b} f^{b a_1 c} + f^{a_3 a_1 b} f^{b a_2 c}\big) f^{c a_4 a_5} \times \partial_{[\mu} A^{a_1}_{\nu]} A^{a_2}_{\rho} A^{a_3 \mu} \frac{1}{\Box} (A^{a_4 \nu} A^{a_5 \rho}) \;= 0$$

Through five points:

- **Feynman diagrams satisfy the color-kinematic duality.**
- **Introduce auxiliary field to convert nonlocal interactions into local three-point interactions.**
- **Take two copies: you get gravity!**

$$A^\mu \tilde{A}^\nu \;\to\; h^{\mu\nu}$$

At each order need to add more and more vanishing terms. General pattern still not understood.

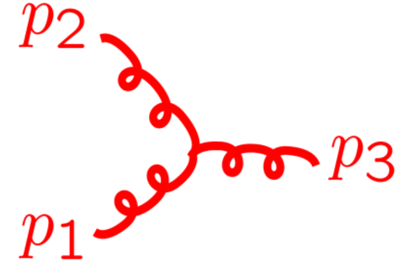
Symmetries

Monteiro and O'Connell

Consider self dual YM. Work in space-cone gauge of Chalmers and Siegel

$$u = t - z$$
$$w = x + iy$$

Generators of diffeomorphism invariance:



$$L_k = e^{-ik\cdot x}(-k_w\partial_u + k_u\partial_w)$$

The Lie Algebra:

YM vertex

$$[L_{p_1}, L_{p_2}] = iX(p_1,p_2)L_{p_1+p_2} = iF_{p_1 p_2}{}^{k} L_k$$

The $X(p_1, p_2)$ are YM vertices, valid for self-dual configurations.

Explains why numerators satisfy Jacobi Identity

YM inherits diffeomorphism symmetry of gravity!

We need to go beyond self dual.

Scattering Equations

Cachazo, He and Yuan

There has been some recent progress based on what are called the scattering equations.

Monteiro and O'Connell

$$\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0$$

σ_a is point on a sphere
k_a is on-shell momentum.
Nontrivial polynomial equations with (n-3)! solutions at n-points

Exactly the saddle point equations found by Gross and Mende in tree level string theory for *s/t* large.

On support of scattering equations a Lie algebra can be defined so that color kinematic duality holds.

$$[\hat{V}^+_A, \hat{V}^+_B] = i X_{A,B} \hat{V}^+_{A+B}$$

V^+_A not known explicitly.
Properties imposed so that BCJ duality works correctly.

These ideas need to be pushed to higher loops.

Adamo, Casali, Skinner

Summary

- A new duality conjectured between color and kinematics. When manifest, it trivially gives us (super) gravity loop integrands.

- Surprisingly good UV behavior of supergravity uncovered.

- For half-maximal supergravity in $D = 5$, 2 loops we know precisely the origin of the "magical UV cancellations": it is *standard magic* that restricts counterterms of nonsusy YM.

- We are beginning to understand the underlying symmetries but more work is needed to properly unravel this.

Key message: duality between color and kinematics is a fundamental tool for understanding perturbative gravity. Its nonnperturbative implication is an important problem that needs further work.