Supplement to Chapter 21: Deriving the Equation of Geodesic Deviation and a Formula for the Riemann Tensor

The equation of geodesic deviation (21.19) relates the acceleration of the separation vector χ between two nearly geodesics to the Riemann curvature exhibited explicitly in (21.20). This supplement works through more of the details of the derivation that was sketched in Sectin 21.2. It parallels the conceptually similar but algebraically simpler derivation of the Newtonian geodesic equation (21.5).

The separation four-vector $\chi(\tau)$ connects a point $x^\alpha(\tau)$ on one geodesic (the fiducial geodesic) to a point $x^\alpha(\tau)+\chi^\alpha(\tau)$ on a nearby geodesic at the same proper time. Since there is no unique way of relating proper time on one geodesic to proper time on another there are many different separation vectors. The exact choice will not be important for us except to assume that the difference is small for nearby geodesics so that χ is small.

The separation acceleration that is the left hand side of the equation of geodesic deviation is

$$\mathbf{w} \equiv \nabla_{\mathbf{u}}\nabla_{\mathbf{u}}\chi = \nabla_{\mathbf{u}}\mathbf{v} \tag{1}$$

where $\mathbf{u}$ is the four-velocity of the fiducial geodesic and $\mathbf{v}$ is the separation velocity, $\mathbf{v} \equiv \nabla_{\mathbf{u}}\chi$. The coordinate basis components of $\mathbf{w}$ and $\mathbf{v}$ can be calculated using (20.54) for the covariant derivatives. Thus, as in (21.17) and (21.18),

$$v^\alpha \equiv (\nabla_{\mathbf{u}}\chi)^\alpha = u^\beta\nabla_\beta\chi^\alpha = \frac{d\chi^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}u^\beta\chi^\gamma, \tag{2}$$

$$w^\alpha \equiv (\nabla_{\mathbf{u}}\mathbf{v})^\alpha = u^\beta\nabla_\beta v^\alpha = \frac{dv^\alpha}{d\tau} + \Gamma^\alpha_{\delta\epsilon}u^\delta v^\epsilon. \tag{3}$$

Here, expressions like $u^\beta(\partial\chi^\alpha/\partial x^\beta)$ have been written $d\chi^\alpha/d\tau$ following the general relation for any function f [cf. (21.14)].

$$\frac{df}{d\tau} = \frac{df(x^\alpha(\tau))}{d\tau} = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} = u^\alpha\frac{\partial f}{\partial x^\alpha}\ . \tag{4}$$

The derivation consists of evaluating w^α by substituting (2) into (3) and using the geodesic equation for the fiducial and nearby geodesic. Substituting (2) into (3) gives

$$w^\alpha = \frac{d^2\chi^\alpha}{d\tau^2} + \frac{d}{d\tau}\left(\Gamma^\alpha_{\beta\gamma}u^\beta\chi^\gamma\right) + \Gamma^\alpha_{\delta\varepsilon}u^\delta\left(\frac{d\chi^\varepsilon}{d\tau} + \Gamma^\varepsilon_{\beta\gamma}u^\beta\chi^\gamma\right)$$
$$= \frac{d^2\chi^\alpha}{d\tau^2} + 2\Gamma^\alpha_{\beta\gamma}u^\beta\frac{d\chi^\gamma}{d\tau} + \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta}\,u^\delta u^\beta\chi^\gamma + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{d\tau}\,\chi^\gamma + \Gamma^\alpha_{\delta\varepsilon}\Gamma^\varepsilon_{\beta\gamma}u^\beta u^\delta\chi^\gamma. \quad (5)$$

Here, (4) has been used and the freedom to relabel dummy indices has been employed to group equal terms together, for instance

$$\Gamma^\alpha_{\beta\gamma}u^\beta\frac{d\chi^\gamma}{d\tau} = \Gamma^\alpha_{\delta\varepsilon}u^\delta\frac{d\chi^\varepsilon}{d\tau}\ . \quad (6)$$

To evaluate (5) note that $x^\alpha(\tau)+\chi^\alpha(\tau)$ obeys the geodesic equation (8.14)

$$\frac{d^2\left(x^\alpha+\chi^\alpha\right)}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}(x^\delta+\chi^\delta)\frac{d(x^\beta+\chi^\beta)}{d\tau}\ \frac{d(x^\gamma+\chi^\gamma)}{d\tau} = 0. \quad (7)$$

When χ vanishes this is the geodesic equation for the fiducial geodesic. Since χ is small, we need to keep only the first order term in (7) in an expansion in χ^α. [Compare the transition from the Newtonian (21.3) to (21.5).] Using $u^\alpha = dx^\alpha/d\tau$, (4), and $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ this is

$$\frac{d^2\chi^\alpha}{d\tau^2} + 2\Gamma^\alpha_{\beta\gamma}u^\beta\frac{d\chi^\gamma}{d\tau} + \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta}u^\beta u^\gamma\chi^\delta = 0. \quad (8)$$

Again, the freedom to relabel dummy indices has been used to group equal terms together.

Eq. (8) can be used to eliminate the second derivative of χ^α from the expression (5) for w^α. The geodesic equation (8.15) can be used to eliminate the derivative of u^β. The result is

$$w^\alpha = -\frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta}\,u^\beta u^\gamma\chi^\delta + \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta}\,u^\beta u^\delta\chi^\gamma - \Gamma^\alpha_{\beta\gamma}\Gamma^\beta_{\delta\varepsilon}u^\delta u^\varepsilon\chi^\gamma + \Gamma^\alpha_{\delta\varepsilon}\Gamma^\varepsilon_{\beta\gamma}u^\beta u^\delta\chi^\gamma. \quad (9)$$

All the terms involving $d\chi^\alpha/d\tau$ have luckily canceled since there is no equation to eliminate that. Using the freedom to relabel dummy indices a common factor $u^\beta\chi^\gamma u^\delta$ can be identified in each term. The result is

$$w^\alpha = -\left(\frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\gamma\varepsilon}\Gamma^\varepsilon_{\beta\delta} - \Gamma^\alpha_{\delta\varepsilon}\Gamma^\varepsilon_{\beta\gamma}\right)u^\beta\chi^\gamma u^\delta \equiv -R^\alpha_{\beta\gamma\delta}u^\beta\chi^\gamma u^\delta. \quad (10)$$

Thus, we derive both the geodesic equation (21.19) and the explicit expression for the Riemann tensor (21.20).