Supplement to Chapter 23:
The Derivation of the Quadrupole Formula

This supplement completes the derivation of the quadrupole formula (23.51) for the power radiated in gravitational waves by a near-Newtonian system in the long wavelength approximation. The necessary tools are already in hand. The propagating ripples in spacetime curvature far from the source are given in terms of the mass quadrupole moment by (23.35)

$$\bar{h}_{ij}(t,r) = \frac{2\ddot{I}_{ij}(t-r)}{r} \ . \tag{1}$$

Monitored far from the source over a small range in angle, this spherical wave is approximately plane. The energy flux of a linearized, plane gravitational wave with frequency ω and amplitude a was given in (16.22) as

$$f_{GW} = \frac{\omega^2 a^2}{32\pi} \ . \tag{2}$$

(An example is the the wave propagating in the z-direction described by (16.2) with $f(t-z) = a\sin[\omega(t-z)]$.) The result (2) was derived in the supplement to Chapter 22: *Stress-Energy Tensor for Short-Wavelength, Linearized Gravitational Waves*. The idea is to use (2) to find an expression for the flux implied by (1) in an arbitrary direction lying along a unit vector n^i and then integrate over all possible directions to find the total power radiated.

We begin by noting from (16.2) that (2) can be expressed more generally as

$$a^2 = \langle h^{TT}_{jk} h^{jk}_{TT} \rangle \ . \tag{3}$$

The sum over the two independent components of h^{ij}_{TT} produces $2a^2$; the time average produces a factor of $1/2$. The energy flux π^i for a plane wave propagating in the direction of a unit vector n^i is then

$$\pi^i_{(GW)} = \frac{\omega^2}{32\pi} n^i \left\langle h^{jk}_{TT} h^{TT}_{jk} \right\rangle \ . \tag{4}$$

The same result could be derived directly from the the $T^{it}_{(GW)}$ component of the effective stress-energy tensor for linearized gravitational waves exhibited in the last equation in the supplement to Chapter 22. We have only to plug (1) into (4) and do the angular integral summing over different directions n^i to find the total power radiated in gravitational waves.

If you have studied electromagnetism, this calculation will seem familiar (cf. Table 23.1). To calculate the power radiated by an oscillating electric dipole $\vec{p}$, for instance, one first evaluates the potentials far from the source $\vec{A} = \dot{\vec{p}}(t-r)/r$. The energy flux is given in terms of these potentials by the Poynting vector $\vec{S} = (\vec{E}\times\vec{B})/4\pi$. Integrating $\vec{S}\cdot\vec{n}\,dA$ over all directions $\vec{n}$ gives the total power radiated.

The only catch with carrying out this program is that the wave amplitudes in the expression for the energy flux (4) are in transverse-traceless gauge *for the direction in which they are propagating*. In particular, h^{TT}_{jk} is transverse to the direction n^i. That is, it has zero components along n^i

$$h^{TT}_{jk}\, n^k = 0\ . \tag{5}$$

The wave amplitude (1) is not generally in transverse-traceless gauge, but must be put in this gauge to evaluate the energy flux.

That is a straightforward matter. As we learned in Section 21.5 , it is simply necessary to set the non-transverse components of $\bar{h}_{ij}$ to zero and subtract out the trace. The transverse-traceless components remain unchanged. Thus the transverse-traceless gauge for a wave propagating in the z-direction is given in terms of a general perturbation by (21.70)

$$h^{TT}_{ij} = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} \frac{1}{2}\left(\bar{h}_{xx}-\bar{h}_{yy}\right) & \bar{h}_{xy} & 0 \\ \bar{h}_{xy} & \frac{1}{2}\left(\bar{h}_{yy}-\bar{h}_{xx}\right) & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} . \tag{6}$$

Note that it doesn't make any difference whether we use h_{ij} or $\bar{h}_{ij}$ on the right hand side of (21.70). The difference cancels out.

With the result (6), the expression (4) for the effective energy flux, and the wave amplitude for large r (1), we can evaluate the energy flux *in the z-direction* in terms of the mass quadrupole moment. Noting that $\ddot{I}_{ij} = -\omega^2 I_{ij}$ for a periodic source, we find:

$$\pi^z_{(GW)} = \frac{\omega^6}{32\pi r^2}\,\left\langle 2\left(I_{xx}-I_{yy}\right)^2 + 8I^2_{xy}\right\rangle\ . \tag{7}$$

To find the flux in the y- or z-directions, just permute x,y, and z appropriately. But to do the angular integrals we need the flux in a general direction n^i.

To find that first note that I_{ij} can be replaced by the reduced quadrupole moment $\mathcal{I}_{ij}$ in (7) because the difference — $1/3\delta_{ij}I^k_{\ k}$ — cancels out. But $\mathcal{I}_{ij}$ has vanishing trace

$$\mathcal{I}^k_{\ k} = \mathcal{I}_{xx} + \mathcal{I}_{yy} + \mathcal{I}_{zz} \ . \tag{8}$$

These facts can be used to rewrite (7) in the form

$$\pi^z_{(GW)} = \frac{\omega^6}{16\pi r^2}\left\langle 2\mathcal{I}_{ij}\mathcal{I}^{ij} - 4\mathcal{I}_{zi}\mathcal{I}^i_z - \mathcal{I}^2_{zz}\right\rangle \ . \tag{9}$$

Write this out using (8), and check that it is the same as (7).

The expression (9) allows us to find the energy flux in any direction n^i:

$$\pi^i_{(GW)}n_i = \frac{\omega^6}{16\pi r^2}\left\langle 2\mathcal{I}_{ij}\mathcal{I}^{ij} - 4\left(\mathcal{I}_{ik}n^k\right)\left(\mathcal{I}^i_{\ \ell}n^\ell\right) - \left(\mathcal{I}_{ij}n^in^j\right)^2\right\rangle \ . \tag{10}$$

When $\vec{n}$ points in the z-direction, (10) reduces to (9). Now we can do the angular integrals over the different directions to give the radiated power

$$\frac{dE}{dt} = \int r^2 d\Omega_{\vec{n}}\pi^i_{(GW)}n_i \ . \tag{11}$$

There are two types of angular integral to do beyond the standard $\int d\Omega_{\hat{n}} = 4\pi$. They are:

$$\mathcal{A}^{ij} \ = \ \int d\Omega_{\hat{n}} n^i n^j \ , \tag{12a}$$

and

$$\mathcal{B}^{ijk\ell} \ = \ \int d\Omega_{\hat{n}} n^i n^j n^k n^\ell \ . \tag{12b}$$

In terms of these

$$\frac{dE}{dt} = \frac{\omega^6}{16\pi}\left\langle 8\pi\mathcal{I}_{ij}\mathcal{I}^{ij} - 4\mathcal{I}_{ik}\mathcal{I}^i_{\ \ell}\mathcal{A}^{k\ell} - \mathcal{I}_{ij}\mathcal{I}_{k\ell}\mathcal{B}^{ijk\ell}\right\rangle \ . \tag{13}$$

One way to do these integrals is component by component using $n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. There are 21 integrals to do. A faster way is to understand the form the answer will necessarily take. It will not depend on the direction $\vec{n}$ — that has been integrated over. It can only be constructed out of δ^{ij}'s in combinations that reflect the symmetries of the integrand. Thus,

$$\mathcal{A}^{ij} = A\delta^{ij} , \tag{14a}$$

$$\mathcal{B}^{ijk\ell} = B\left(\delta^{ij}\delta^{k\ell}+\delta^{i\ell}\delta^{jk}+\delta^{ik}\delta^{j\ell}\right) \tag{14b}$$

where A and B are numerical coefficients. Two integrals suffice to evaluate these. The simplest way to proceed is to contract all indices in both (14) and (12). For example, from (12a)

$$\mathcal{A}^i_i = \int d\Omega_{\vec{n}} = 4\pi . \tag{15}$$

But also from (14a)

$$\mathcal{A}^i_i = 3A . \tag{16}$$

The result is $A = 4\pi/3$. Proceeding in this way we find

$$\mathcal{A}^{ij} = \frac{4\pi}{3}\,\delta^{ij} , \tag{17a}$$

$$\mathcal{B}^{ijk\ell} = \frac{4\pi}{15}\left(\delta^{ij}\delta^{k\ell}+\delta^{i\ell}\delta^{jk}+\delta^{ik}\delta^{j\ell}\right) . \tag{17b}$$

Equations (17) can now be used to evaluate (13). The final form is evident. It can only be proportional to $I\!\!\!{-}^{ij}I\!\!\!{-}_{ij}$. It might involve $I\!\!\!{-}^2$ but that is zero from (8). In the end, one finds

$$\frac{dE}{dt} = \frac{\omega^6}{5}\,\left\langle I\!\!\!{-}_{ij}I\!\!\!{-}^{ij}\right\rangle . \tag{18}$$

This is the quadrupole formula (23.51).