

Errata for Instructor's Solutions Manual for Gravity, An Introduction to Einstein's General Relativity *Versions 1.0 (first printing) and 1.1*

Updated 11/20/2006

(Thanks to Ted Jacobson, John Friedman, Don Page, and Mario Serna who provided most of these.)

Problem 2.2:

Replace Solution with the following:

Gauss' triangle was located near the surface of the Earth. The relevant radius in the expression (2.1) is the distance of the triangle from the center of attraction. Eq (2.1) gives the approximate size of the effect of the Earth where $R_{\oplus} = 6378$ km is the radius of the Earth. However for Sun the relevant radius is the distance of the triangle from the center of the Sun, which approximately the size of the Earth's orbit $r_{\oplus} \approx 1.4 \times 10^8$ km. Then, from (2.1) the ratio of the effect of the Sun to that for the Earth can be written

$$(\text{ratio})_{\text{Sun to Earth}} \sim \left(\frac{GM_{\odot}}{c^2} \right) \left(\frac{c^2}{GM_{\oplus}} \right) \left(\frac{R_{\oplus}}{r_{\oplus}} \right)^3 .$$

For the Earth $GM_{\oplus}/c^2 = .443$ cm and for the Sun $GM_{\odot}/c^2 = 1.48$ km. For the ratio we get

$$(\text{ratio})_{\text{Sun to Earth}} \sim 10^{-19} \quad (!) .$$

The effect of the Sun is therefore much smaller than the effect of the Earth.

Problem 2.9:

In the 5th printing and later the solution has to be revised in an obvious way to reflect the minor changes in the data in the problem.

Problem 4.4:

In the last equation it should be -3.4 instead of $+3.4$.

Problem 4.14:

The solution given is when \vec{V} and \vec{v} are colinear. The general case is not too difficult and will appear in the next version of this manual.

Problem 5.17:

In the fourth printing and onward the problem is replaced with the following clearer statement and with the corresponding changes in notation in the solution:

[C] (*Relativistic Beaming*) A body emits photons of frequency ω_* at equal rates in all directions in its rest frame. A detector at rest in this frame a large distance away (compared to the size of the body) receives photons at a rate per unit solid angle $(dN/dtd\Omega)_*$ [photons/(sec·sr)] that is independent of direction. In an inertial frame (t', x', y', z') in which an observer is at rest the body is moving with speed V along the x' -axis.

1. Derive (5.75) relating a photon's direction of propagation in the rest frame to the direction of propagation in the observer's frame.
2. Find the rate at which photons are received per unit solid angle $dN/dt'd\Omega'$ a large distance away in the observer's frame as a function of angle α' from the x' -axis. [Hint: Remember that the time interval between the reception of two photons by a stationary observer is not the same as the time interval between their emission if the source is moving.]
3. Find the luminosity per unit solid angle $dL'/d\Omega'$ [erg/(sec·sr)] a large distance away as a function of the angle α' in the observer's frame.
4. Discuss the beaming of number and energy in the observer's frame as the velocity of the source approaches the velocity of light.

Solution

1. Suppose a photon is emitted in the rest frame making an angle α with the x -axis. The components of its four momentum \mathbf{p} in this frame are:

$$\mathbf{p} = (p, p \cos \alpha, p \sin \alpha, 0), \quad p = \hbar \omega_*$$

In the observer's frame the components of \mathbf{p} are:

$$p^{t'} = \gamma(p + Vp \cos \alpha), \quad (1)$$

$$p^{x'} = \gamma(p \cos \alpha + Vp), \quad (2)$$

$$p^{y'} = p \sin \alpha. \quad (3)$$

The angle α' made by the photon with the x' axis is then

$$\cos \alpha' = \frac{p^{x'}}{p^{t'}} = \frac{\cos \alpha + V}{1 + V \cos \alpha}.$$

The inverse of this obtained by replacing V by $-V$ is also useful:

$$\cos \alpha = \frac{\cos \alpha' - V}{1 - V \cos \alpha'}. \quad (4)$$

2. The number of photons emitted in a time dt and solid angle $d\Omega$ in the rest frame must be the same as the number emitted in a corresponding interval dt'_e and solid angle $d\Omega'$ in the frame in which the source is moving. We use the notation dt'_e for the time interval between photons at *emission* to reserve dt' for the time interval between photons when they are *received*. $2\pi \sin \alpha' d\alpha' dt' (dN/dt'_e d\Omega')$ is the number of photons emitted at angle α' into an annulus of angular width $d\alpha'$ in time dt'_e . This must be the same as the number emitted in the corresponding annulus in the rest frame — $2\pi \sin \alpha d\alpha dt (dN/dt d\Omega)$ in the corresponding time dt . The time intervals dt and dt'_e are connected by time dilation — $dt'_e = \gamma dt$. The connection between angles (4) allows us to compute:

$$\frac{d(\cos \alpha)}{d(\cos \alpha')} = \frac{1}{\gamma^2 (1 - V \cos \alpha')^2}$$

The result is:

$$\frac{dN}{dt'_e d\Omega'} = \left(\frac{dN}{dt d\Omega} \right) \frac{1}{[\gamma(1 - V \cos \alpha')]^2} \quad (5)$$

Now we connect the time interval for emission dt'_e with the time interval for reception dt' — *both in the same frame where the source is moving*. Suppose one photon is emitted at an angle α' to the motion and travels a distance d to reception. When the second photon is emitted a time dt'_e

later the source has travelled a distance Vdt'_e in the direction of motion. A little geometry shows that the distance this photon travels is shorter by $(Vdt'_e) \cos \alpha'$ when d is large. Thus

$$dt' = dt'_e(1 - V \cos \alpha') \quad (6)$$

Combining this result with (5) we find finally:

$$\frac{dN}{dt'd\Omega'} = \left(\frac{dN}{dt d\Omega} \right) \frac{1}{[\gamma(1 - V \cos \alpha')]^3} \quad (7)$$

3. The energy of the photons emitted at angle α' is, from the inverse of (5.73) or from (1):

$$\hbar\omega_* = E' \gamma(1 - V \cos \alpha')$$

Solving for E' and using the result of part (b) the luminosity per unit solid angle is

$$\frac{dL'}{d\Omega'}(\alpha') = E'(\alpha') \frac{dN}{dt'd\Omega'}(\alpha') = \frac{(dL/d\Omega)}{[\gamma(1 - V \cos \alpha')]^4}$$

4. The ratio of luminosity in the forward to backward direction is

$$\frac{(dL'/d\Omega')(0)}{(dL'/d\Omega')(\pi)} = \left(\frac{1+V}{1-V} \right)^4 \quad (8)$$

which as $V \rightarrow 1$ becomes very large, meaning most of the radiation is beamed forward.

For a solution to the version of this problem in printings 1-3 replace $(dN/dtd\Omega)_*$ with f_* , $dN/dt'd\Omega'$ by $f'(\alpha')$, and $dL'/d\Omega'$ by $L'(\alpha')$.

Further comments: An easy way to remember this result is that the combination:

$$\omega^{-3} \frac{dN}{dt d\Omega}$$

is the same in both frames — an invariant.

As $V \sim 1$, $1/(1 - V) \sim 2\gamma^2$ and the forward/backward ratio (8) becomes $\sim 256\gamma^8$. Since $\gamma \sim 10$ for some matter in active galactic nuclei jets the forward/backward difference in luminosity can be very large.

Solution:

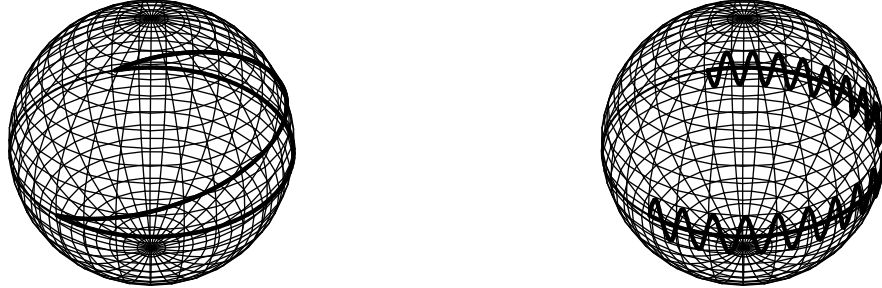
Solution to Problem 6.9:

In (2) change “5.2” to “5.3” and in (3) change “4.5” to “4.6”.

Solution to Problem 6.10: In the 5th printing the problem was revised so that the radioactive species had an *exponential decay time* of 6.5 billion years like ^{238}U implying obvious changes in the solution. In at the end of the problem the exponential decay time is erroneously labeled the half-life although the calculations are o.k.

Solution to Problem 6.12: In the 5th printing this problem was modified slightly. Replace the solution with the following:

There is one great circle through any two points on a sphere but it defines *two* curves of extremal distance connecting the two points. The shorter segment of the great circle is the path of shortest distance between the two points. But the longer segment around the other way is also an extremal curve. However, it provides neither the longest or shortest distance when compared with other nearby paths. To see that there is a *shorter* nearby path imagine the two points are on the equator and slide the long segment up a bit toward the north pole (left figure below). It gets shorter. To see that there is a *longer* nearby path, imagine a path which wiggles a little up above the long segment and below it many many times (right figure below). That will be a longer path. There is no longest path connecting the two points. Imagine for example taking paths that start at one endpoint and circle the globe 10 times, 1000 times, 10,000 times, etc before connecting to the other endpoint. Those are a sequence of increasingly longer paths, and there is no limit to how long they can be.



Solution to Problem 6.14:

Replace with following which has more comments than the earlier solution:

To order $1/c^2$ accuracy the proper time along any of these curves is given by (6.25) so

$$\Delta\tau = P - \frac{1}{c^2} \int_0^P dt \left[\frac{\vec{V}^2}{2} - \Phi \right]$$

in an inertial frame in which the center of the Earth is approximately at rest.

1. For a circular orbit of period P , $\Phi = -GM/R$ where R is related to P by Kepler's law $P^2 = (4\pi^2/GM)R^3$. Further, $V^2/R = GM/R^2$. The net result for the above integral is

$$\Delta\tau = P \left(1 - \frac{3}{2} \frac{GM}{Rc^2} \right)$$

which can be entirely expressed in terms of P and M using Kepler's law.

2. For a stationary observer $\vec{V} = 0$

$$\Delta\tau = P \left(1 - \frac{GM}{Rc^2} \right)$$

which is a longer proper time than a). Therefore, the circular orbit, although an extremal curve, is not a curve of *longest* proper time.

3. There is zero elapsed proper time. A circular orbit is not a curve of shortest proper time either.

There are many other extremal world lines connecting the two points. For instance, there is the world line followed when a ball is thrown radially outwards with the right velocity so that it falls back in time P . More generally the elliptical orbits with the same period P that pass through the radius R will be alternate extremal curves.

Comment: By calculating the second variation of the proper time the circular orbit can be shown to have the longest proper time with respect to *nearby* world lines connecting A and B but part (b) shows that the proper time is not the longest when compared to *any* world line connecting the two points. The elliptical orbits mentioned above with the same period and semi-major axes close to R will be nearby the circular orbit. They are therefore extremal world lines connecting A and B with *shorter* proper time. Conversely for any one of these elliptical orbits, the circular one is a nearby world line with *longer* proper time. There is always nearby world line of *shorter* proper time made up of small lightlike segments. The elliptical orbits are therefore examples of saddle points, extremal but neither the longest or shortest when compared with nearby world lines. The problem can be extended along these lines.

Solution to Problem 7.5:

Change $1/(2x)$ to $2/x$.

Solution to Problem 8.1:

Strictly speaking the case $A = 0$ should be considered separately. It is a vertical straight line.

Solution to Problem 9.11:

In the denominator the factor $2\ell^2(r_{\max} - 6M)$ should be $\ell^2(6M - r_{\max})$.

Solution to Problem 9.12: Replace (7.48) with (9.46).

Solution to Problem 9.12:

Delete version two. Its incorrect. In version one replace the equation and text

after “Solving (5) and (6)” with:

$$V = \left(\frac{2M}{R} \frac{1}{1 - R^2/b^2} \right)^{1/2} .$$

This is coincidentally the same as the formula for V in Newtonian theory.

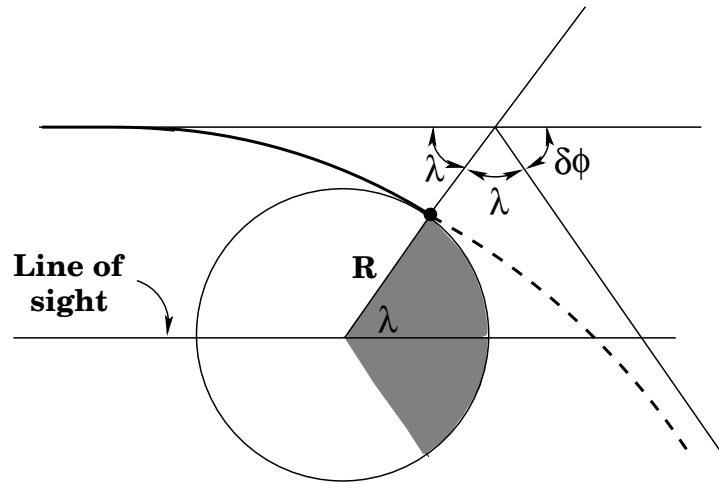
You can also derive the same relation for V by starting, not from (2), but from the relation that at the turning point $V = u^{\hat{\phi}}/u^{\hat{t}}$ in the orthonormal basis associated with the stationary observer.

Problem and Solution to 9.13: Replace $\ell = 4.6$ with $\ell/M = 4.6$. (Correct units.)

Problem and Solution to 9.21: The reference to “latitude” can confuse. Replace with the following”

[E] Suppose a neutron star were luminous so that features on its surface could be viewed with a telescope. The gravitational bending of light means that, not only could the hemisphere facing us be seen, but also a part of the far hemisphere. Explain why and *estimate* the angle measured from the line of sight on the far side above which the surface could be seen. This would be $\pi/2$ if there were no bending, but less than that because of the bending. A typical neutron star has a mass of $\sim M_{\odot}$ and a radius of ~ 10 km.

Solution:



The above figure shows the geometry relevant to the problem. The telescope and observer are off to the far left along the line of sight. The solid line is the trajectory of the light ray that leaves the surface almost tangent to it, but reaches the observer because of light bending. The observer can see features on the part of the surface bounding the unshaded part of the figure, and cannot see features on the part of the surface bounding the shaded part. The angle λ defined in the problem, and dividing the seen from unseen parts, is shown together with its connection to the deflection angle $\delta\phi_{\text{def}}$. For a neutron star $M_{\odot}/R \sim (1.5\text{km})/(10\text{km}) = .15$. This is small enough that a reasonable estimate for the angle λ can be obtained from (9.83) for the deflection angle and the geometry of the above figure. Note that by symmetry light rays from the smallest λ visible on the far side will be ones moving at constant azimuthal angle with respect to the line of sight. Evidently $2\lambda + \delta\phi_{\text{def}} = \pi$ so $\lambda = (\pi - \delta\phi_{\text{def}})/2 = \pi/2 - 2M/R \approx 1.3$ radians $\approx 73^\circ$.

For a further problem try and figure out what such an image would look like.

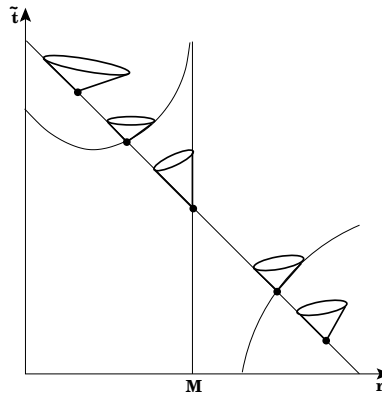
Solution to Problem 11.4: Delete a factor of 2 from the final result.

Solution to Problem 12.4:

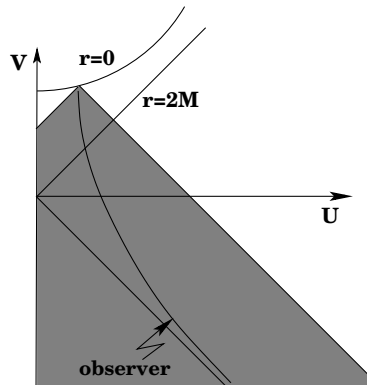
On p. 158 in two the two equations specifying $d\tilde{t}/dv$ the left hand sides should be $d\tilde{t}/dr$ not $d\tilde{t}/dv$. The text beginning “This information...” and the figure can be

replaced by the following improved versions

This information on slopes is enough to give the following (\tilde{t}, r) spacetime diagram. Three radial light rays and some light cones are sketched qualitatively. The size of the cones is arbitrary. One light ray is at $r = M$, another escaping to infinity is at $r > M$, and one confined inside the horizon is at $r < M$.



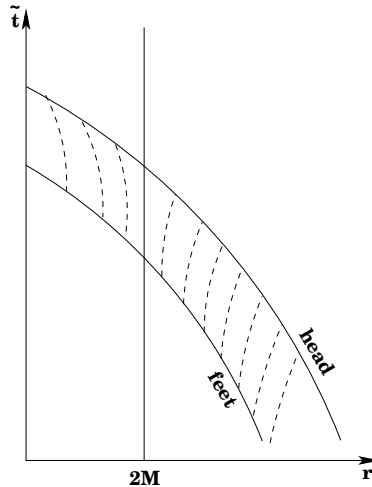
Solution to Problem 12.8: Improved text and corrected figure:



An observer who falls into the black hole can in principle receive information from any point in the shaded region between crossing the horizon at $r = 2M$ and destruction at $r = 0$. This is the region of points which can be connected to the world line of the observer between $r = 2M$ and $r = 0$ by null or timelike world lines. (Not necessarily radial ones.) This includes information about events out-

side $r = 2M$ but not all of the region outside $r = 2M$.

Solution to Problem 12.13: Improved figure:



Replace the solution to (a) with the following, its a bit clearer.

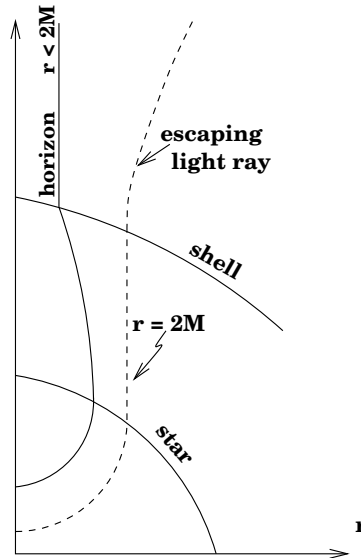
The figure above shows an Eddington-Finkelstein diagram with schematic world lines of the observer's head and feet. At a given \tilde{t} her feet are at a smaller radius than her head because she is falling in feet first. Radial rays emitted originating at her feet are shown. (These are segments of light rays illustrated in Figure 12.2.)

There is no instant when she is not receiving a light ray from her feet. She sees them always. When her head crosses the horizon she sees her feet at the same radius, because the horizon is generated by light rays.

When her head hits the singularity she still sees light from her feet that was emitted earlier but is falling into the singularity as well. But, she never sees her feet hit the singularity because her head and feet meet the singularity at spacelike separated points. When close to the singularity light rays from her feet fall into the singularity before they intersect her world line, as the figure shows. (Some students interpret this question to ask if she sees her feet *when* they hit the singularity. But there is no invariant meaning to "when". For some it is made clearer in a Kruskal diagram.)

Solution to Problem 12.14: In the second line of the expression for τ the sign following e^2 should be a +.

Solution to Problem 12.19: Improved figure:



Solution to Problem 13.5: Change 3×10^5 yr to 4×10^5 yr, 3×10^{-16} cm to 4×10^{-16} cm and 4×10^{12} g to 5×10^{12} g. This makes the solution consistent with the statement of the problem.

Solution to Problem 13.7: Replace with the following simpler solution:

Let M_0 denote the present mass of a black hole going to explode 1s after the present time. From (13.19) we can find its mass by putting $t_* - t = 1$ s with the result

$$M_0 = 1.7 \times 10^{-20} \text{ cm} = 2.3 \times 10^8 \text{ g}.$$

If the black hole evaporates completely the energy emitted in the next second will be $M_0 c^2 = 4.2 \times 10^{29}$ erg. Therefore, the total energy per unit area received at Earth in that 1s will be $M_0 c^2 / (4\pi r^2)$ where r is the distance to the evaporating black hole. In the same 1s the energy per unit area received from a star with the solar luminosity L_\odot a distance 10 pc away will be $(L_\odot \cdot (1 \text{ sec})) / (4\pi (10 \text{ pc})^2)$.

These will be equal when

$$\left(\frac{r}{10\text{pc}}\right) = \frac{M_0 c^2}{L_\odot \cdot 1\text{s}}$$

Thus r is 0.007 of 10 pc or 2.2×10^{17} cm or 1.5×10^4 AU. The black hole would be within the region called the ‘‘Oort cloud’’ containing the debris from the formation of the solar system that supplies some of the comets.

Solution to Problem 18.6: In the final sentence replace $\tilde{t} = t_0$ with $\tilde{t} = \tilde{t}_0$.

Problem 18.25: In the 5th printing the problem was modified slightly as follows. The solution here is for that, but is also a better solution to the version that appeared in earlier printings:

Problem:[C] Is there a value of Ω_ν that would allow the universe to bounce at a small radius, but still reach a temperature $T \sim 10^{10}$ K such that nucleosynthesis could occur? Assume $\Omega_r = 8 \times 10^{-5}$ and $\Omega_m = .3$.

Solution: For the temperature at the bounce to be greater than $\sim 10^{10}$ K, the value of \tilde{a} at the bounce \tilde{a}_b must be less than $\sim 10^{-10}$ [cf. (18.26)]. For simplicity let us suppose that the bounce occurs when the temperature is exactly 10^{10} K corresponding to a redshift z of approximately 3×10^9 and an $\tilde{a}_b \approx 3 \times 10^{-10}$.

A necessary condition for a bounce [cf. Fig 18.9] and (18.78) is

$$\frac{1}{2}\Omega_c = U_{\text{eff}}(\tilde{a}_b) = -\frac{1}{2} \left[\Omega_\nu \tilde{a}_b^2 + \frac{\Omega_m}{\tilde{a}_b} + \frac{\Omega_r}{\tilde{a}_b^2} \right]. \quad (9)$$

Using $\Omega_c = 1 - \Omega_\nu - \Omega_m - \Omega_r$ from (18.76) this can be solved for Ω_ν in terms of the other cosmological parameters and \tilde{a}_b . The result is

$$\Omega_\nu = \frac{1}{1 - \tilde{a}_b^2} \left[1 + \Omega_m \left(\frac{1}{\tilde{a}_b} - 1 \right) + \Omega_r \left(\frac{1}{\tilde{a}_b^2} - 1 \right) \right]. \quad (10)$$

For the given values of Ω_m and Ω_r and $\tilde{a}_b \approx 3 \times 10^{-10}$ this evaluates to

$$\Omega_\nu \sim 10^{15} \quad (!), \quad (11)$$

the radiation term in (10) being the dominant one. (This value is wildly inconsistent with current observations as we will see in Chapter 19, but let’s first see if it even corresponds to a bounce.)

While the condition (9) is necessary for a bounce, it is not sufficient. That is because, as Figure 18.9 makes clear, (9) is satisfied both by cosmological models where \tilde{a}_b is a minimum of $\tilde{a}(\tilde{t})$ (bounces) and where it is a maximum of the expansion. For a bounce solution, \tilde{a}_b lies above the value of the maximum of $U_{\text{eff}}(\tilde{a})$. Let's check whether this is satisfied for (11).

At the time of the putative bounce when $T \sim 10^{10}\text{K}$ the matter can be neglected. The maximum of $U_{\text{eff}}(\tilde{a})$ occurs at

$$\tilde{a}_{\text{max}} = \left(\frac{\Omega_r}{\Omega_v} \right)^{\frac{1}{4}} \sim 10^{12} \quad (12)$$

for (11). This is vastly bigger than $\tilde{a}_b \approx 3 \times 10^{-10}$. The maximum of $U_{\text{eff}}(\tilde{a})$ is above the value of \tilde{a}_b , not below it as would be required for a bounce.

We conclude that there is *no* value of Ω_v that is that would lead to a bounce at a temperature higher than 10^{10}K .

Solution to Problem 15.10: Replace with the following (simpler, more closely related to the discussion in the text):

There are a number of different ways of solving this problem. We give two.

1) We can follow the demonstration given in Section 15.3 that the surface $r = R$ is a stationary null surface. Tangent vectors \mathbf{t} in the surface have the general form

$$t^\alpha = (t^t, 0, t^\theta, t^\phi) . \quad (13)$$

The surface is null if, at each point, a null tangent vector ℓ can be found along with two orthogonal (to ℓ and each other) spacelike tangent vectors. (See Section 7.9.) The condition $\ell \cdot \ell$ for a tangent vector reads

$$\ell \cdot \ell = \left(1 - \frac{r^2}{R^2} \right) (\ell^t)^2 + r^2 (\ell^\theta)^2 + r^2 \sin^2 \theta (\ell^\phi)^2 .$$

On $r = R$ we see that $\ell^\alpha = (1, 0, 0, 0)$ is a null tangent vector, (indeed its the unique one up to a multiplicative constant). Further, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are two spacelike tangent vectors orthogonal to each other and to ℓ

The null surface at $r = R$ has the one-way property discussed in Section 7.9. Once through it you cannot come back.

2) Its possible to worry about the above solution because the t coordinate is singular at $r = R$ in much the same way that the Schwarzschild t coordinate is singular in the Schwarzschild metric. We can both demonstrate this and understand the nature of the $r = R$ three-surface by transforming to coordinates analogous to Eddington-Finkelstein coordinates. Following (12.1) we transform from t to a new coordinate u defined by

$$t = u + F(r) \quad (14)$$

where

$$\frac{dF}{dr} = \left(1 - \frac{r^2}{R^2}\right)^{-1}, \quad F(r) = \frac{R}{2} \log \left(\frac{R+r}{R-r}\right). \quad (15)$$

Then the metric takes the form:

$$ds^2 = - \left(1 - \frac{r^2}{R^2}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (16)$$

In this non-singular form of the line element, its clear that $r = R$ is a null surface. Furthermore, because of the choice of signs in (14), its an outgoing null surface — fixed u means r has to get larger as t gets larger. It therefore has the property that once crossed its impossible to return. (See the discussion in Section 7.9.)

Comment: The given metric is one form of the metric of deSitter space which is the maximally symmetric solution of Einstein's equation with a cosmological constant. Another form, covering a different patch, is given by (18.1) and (18.39). The surface $r = R$ is called the “deSitter horizon”.

Solution to Problem 15.13: Delete the sentence “ C vanishes at null.’ Its meaningless.