Symmetry and Aesthetics in Contemporary Physics

CS-10, Spring, 2016

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CLASS 4: SYMMETRY AND GROUPS: THE PLOT THICKENS

Today's plan:

1. Go over Livio chapters
2. More on symmetry, groups, and physical laws
3. ~Break~
4. Presentation of your symmetry demos

News:

May 27th: Dr. Andrea Puhm,
post-doc with Prof. Gary Horowitz
will talk to us about String Theory!

Who is Dr. Mario Livio?

- **Astrophysicist, works at STScI**
- **Author of multiple popular books on physics and math**
- **Art collector**

https://www.youtube.com/watch?v=MpXxNrcZuqs

Discussion of two chapters by Mario Livio from his book The Equation that Couldn’t be Solved

Comments?
Reactions?
Points you like/agree with?
Points that particularly interested you?
Points with which you might disagree?
Questions he left you with?

You can learn more about Dr. Mario Livio on his…

Blog: **http://www.huffingtonpost.com/mario-livio/**

face book page: **https://www.facebook.com/mariolivio**

twitter feed: **https://twitter.com/Mario_Livio**

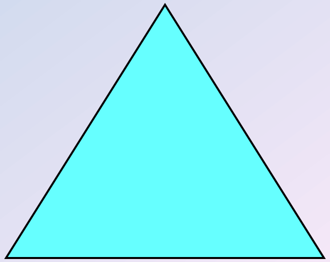
SYMMETRY AND GROUPS: THE PLOT THICKENS

Recall: Symmetry = when you perform a transformation on a system, and the system remains the same

Groups are represented by symmetry transformations

Defining characteristics of groups:

- Closed under their symmetry transformations
- Successive symmetry transformations are associative
- Every element has an inverse
- There is an identity element



Classifying objects by their group of possible transformations. For example:

The “Jean” Group

Let X = turn the jeans backwards

Let Y = turn the jeans inside-out

Let Z = do X, followed by Y

Elements of the Jean Group: X, Y, Z, and I (do nothing)

Show that the Jean Group is closed under these operations.

○	I	X	Y	Z
I				
X				
Y				
Z				

Work with a partner – fill in the table.

The Jeans Group is closed under these symmetry operations.

It is also *Abelian* because these operations are commutative.

Neils Henrik Abel

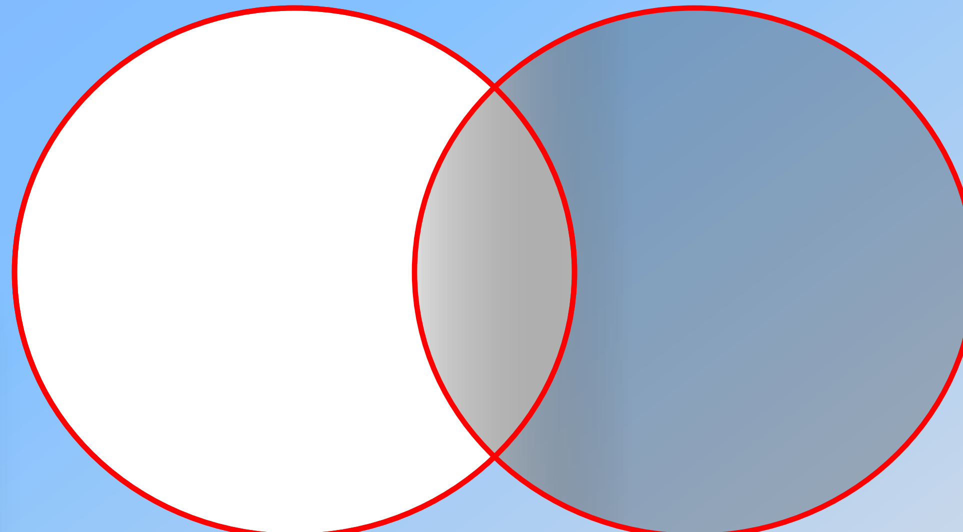
○	**I**	**X**	**Y**	**Z**
I	I	X	Y	Z
X	X	I	Z	Y
Y	Y	Z	I	Y
Z	Z	Y	X	I

Let's try another…

A Chicken or a Cow but not Both

Let X = the set with one chicken
Let Y = the set with one cow
Let Z = the set with two animals,
one chicken + one cow
Let I = the empty set, no members

Let ∆ = operation that creates a new set with either a chicken or a cow, but not both

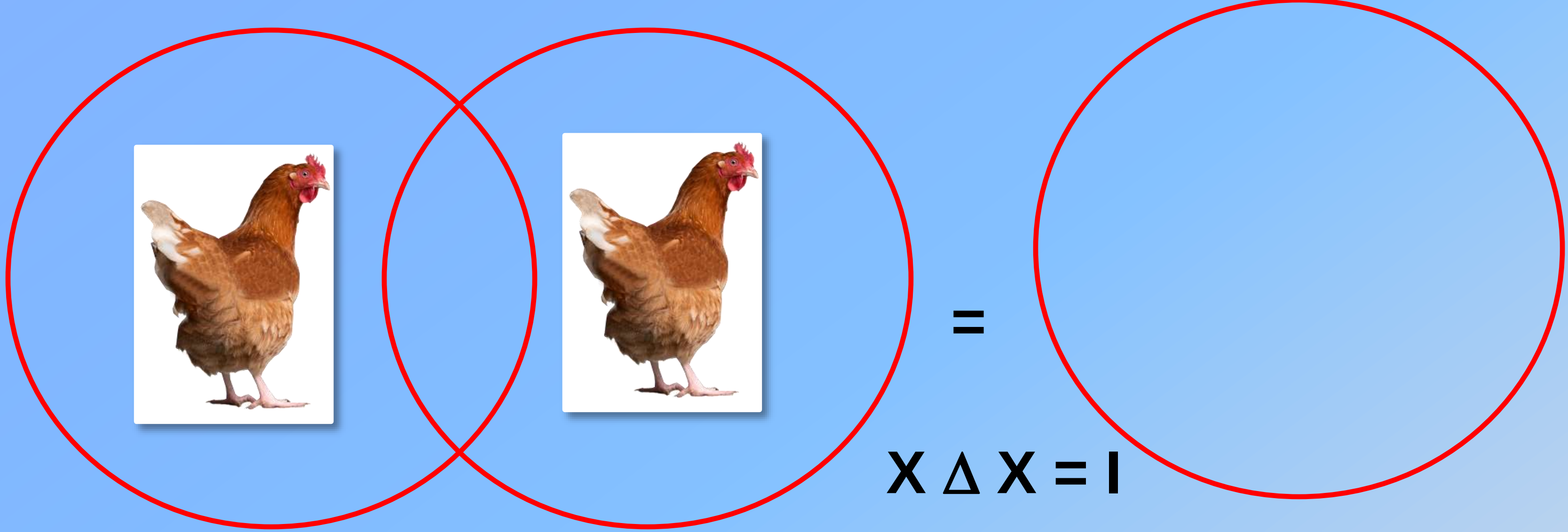


=

$X \Delta Z = Y$

=

$X \Delta Y = Z$



$X \Delta X = I$

Working with a partner, fill in the rest of the table.

Δ	I	X	Y	Z
I				
X		I		
Y		Z		
Z		Y		

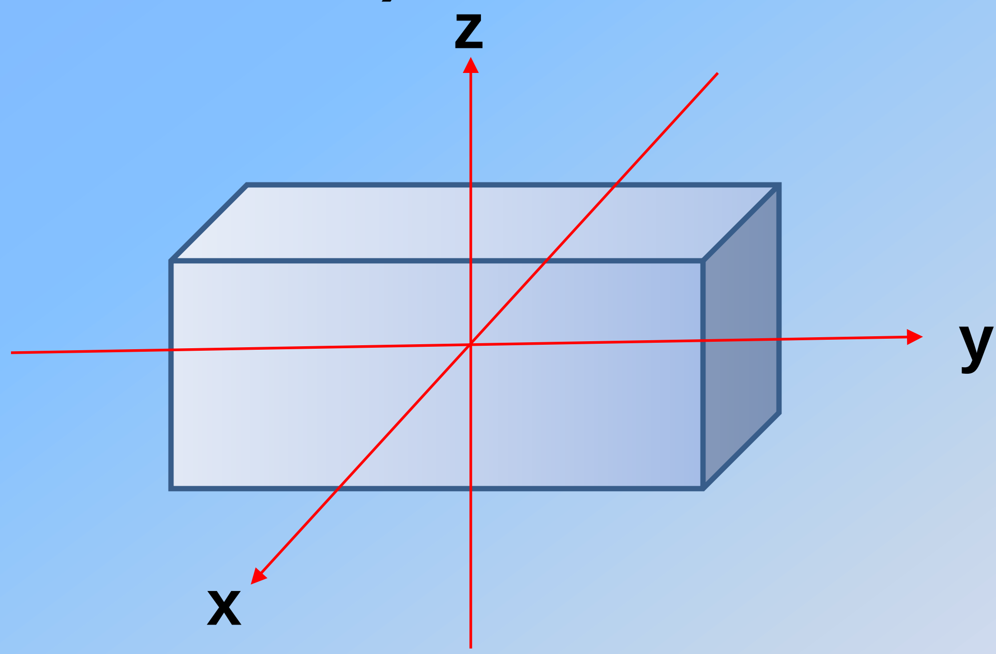
  

Chicken or Cow but not Both Group

Δ	I	X	Y	Z
I	I	X	Y	Z
X	X	I	Z	Y
Y	Y	Z	I	X
Z	Z	Y	X	I

This is the same as the table for the Jean Group

The Jean Group and the “Chicken or Cow but not Both” Group are *isomorphic* to each other, and also to the group of <u>180^0</u> Rotations of a quadrilateral.
(Note: Not 90^0 rotations!)



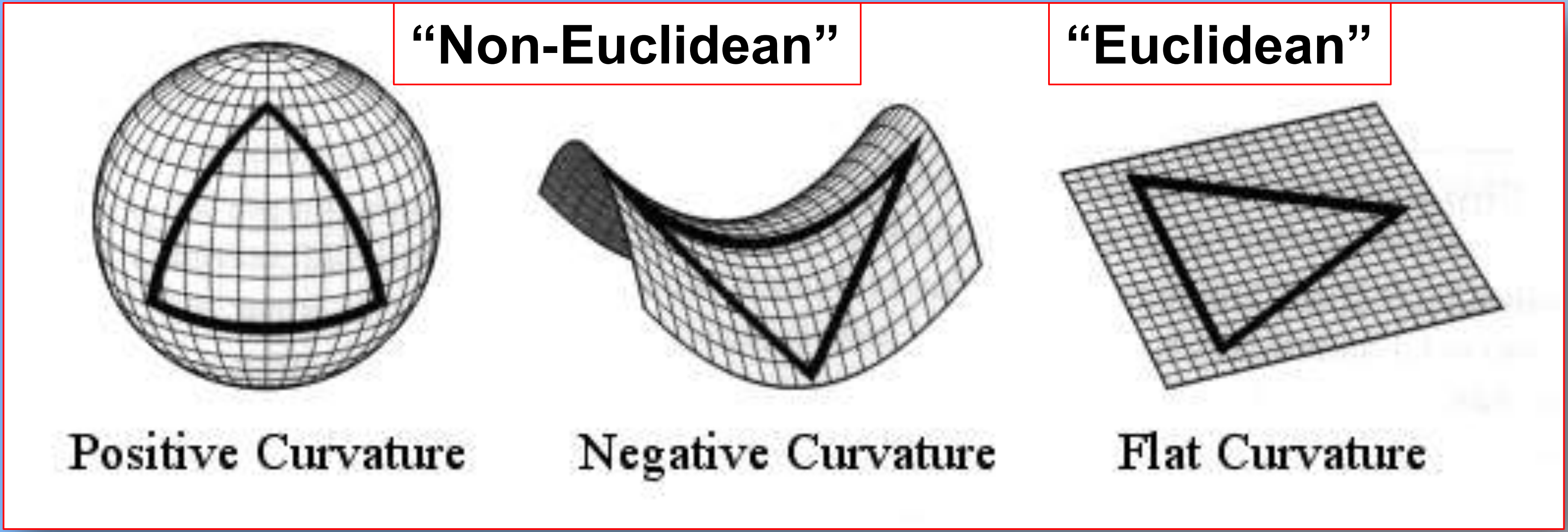
X = 180^0 rotation about x axis
Y = 180^0 rotation about y axis
Z = 180^0 rotation about z axis
I = identity “do nothing” element

X ° Y = Z
X ° X = I
Y ° X = Z … etc.! You get the same table!

All three are representations of the Klein-4 Group, discovered by Felix Klein (1849-1925)

German mathematician whose unified view of geometry as the study of the properties of a space that are invariant under a given group of transformations, known as the *Erlangen Program*, profoundly influenced mathematical developments.

Felix Klein: "Geometry is characterized and defined not by objects, but by the group of transformations that leaves it invariant."



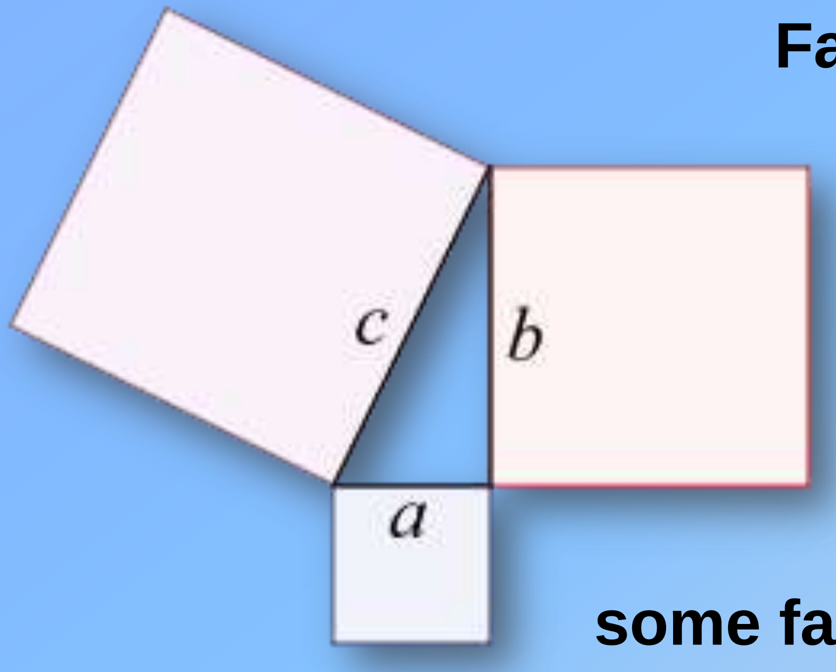
sum of interior angles $> 180^0$

sum of interior angles $< 180^0$

sum of interior angles $= 180^0$

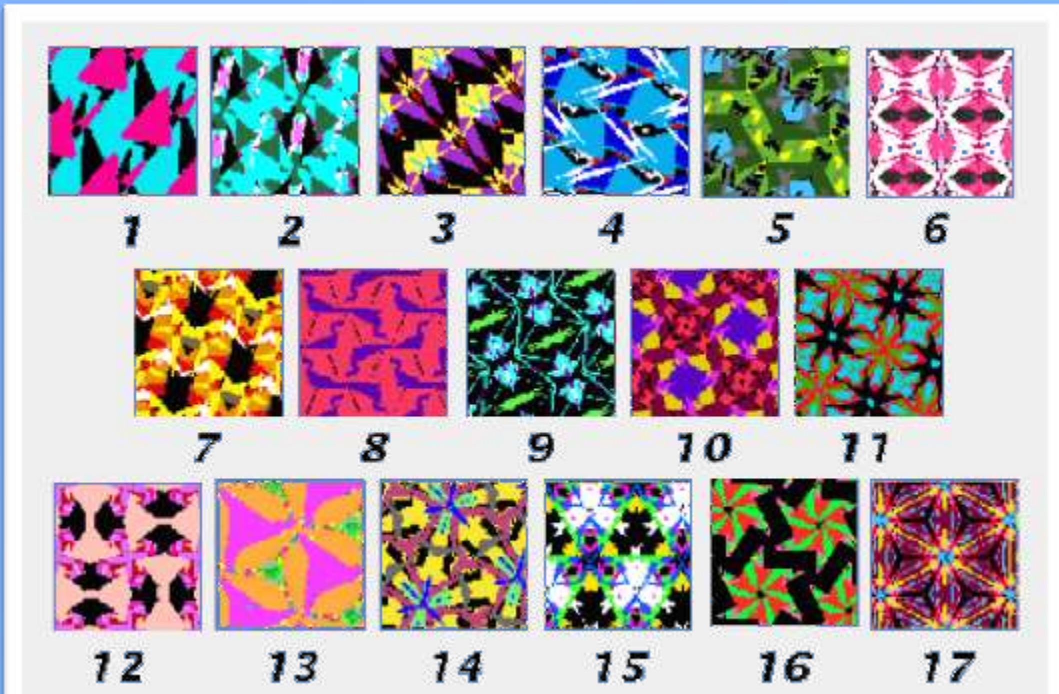
Different spaces are characterized by different symmetry groups.

Familiar Euclidean geometry



some familiar characteristics:

- parallel lines meet at infinity
- the sum of interior angles of a triangle is 180^0
- Pythagorean theorem
- “alternate interior angles”
- etc…



17 wallpaper groups

$$SO(2,\theta) = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$$

determinant = 1

Continuous Symmetry operations in a plane: rotations and reflections described by O(2), orthogonal group of order 2

definition of orthogonal:
det(R^T) = det(R) →
det(R^T)^2 = 1 so
det(R) = +/- 1
→ R^T = I

$$SO(2) = \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$$

- **Special Orthogonal Group of order 2 describes continuous rotations in a plane**
- **subgroup of O(2)**

- **Special Orthogonal Group: det(R) = +1**

the story so far:

- **matrix multiplication characterizes a group**
- **groups that look different, but which are characterized by the same symmetry transformations are isomorphic (the same)**
- **the group of symmetry transformations that preserves lengths and angles of objects in a space defines that space**

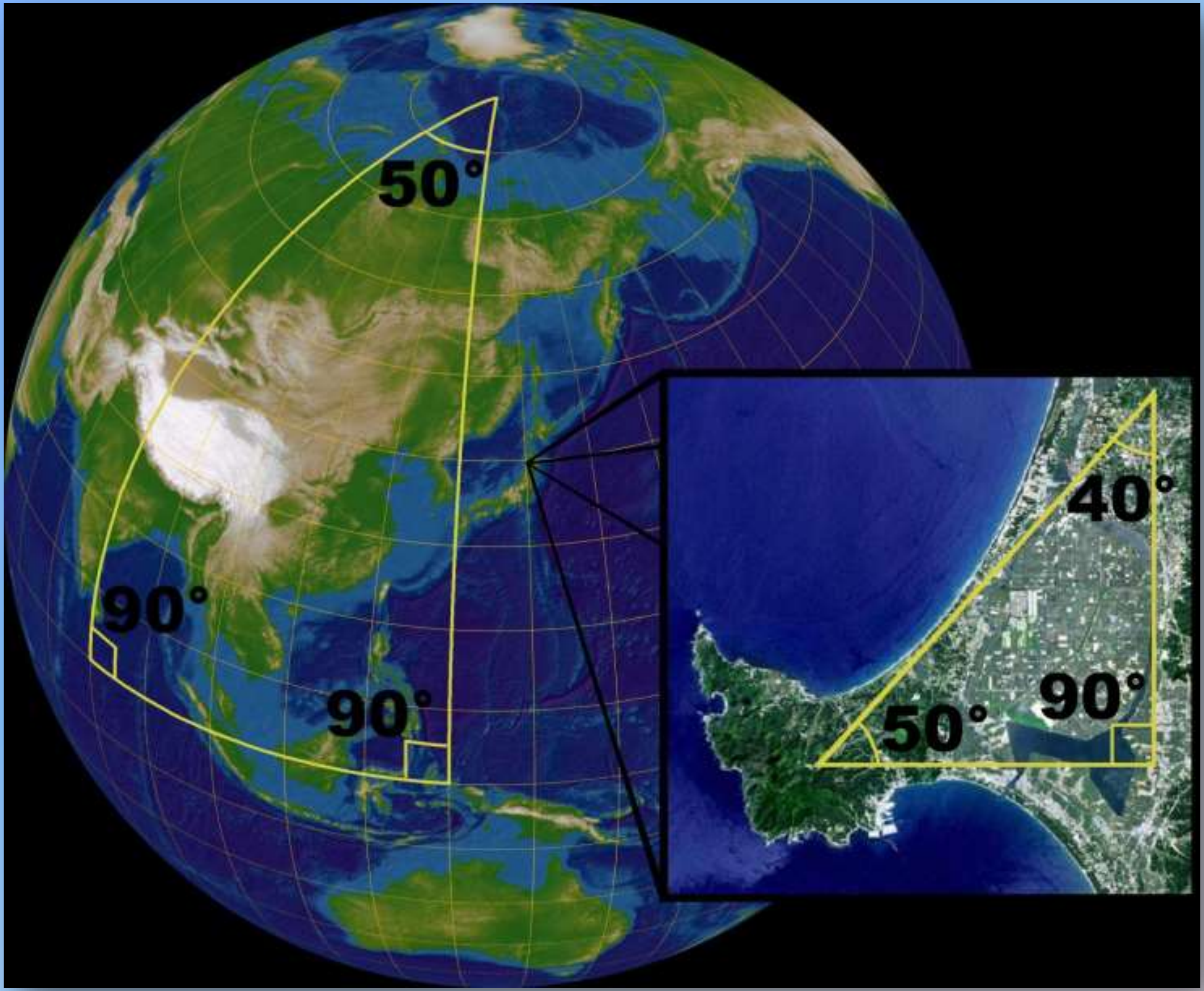
Platonic solids

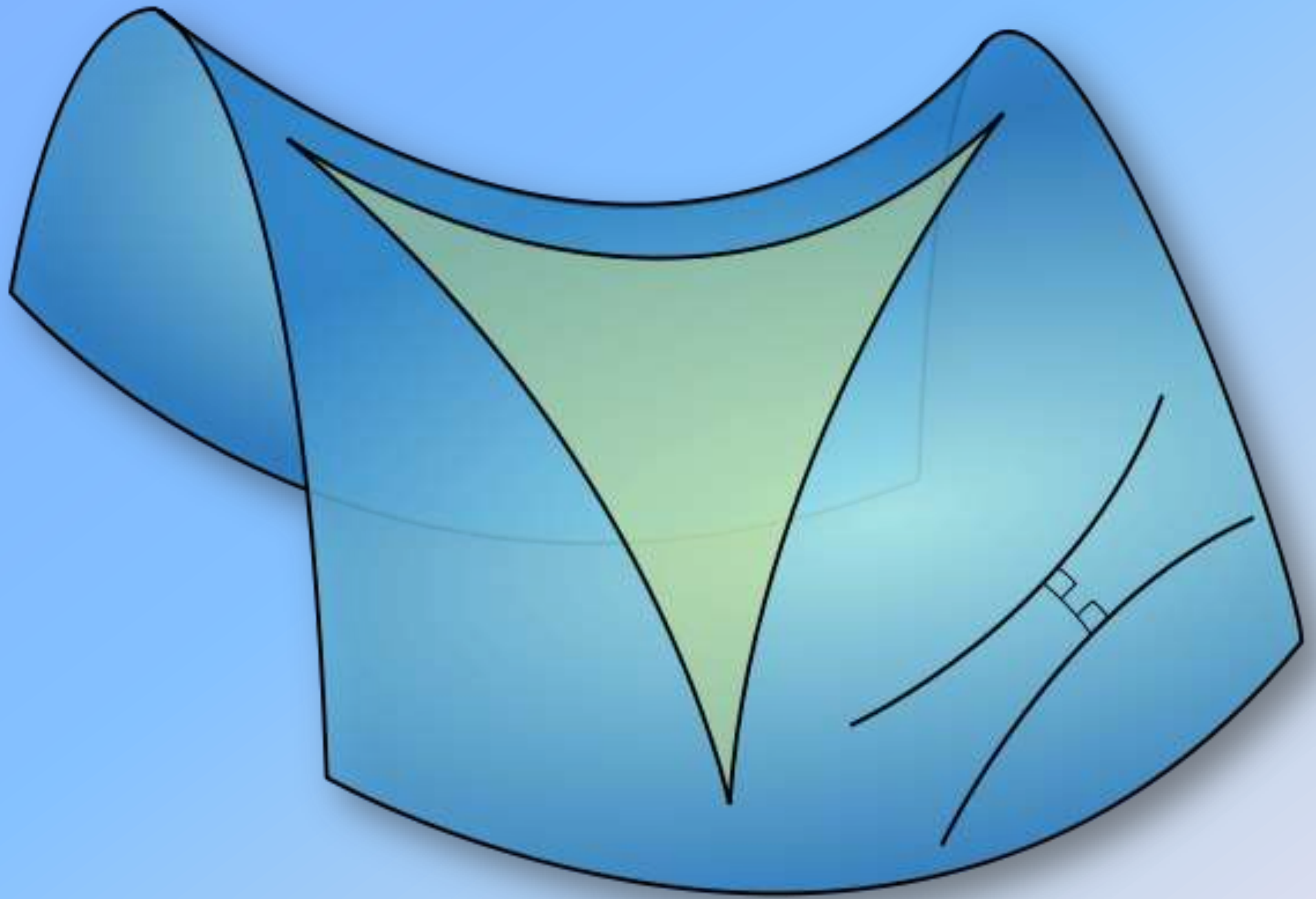
Symmetry transformations that preserve lengths and angle relationships are rotations about an axis, and reflections about a line in 3-space.

These define "Euclidean space" – 3 spatial dimensions

Newtonian physics describes the motion of rigid bodies in Euclidean space.

Locally 'flat' globally curved

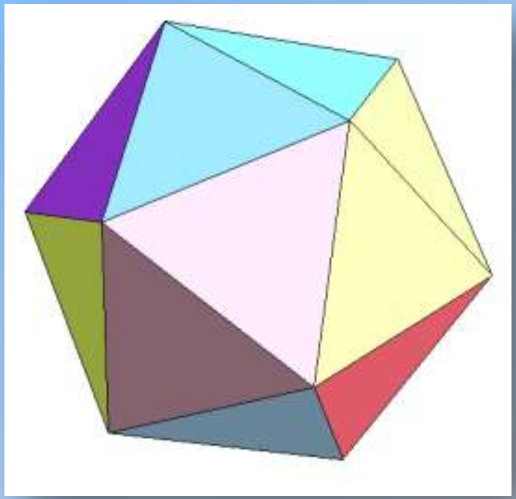




Hyperbolic space: As we will show next time, this is the geometry of Special Relativity – the geometry that defines spacetime without gravity: (*x, y, z, -ict*)

Evariste Galois, originator of Group Theory – the first to find that the symmetries of regular polygons are related to the symmetries of algebraic equations

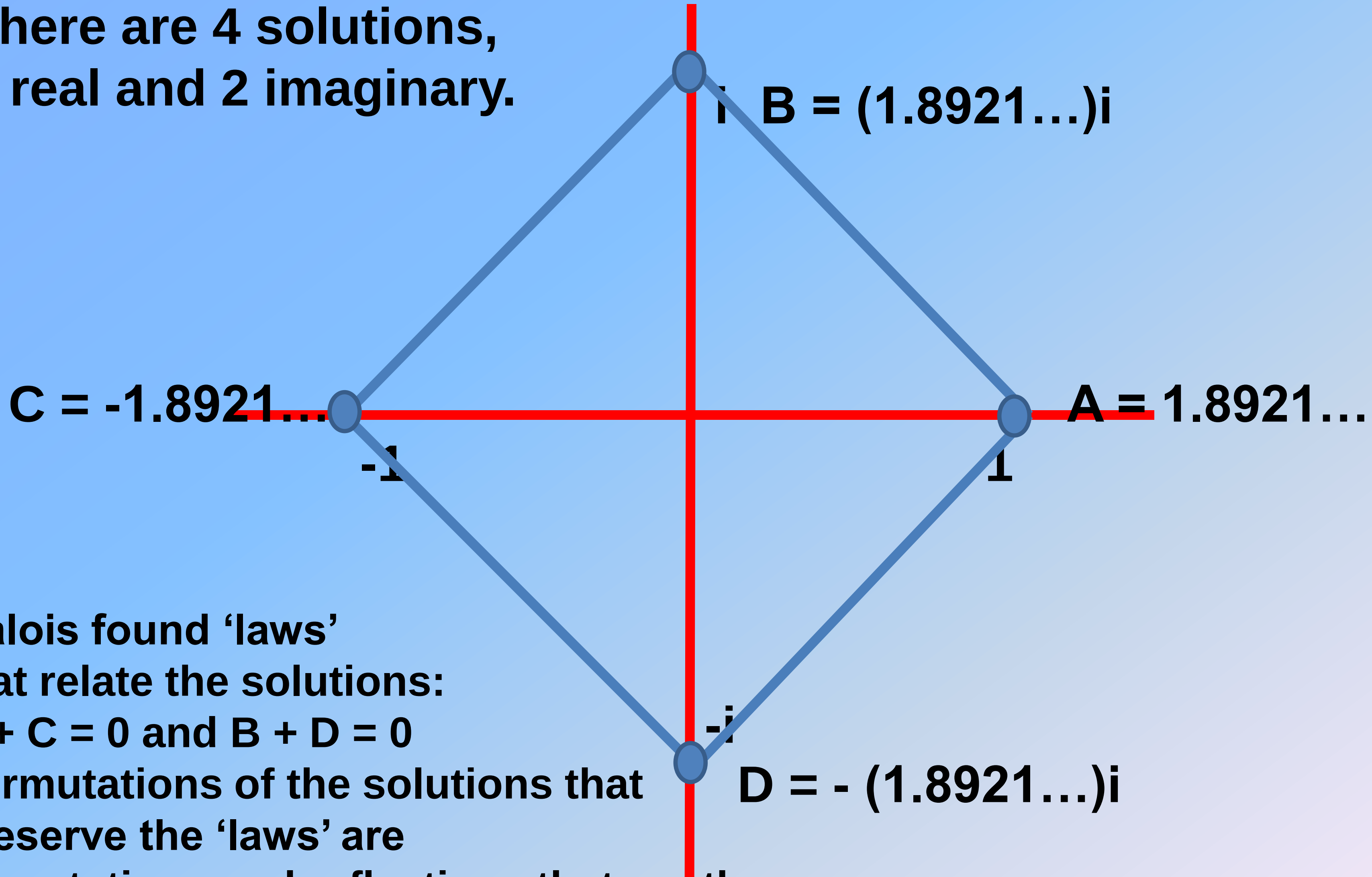
The symmetries of the solutions to equations of order n are linked to the symmetries of certain shapes.



$$x^5 + 6x + 3 = 0$$

Born	Oct. 25, 1811 Bourg-la-Reine, French Empire
Died	May 31, 1832 (at age 20) Paris, Kingdom of France

Galois considered the four solutions of a quartic equation.
Take, for example: $x^4 = 2$
There are 4 solutions,
2 real and 2 imaginary.



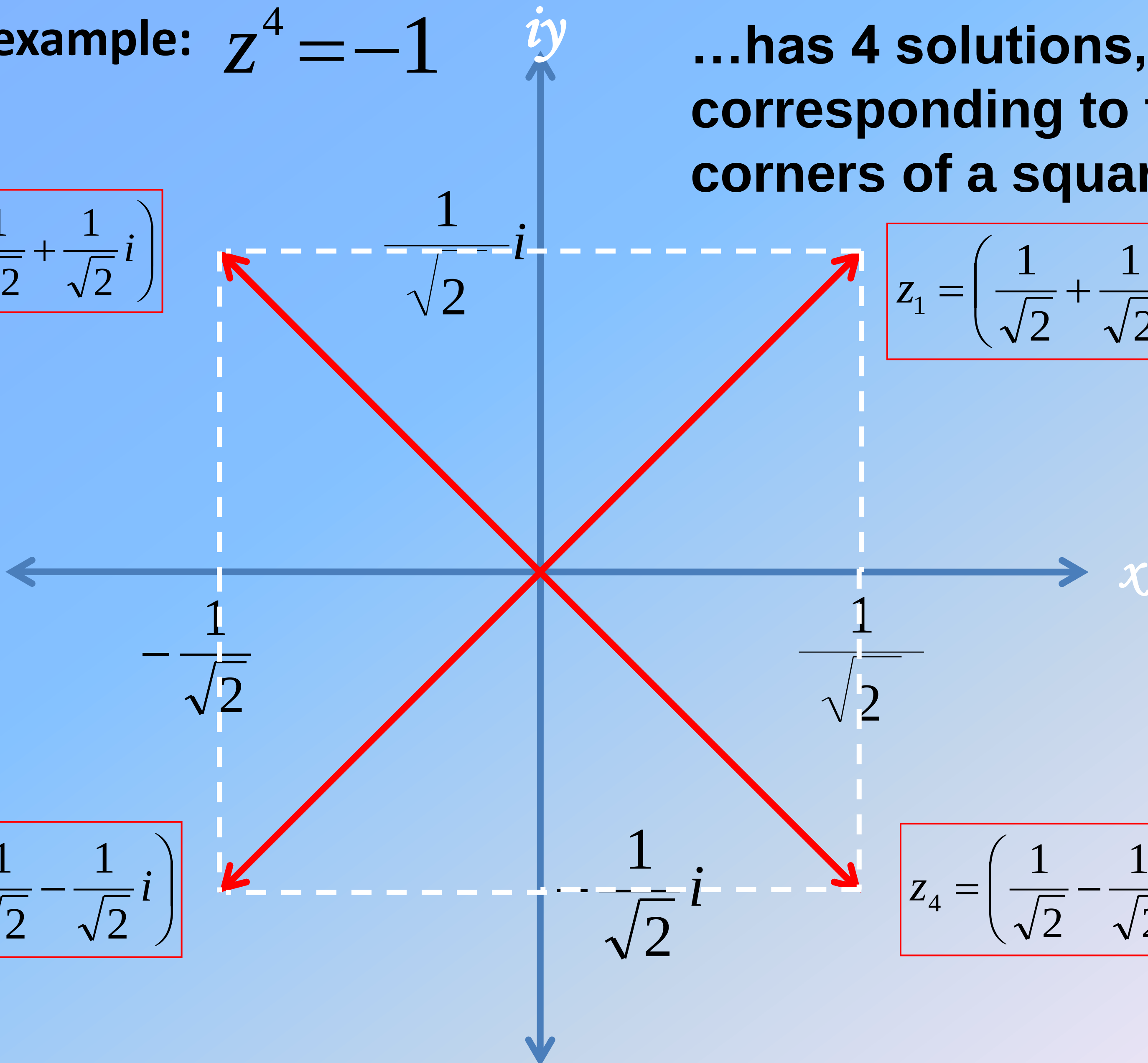
Galois found ‘laws’
that relate the solutions:
A + C = 0 and B + D = 0
Permutations of the solutions that
preserve the ‘laws’ are
the rotations and reflections that are the
symmetry operations of a square.

Another example: $z^4 = -1$

…has 4 solutions, corresponding to the corners of a square.

$$z_2 = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right)$$

$$z_1 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right)$$

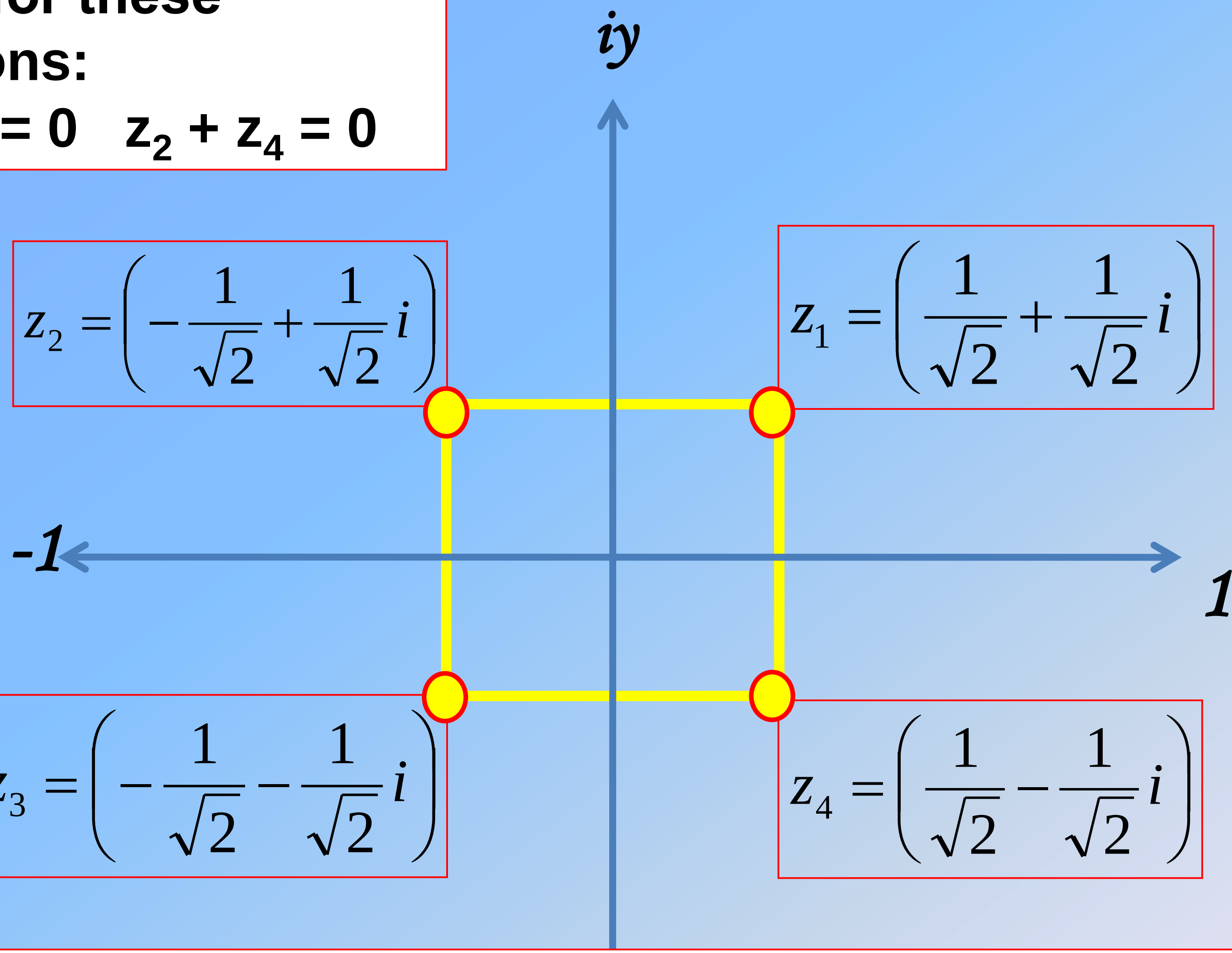


$$z_3 = \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right)$$

$$z_4 = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right)$$

‘laws’ for these solutions:

$z_1 + z_3 = 0 \quad z_2 + z_4 = 0$



$$z_2 = \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right)$$

$$z_1 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right)$$

$$z_3 = \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right)$$

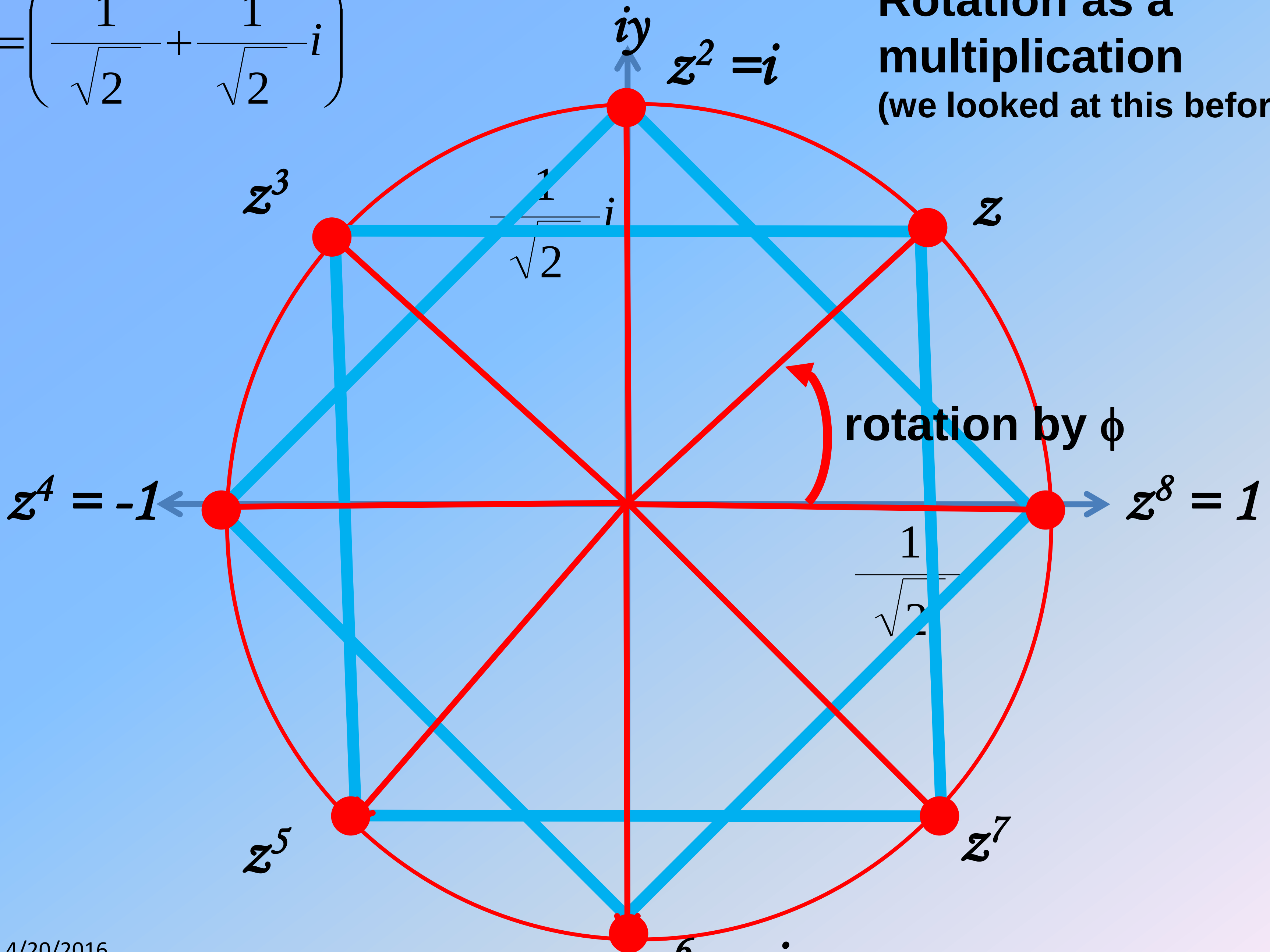
$$z_4 = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i \right)$$

The ways you can interchange the roots of the equation correspond to the rotations and reflections of a rigid square.

$$z = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right)$$

Rotation as a multiplication

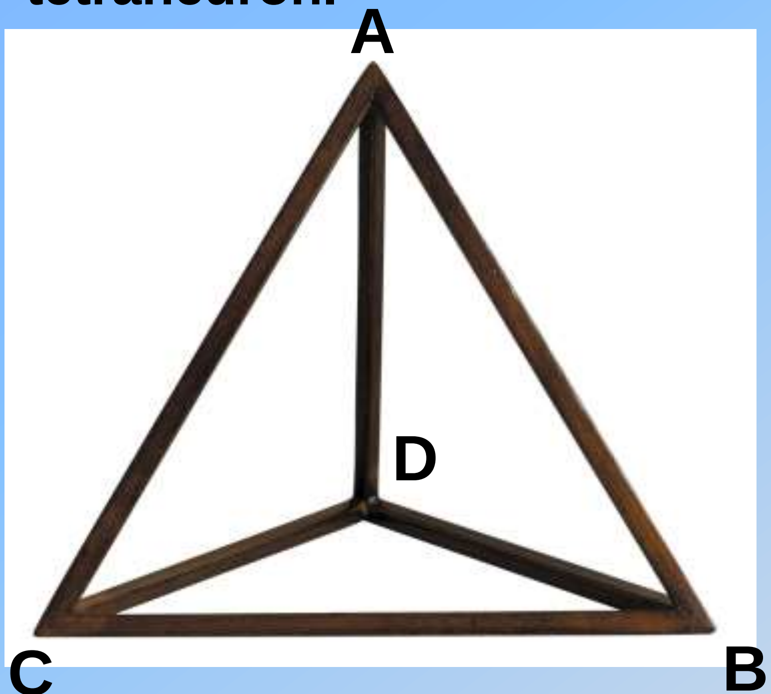
(we looked at this before)



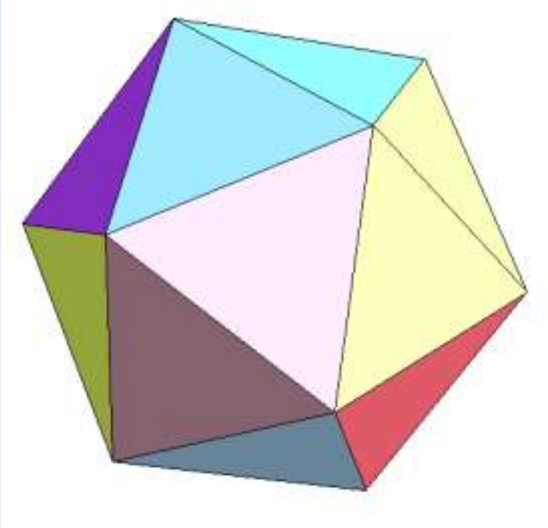
“ How the roots of an equation behave as you permute it suggests that each equation has a certain symmetrical object associated with it, and it is the individual properties of these symmetrical objects that holds the key to how to solve individual equations.”

From Symmetry, by Marcus du Sautoy, p. 171

Galois found that the symmetries of the four solutions to
$x^4 - 5x^3 - 2x^2 - 3x – 1 = 0$
correspond to the 24 symmetries of the tetrahedron.



and that the symmetries of the 5 solutions to $x^5 + 6x + 3 = 0$ correspond to the symmetries of the icosahedron.



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Lorentz symmetry (translations, rotations, and boosts) defines Minkowski Space – Spacetime without gravity.

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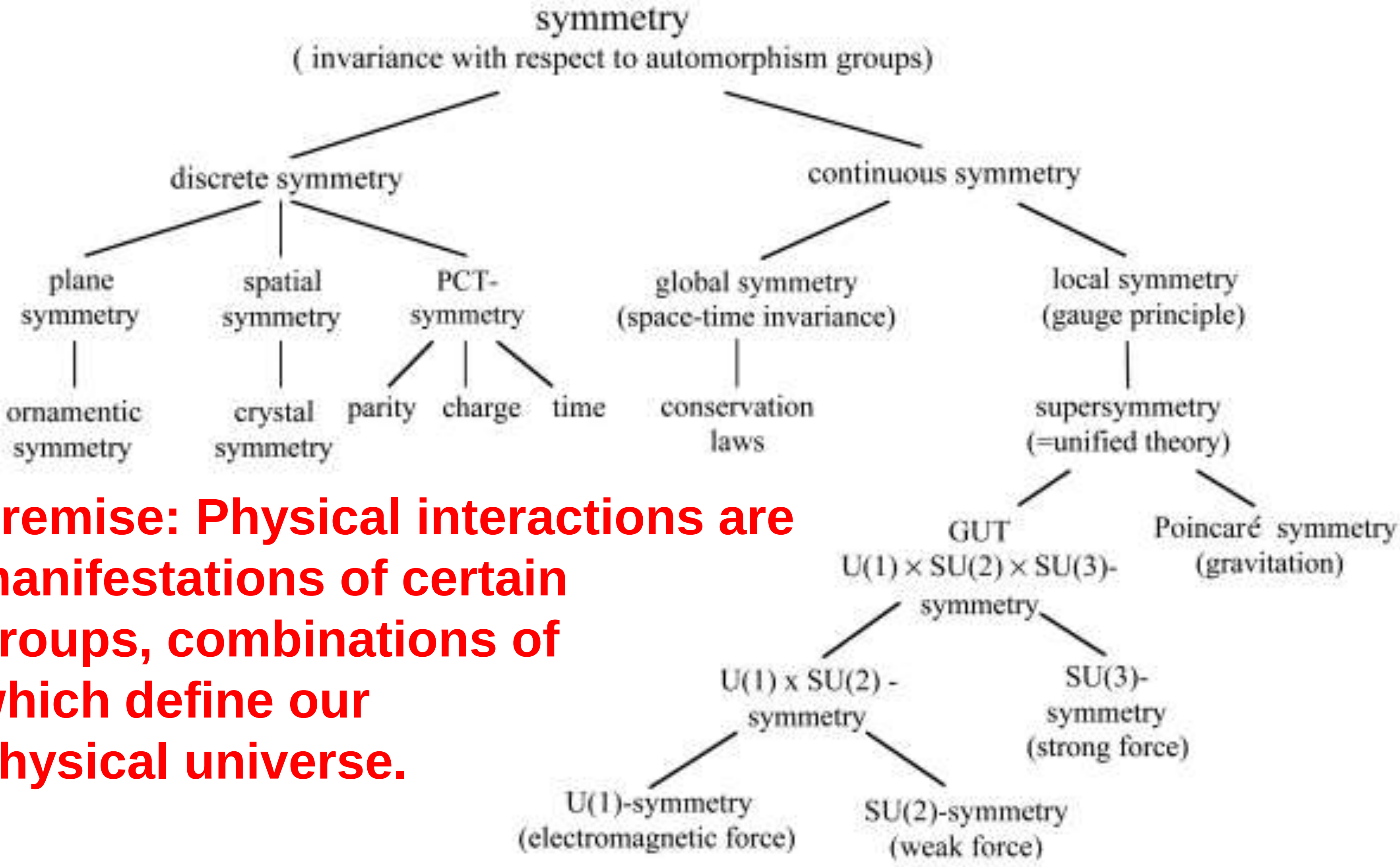
Lorentz symmetry (translations, rotations, and boosts) defines Minkowski Space – Spacetime without gravity.

Lorentz invariance is the basis for Special Relativity.

SO(n): Special Orthogonal Group of order n

n-dimensional rigid body is defined by the rigid transformation, $[T] = [A, d]$, where d is an n-dimensional translation and A is an $n \times n$ rotation matrix, which has n translational degrees of freedom and $n(n - 1)/2$ rotational degrees of freedom.

Group	Representations	Degrees of freedom
SO(2) Motion in a plane Euclidean geometry	**Circle, motion in a plane**	**[2(2-1)/2] = 1 angle n= 2 directions of translation**
SO(3) Newtonian mechanics Euclidean geometry	**Rotations on a sphere**	**[3(3-1)/2] = 3 3 rotation angles, n = 3 directions of translation**
SO(4) Minkowski spacetime 3 spatial directions + imaginary time "Poincare Group"	**Spacetime**	**[4(4-1)/2] = 6 6 rotation angles n = 4 directions of translation**



Premise: Physical interactions are manifestations of certain groups, combinations of which define our physical universe.

Figure 9. Classification of symmetry

Symmetry and complexity in dynamical systems, Manzier, K. (2005), European Review, Vol. 13, Supp. No. 2, 29–48

Break time!

symmetry demos

Next week:
Illustrate symmetry in some artistic representation – can be visual art, computer simulation, musical composition, dance, poetry, or other form.

Describe the symmetry operations you are illustrating, and the group that your piece represents.