Symmetry and Aesthetics in Contemporary Physics

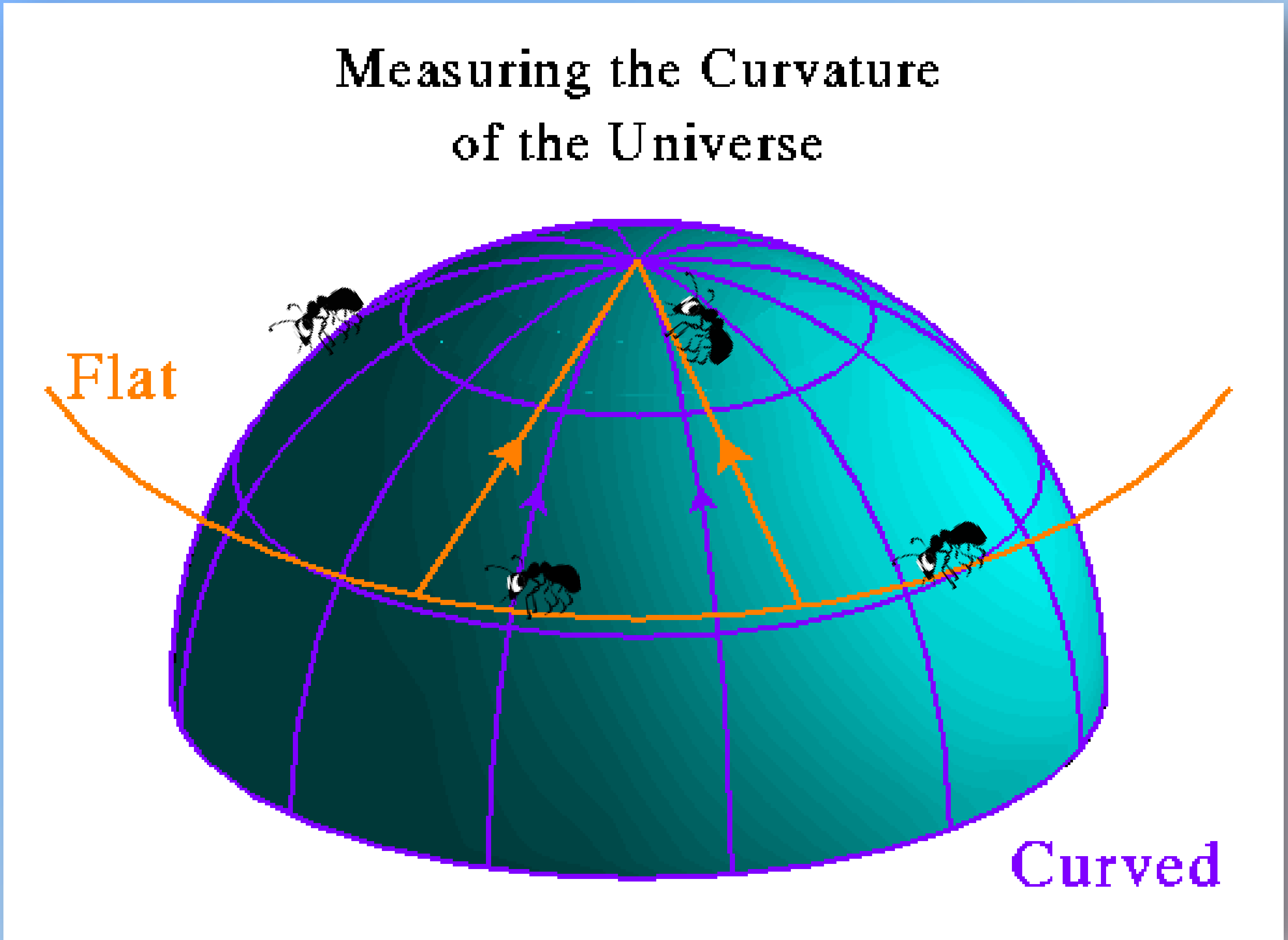
CS-10, Spring 2016

Dr. Jatila van der Veen

CLASS 7:

CURVED SPACETIME

Discussion of Feynman, Chapter 42: questions ~ comments ~ opinions?



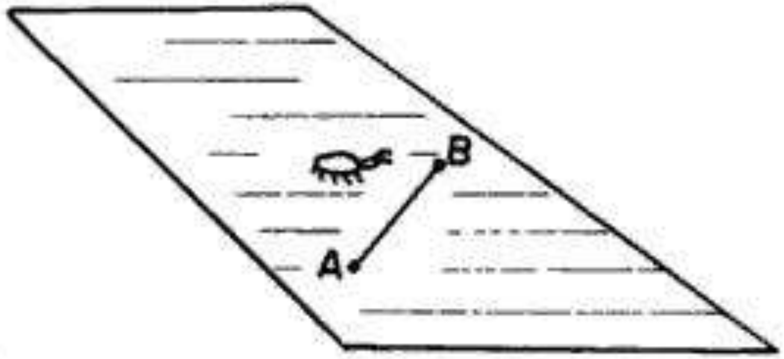


Figure 6-4. Making a "straight line" on a plane.

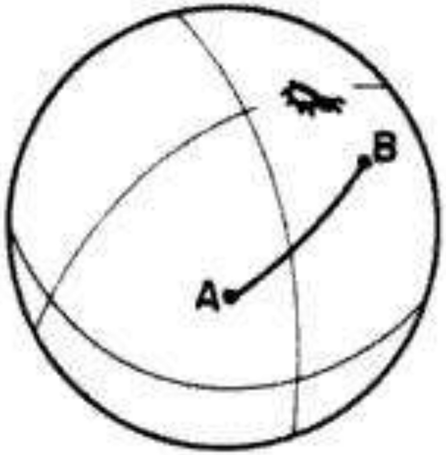


Figure 6-5. Making a "straight line" on a sphere.

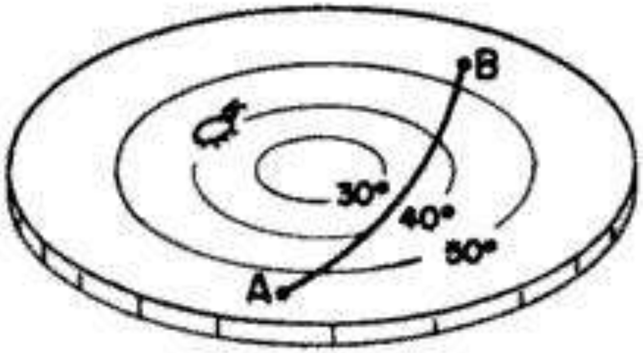


Figure 6-6. Making a "straight line" on the hot plate.

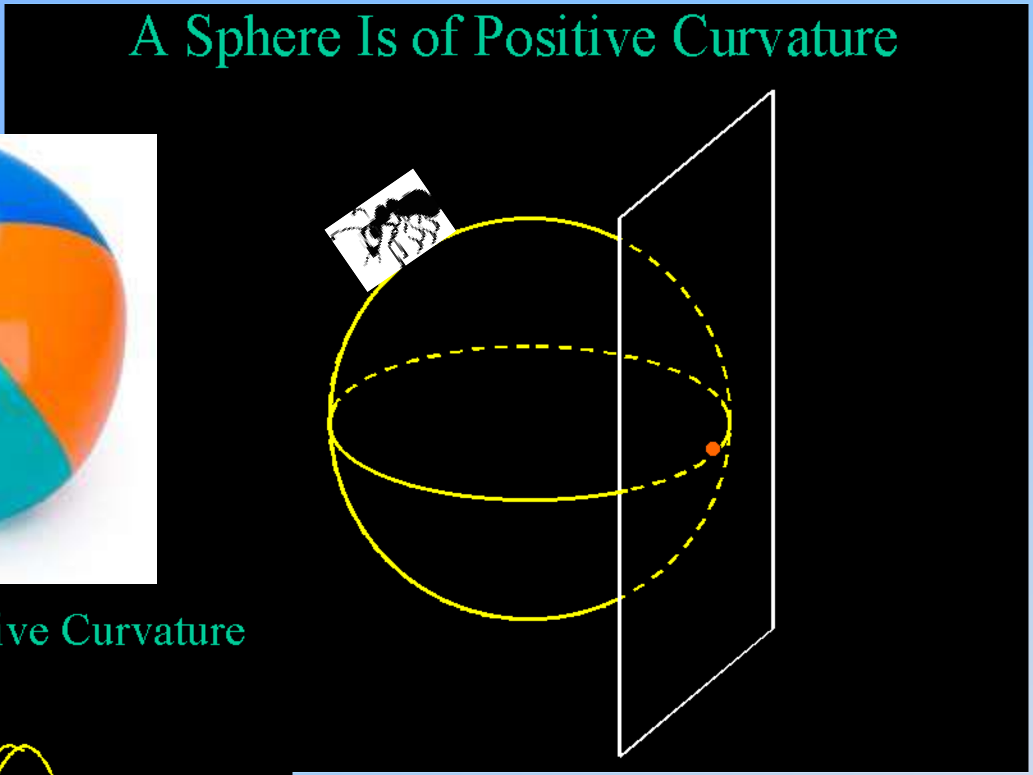
Feynman's bugs

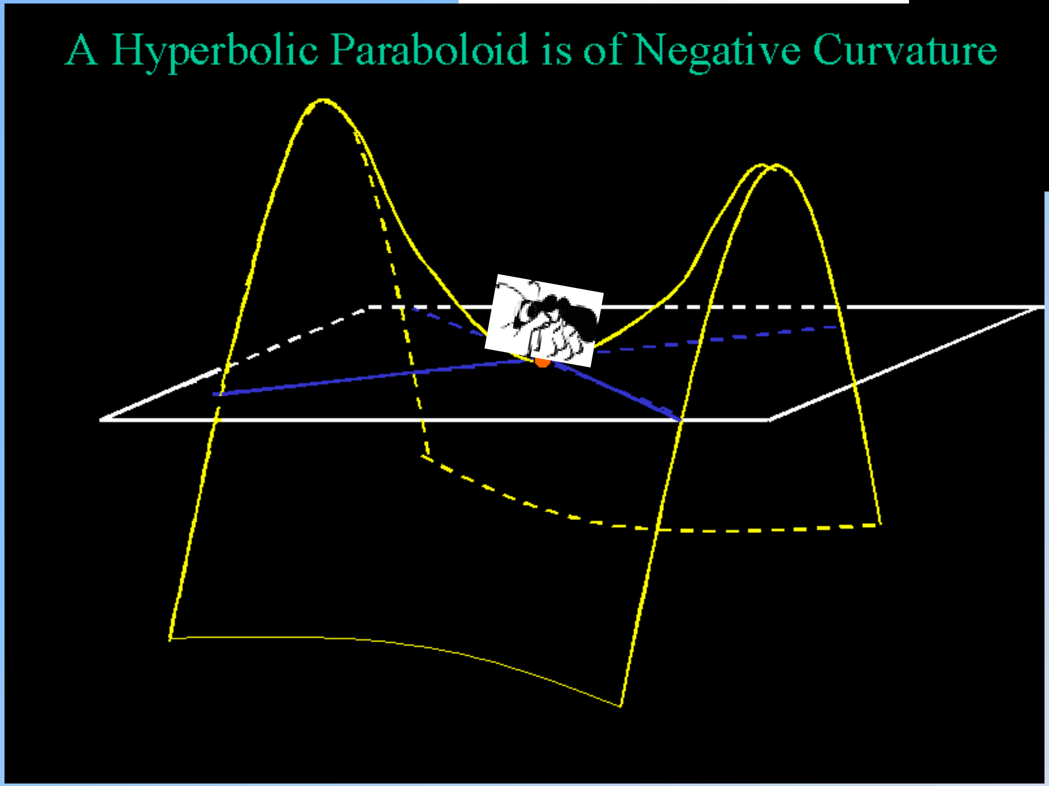
Possible shapes of spacetime:

closed

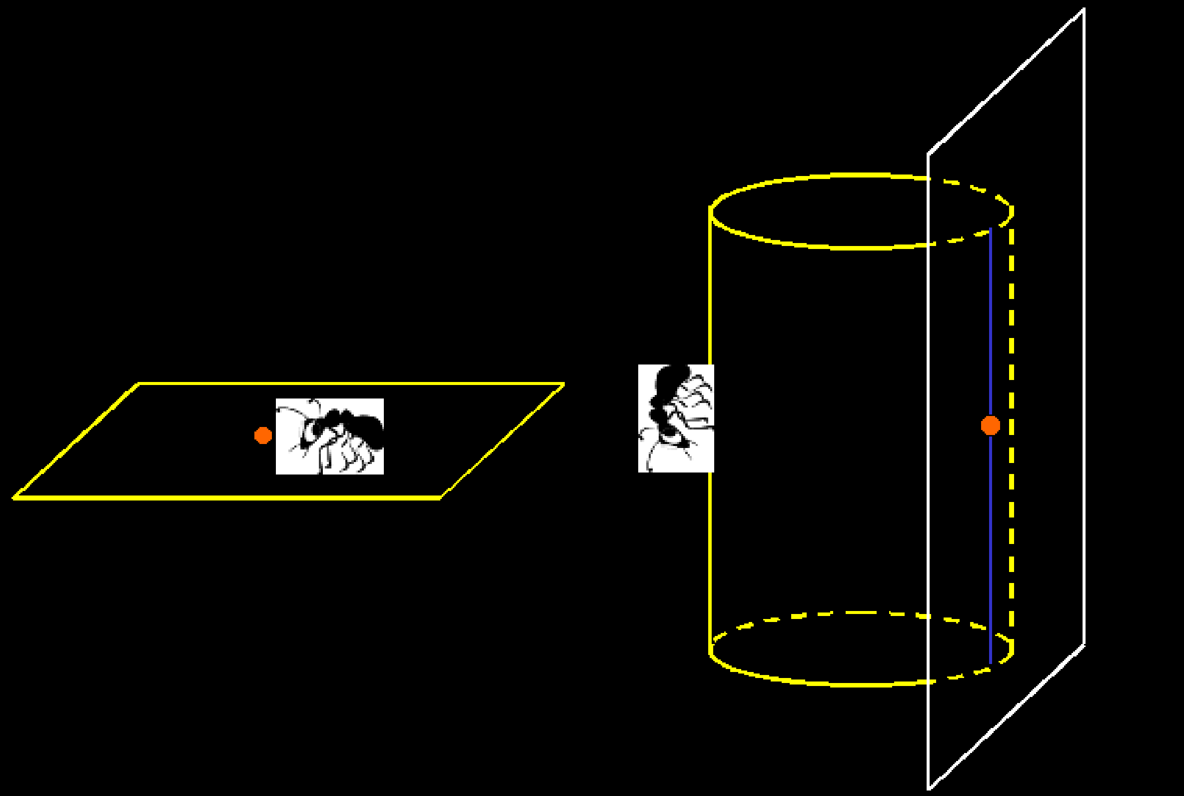
open

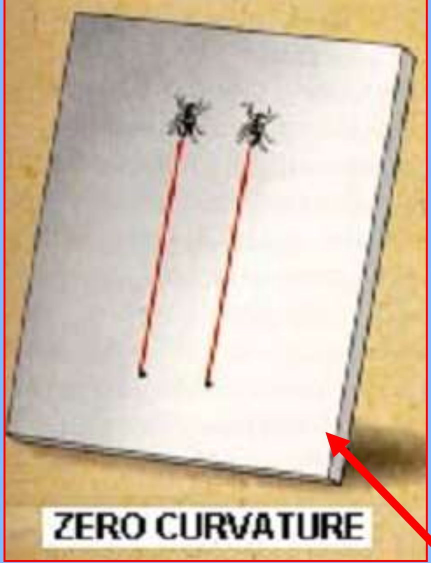
flat



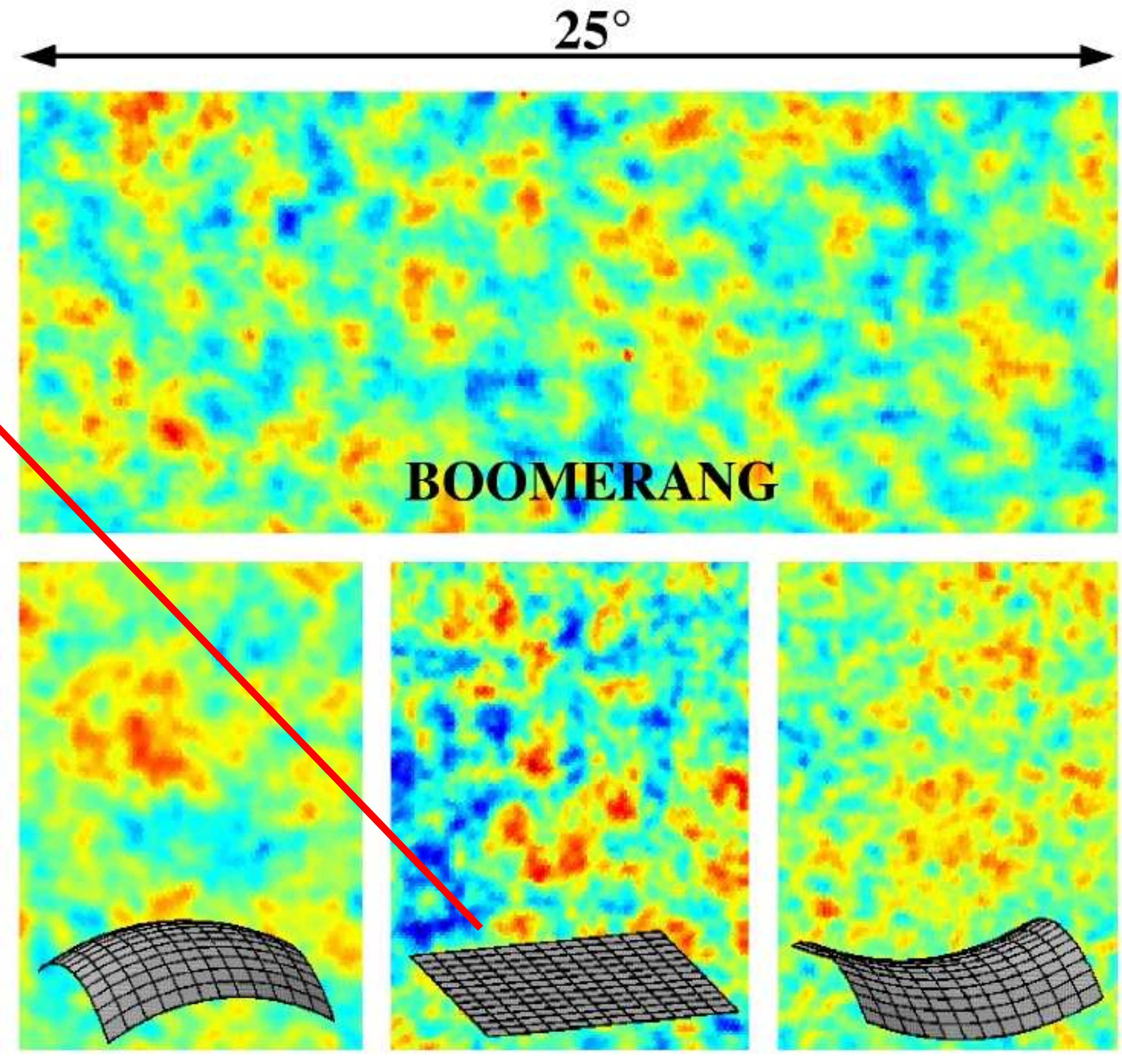


Planes and Cylinders are of Zero Curvature



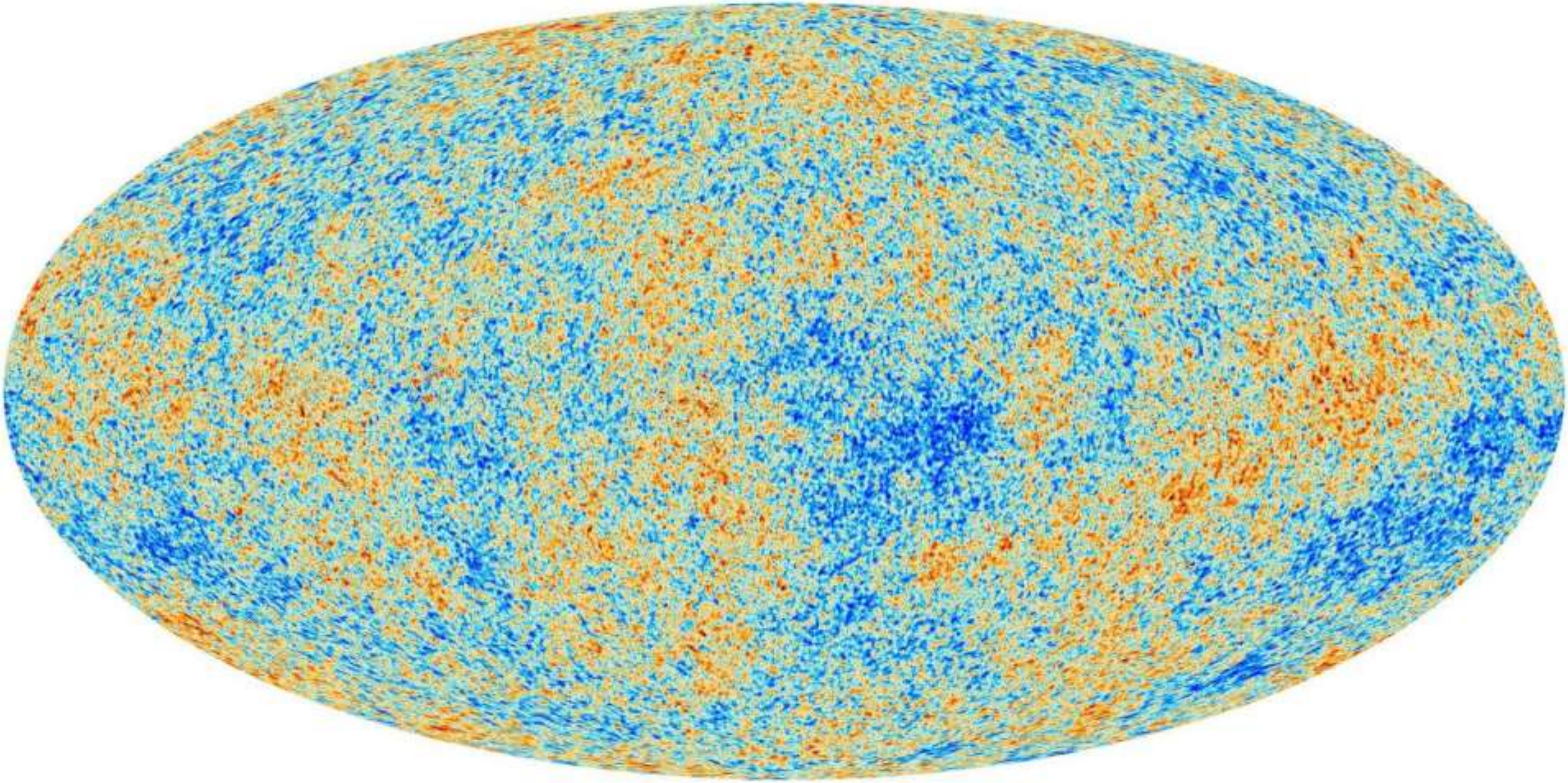


Measurements indicate that on very large scales the geometry of the universe is very nearly FLAT.



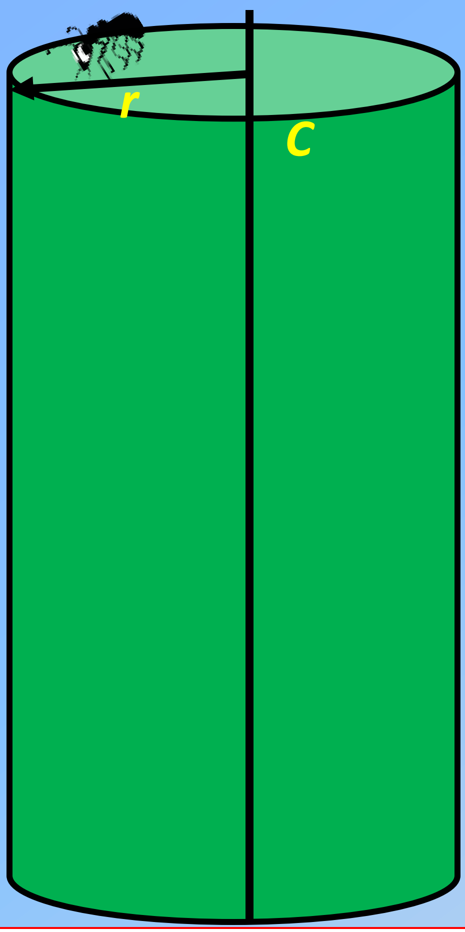
Boomerang satellite mapped the CMB over Antarctica in 2001

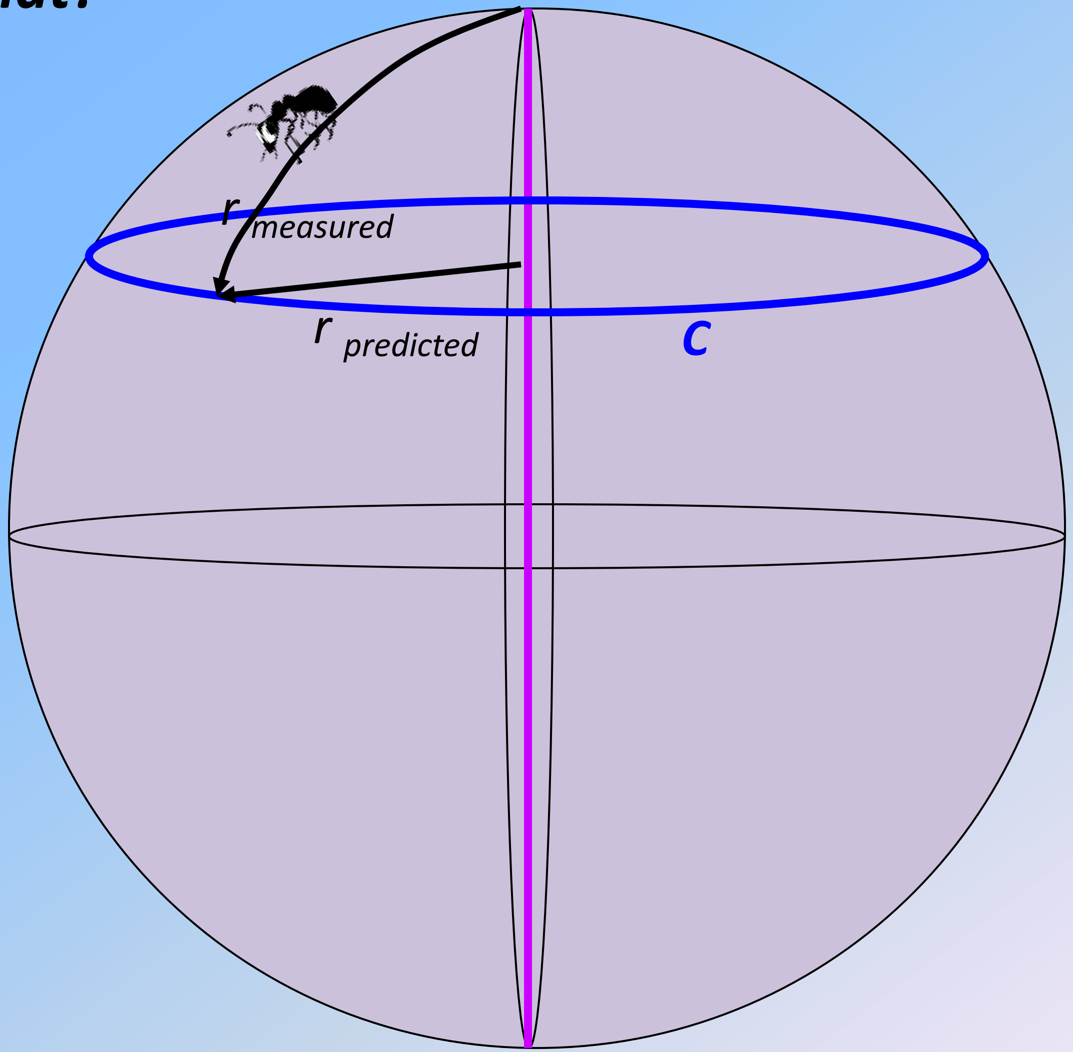
Planck detailed map of the CMB with foreground removed, 2015



$$\frac{C}{2\pi} = r_{predicted}$$

How can you tell if the surface you live on is curved or flat?





$$r_{predicted} = \frac{C}{2\pi} = r_{measured}$$

$$r_{predicted} = \frac{C}{2\pi} < r_{measured}$$

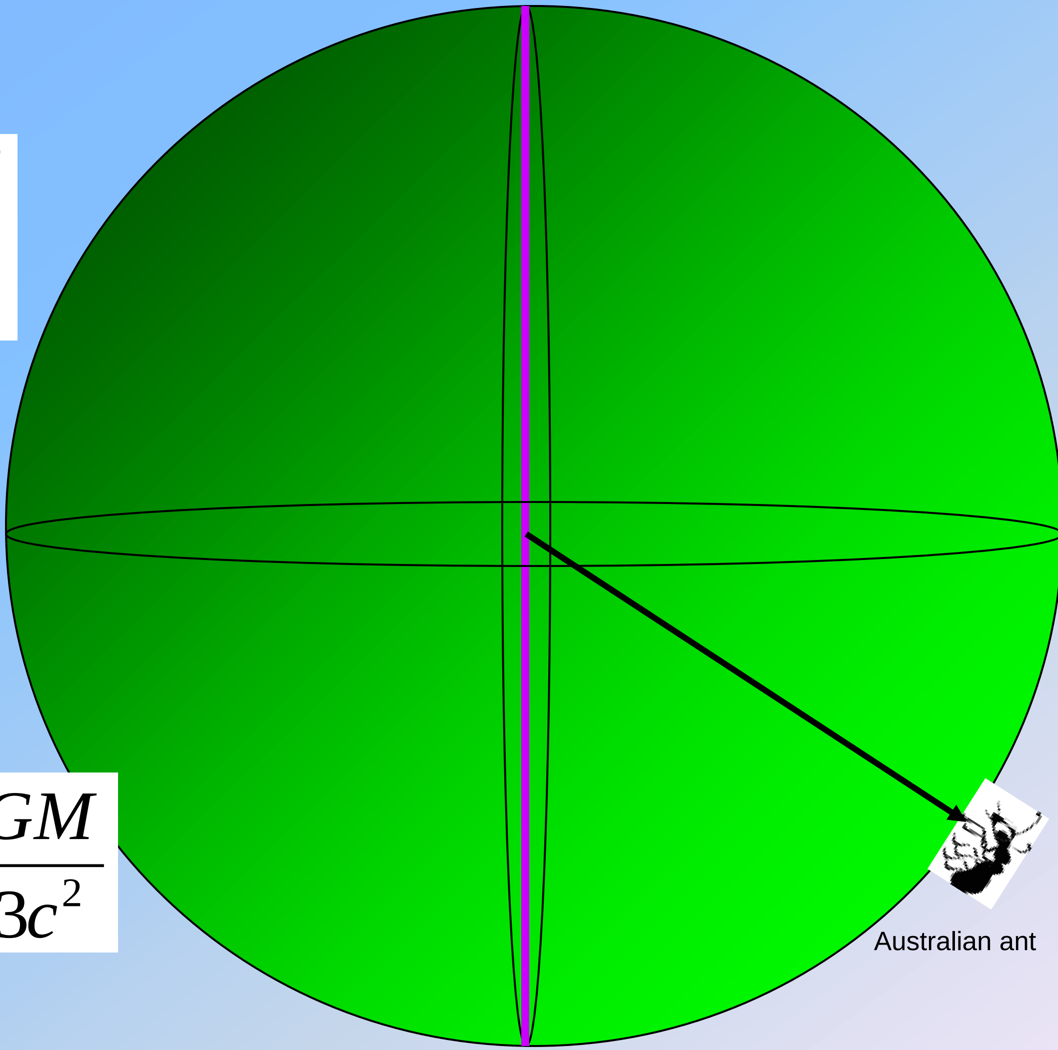
For a sphere: $$Area = 4\pi R^2$$

Predicted radius is thus:

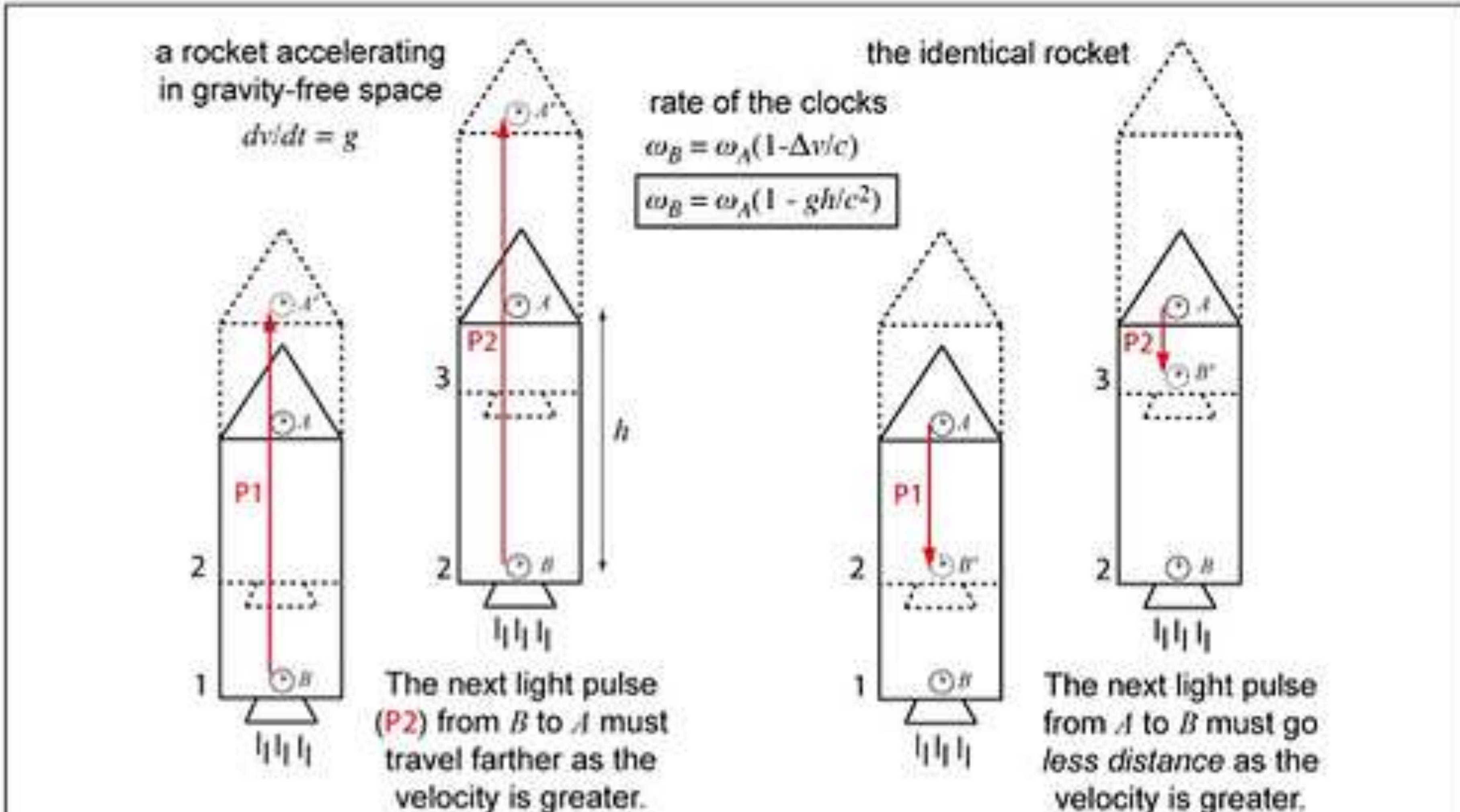
$$R_{predicted} = \sqrt{\frac{A}{4\pi}}$$

Now, if you dig a hole and measure the actual radius of the earth directly, you find that $R_{predicted} > R_{measured}$!

$$\Delta R = R_{predicted} - R_{measured} = \frac{GM}{3c^2}$$

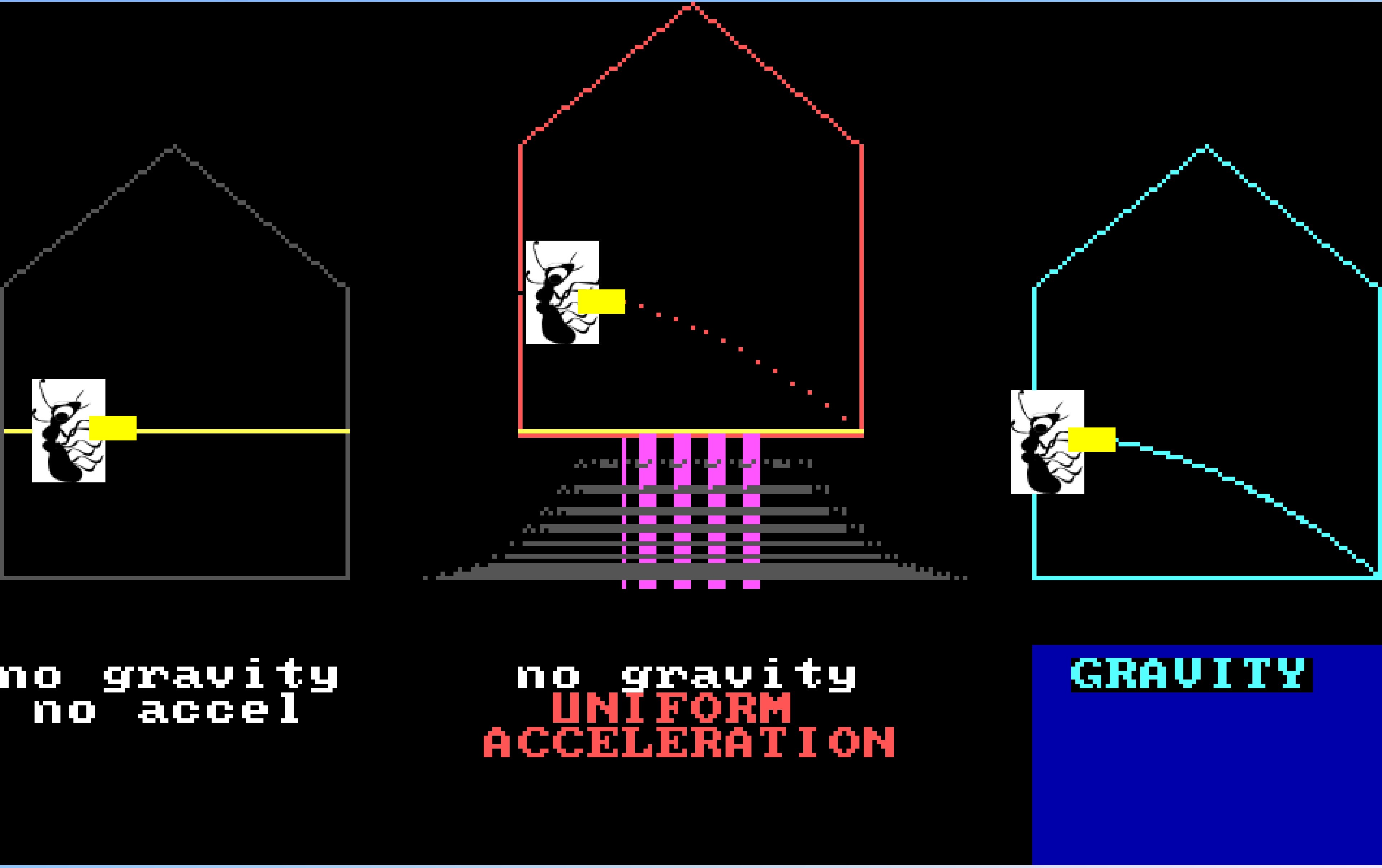


The radius of the Earth is smaller than you would predict if you calculate it from measuring the area!

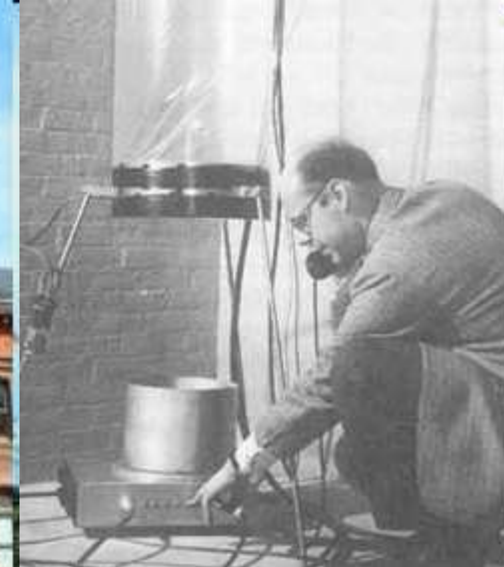


Observers at the nose and tail of an accelerating rocket observe different times for light pulses emitted by each other's light clocks.

Einstein's Principle of equivalence



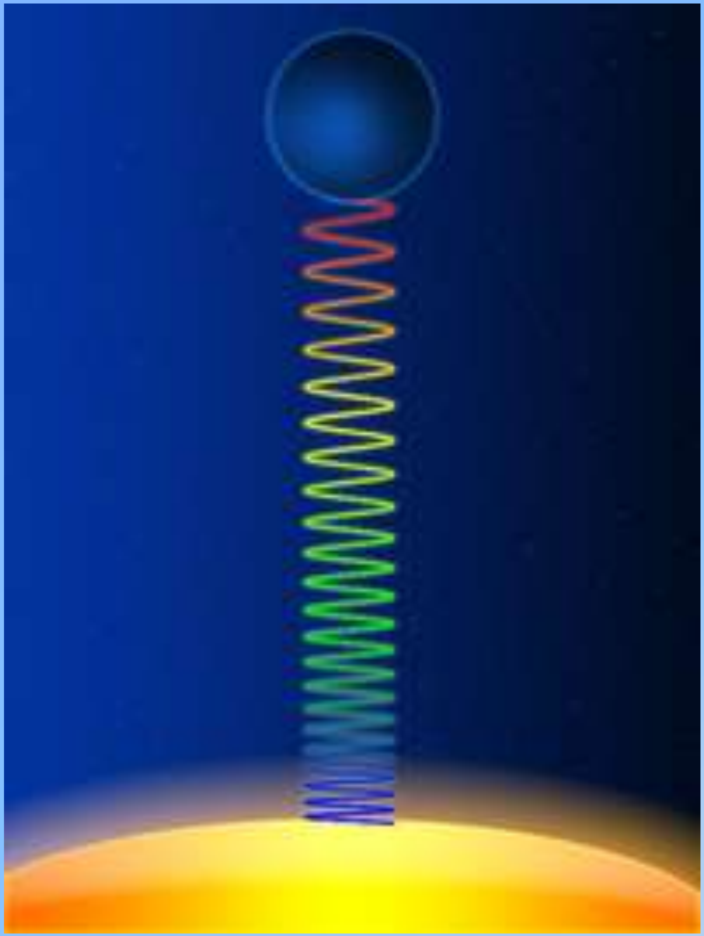
Experiment by Pound & Rebka

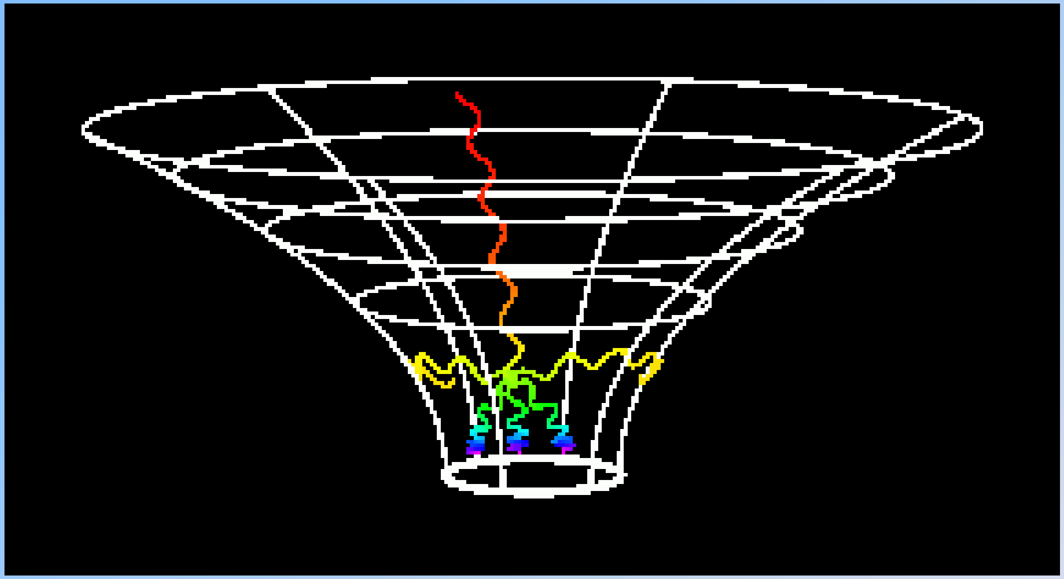


$$\frac{\Delta\omega}{\omega_0} = \frac{gh}{c^2} = \frac{\Delta E}{E_0}$$

$$\left.\frac{\Delta E}{E_0}\right]_{down} - \left.\frac{\Delta E}{E_0}\right]_{up} = 5\times10^{-15}$$

Stretching of light waves due to the curvature of spacetime.





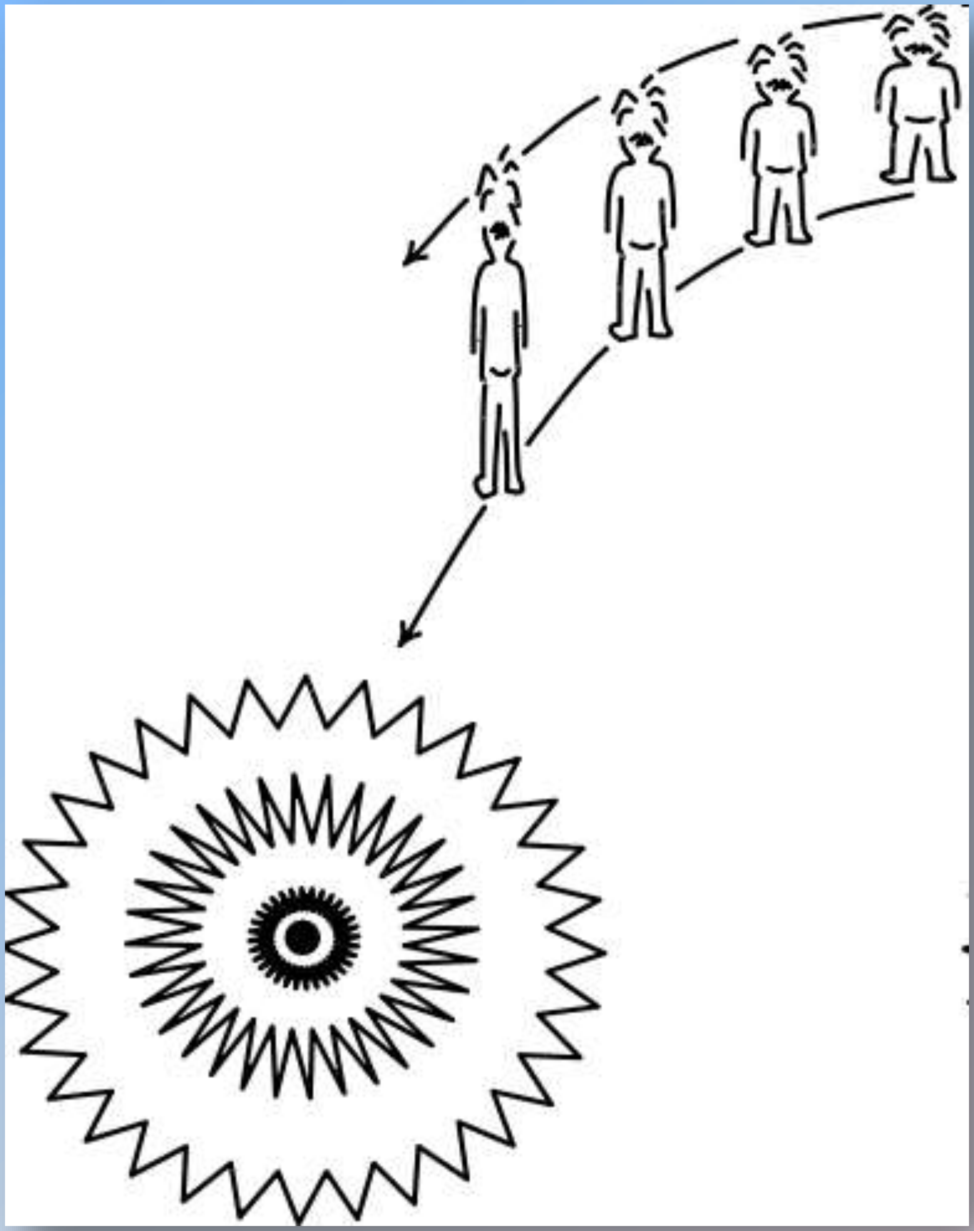
**photon climbing out loses energy,
is red-shifted according to an observer at the ‘top’**

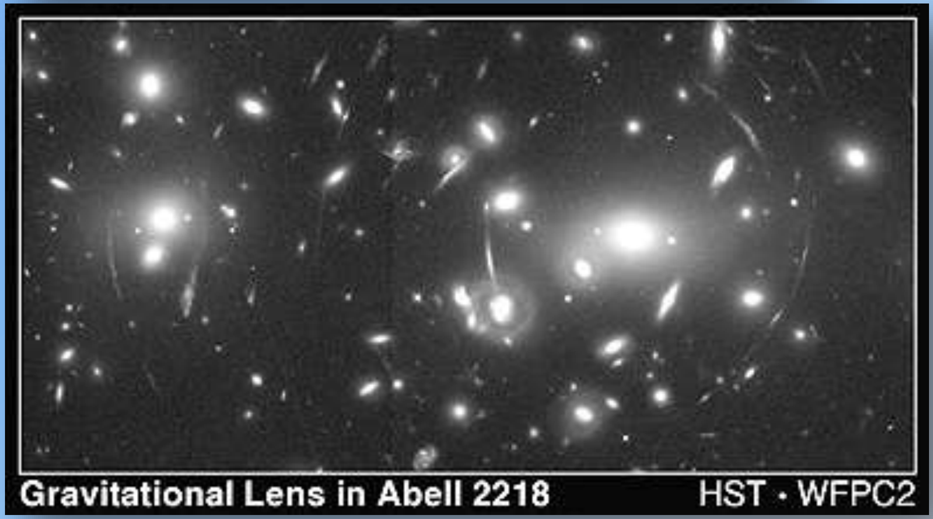
**photon falling in gains energy,
is blue-shifted according to an observer at the ‘bottom’**

Extreme tidal forces close to a black hole

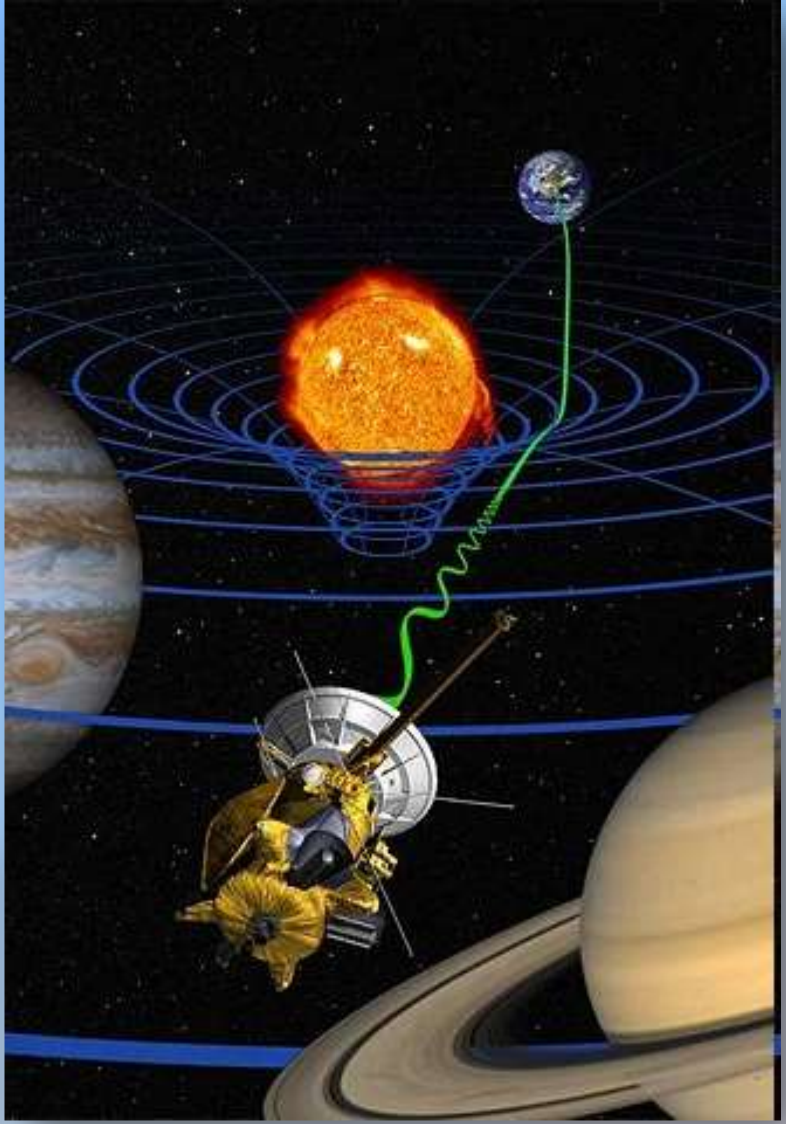
https://vimeo.com/14145244?from=outro-embed

falling through the event horizon – solutions to equations visualized



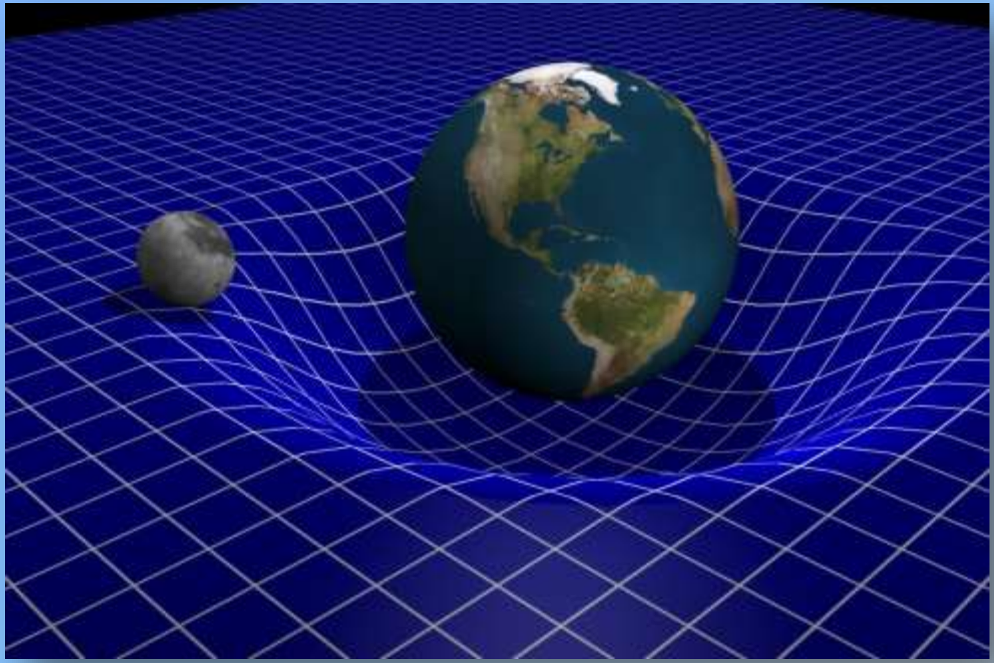


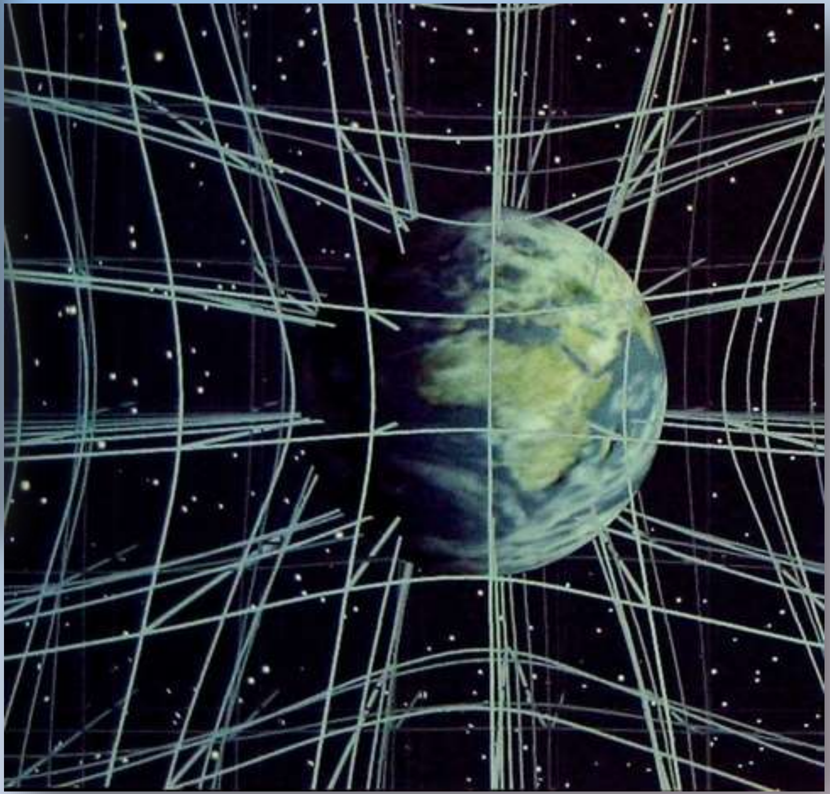
Gravitational lenses

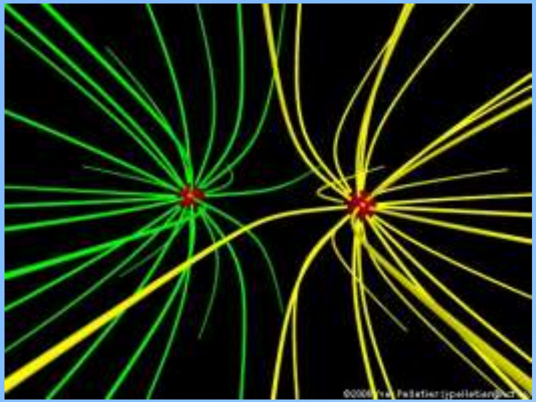


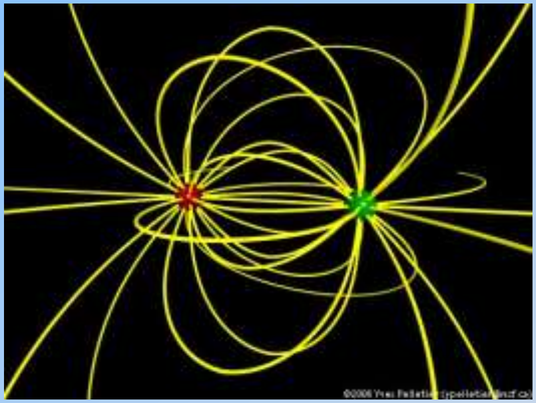
Cassini probe measured gravitational redshift of signal sent to Earth by the gravitational field of the Sun

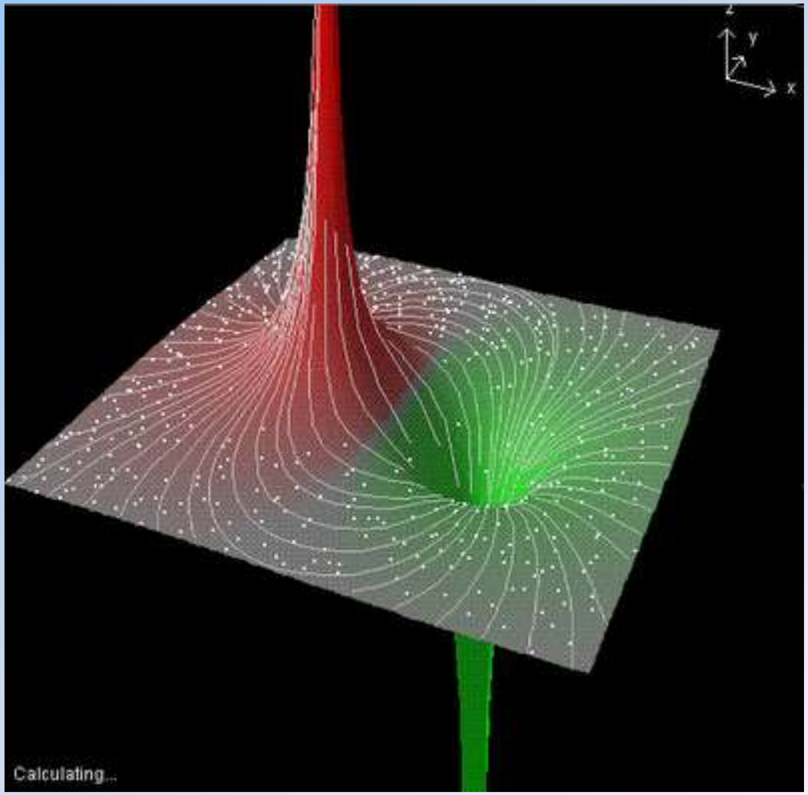
Spacetime is curved. But --
How can space itself be *curved*?
What does this even mean?
What does 4D spacetime curve *into*?
How can we truly visualize this?







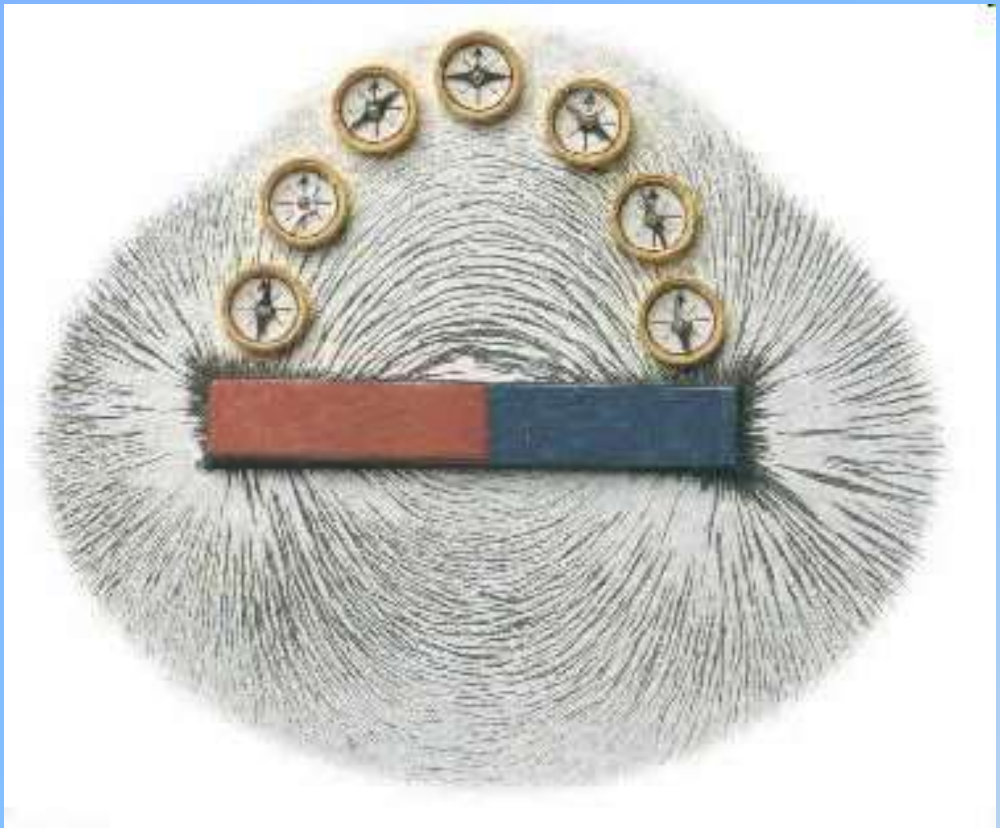


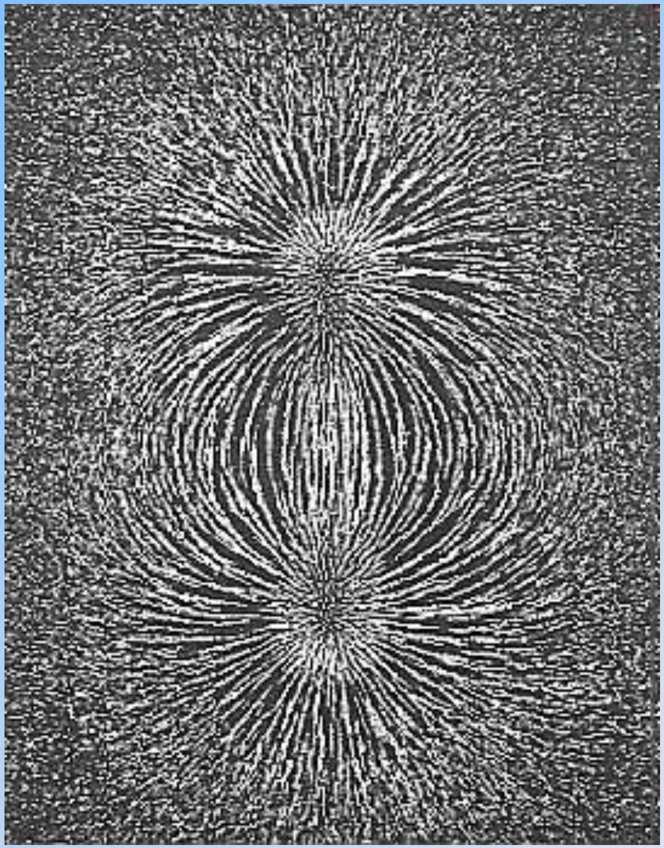


Michael Faraday in his lab
Painting by Harriet Jane Moore

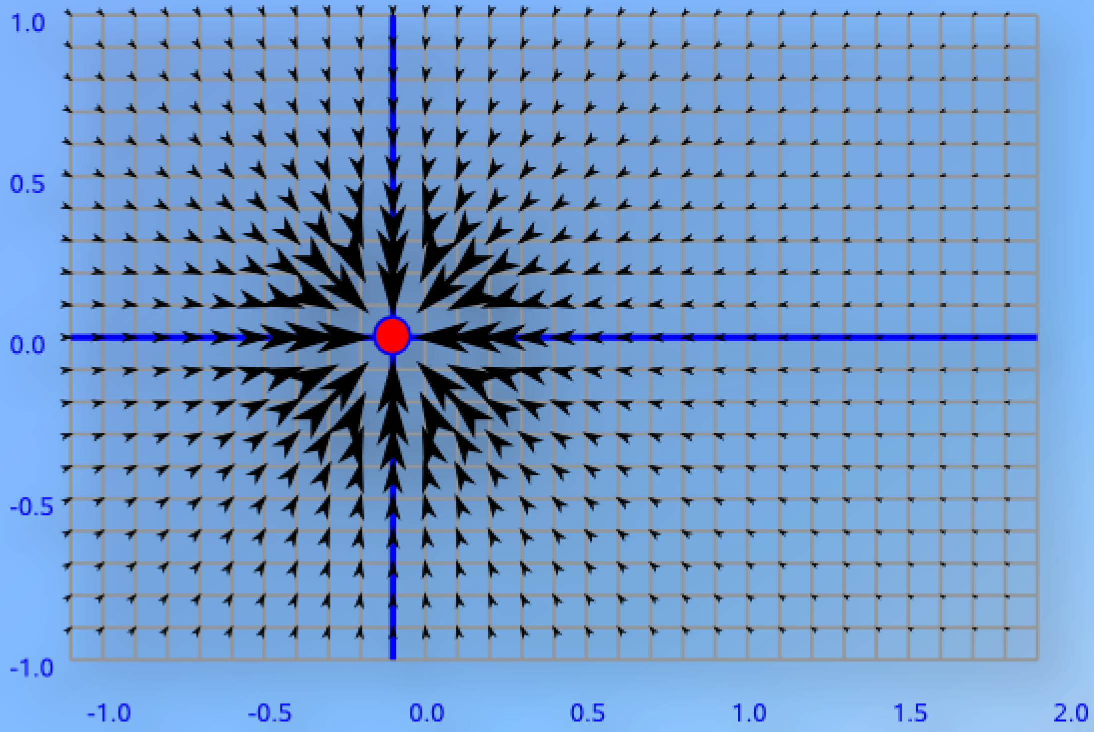
Faraday's Field Lines:
The first idea that a charge creates a field which influences the shape of the space around it, and effects other charges.

Iron filings follow magnetic field lines of a dipole magnet



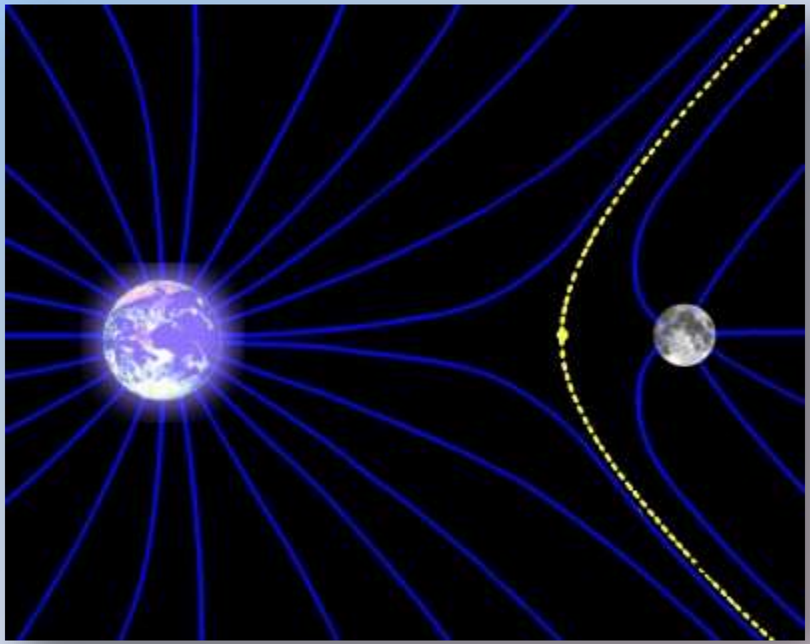


An electron *at rest* in a magnetic field does not “feel” a magnetic force. Only an electron moving in a magnetic field is deflected by a force: F = qv x B.

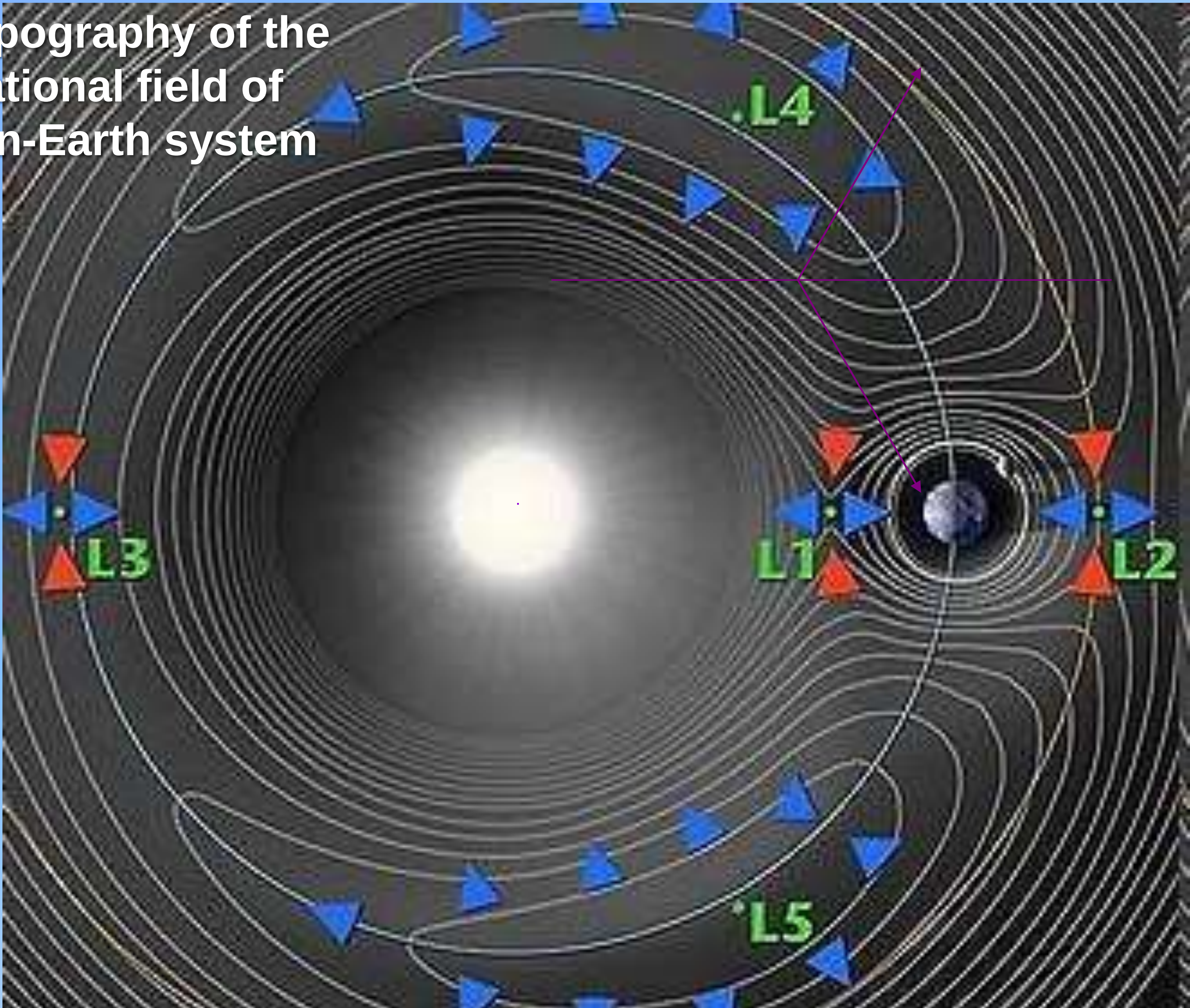


Visualizing gravitational fields

Analogous to the way an electric charge “disrupts” the space around it with an electric field, a mass disrupts the spacetime around it with a gravitational field.



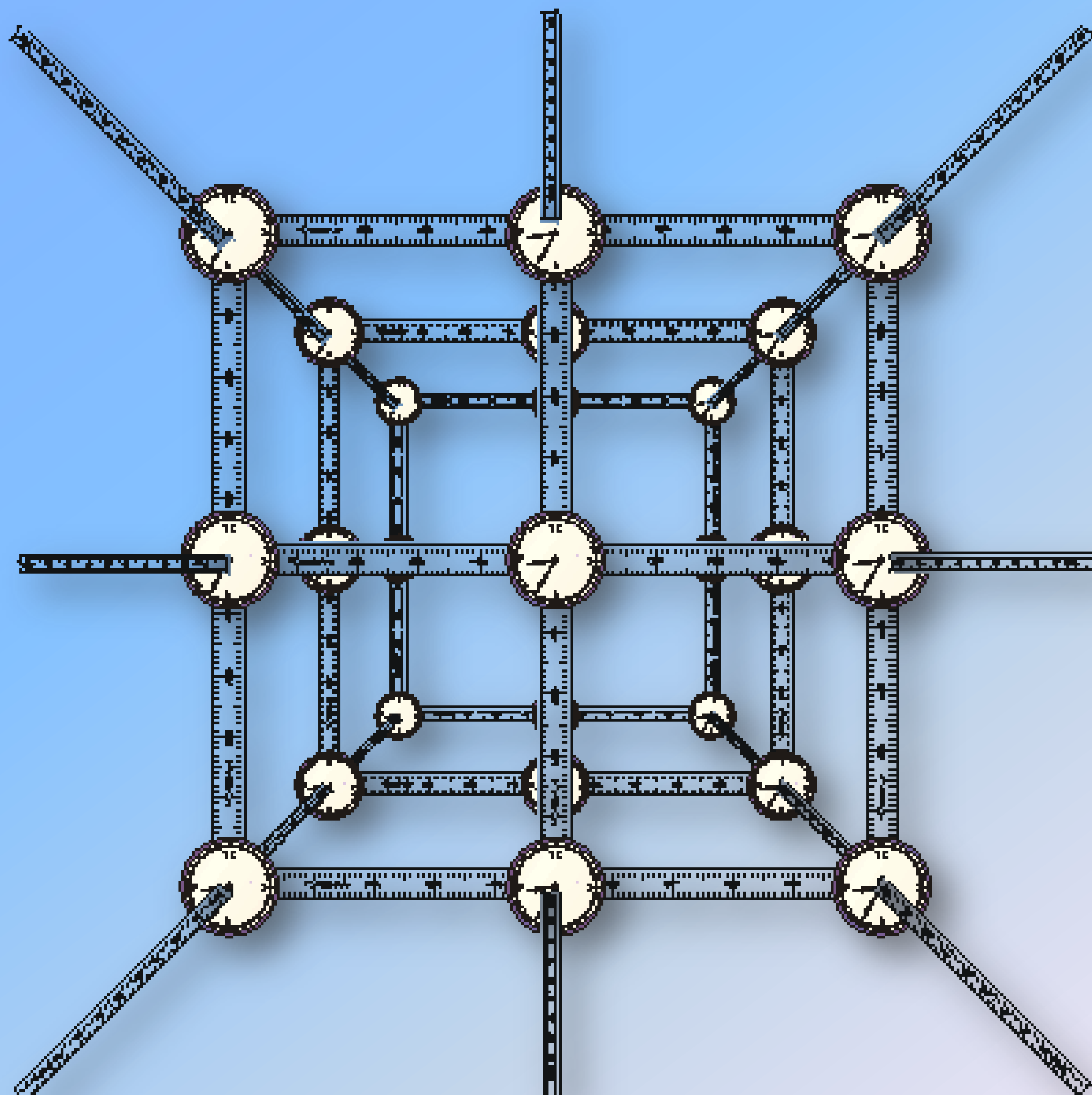
The topography of the gravitational field of the Sun-Earth system



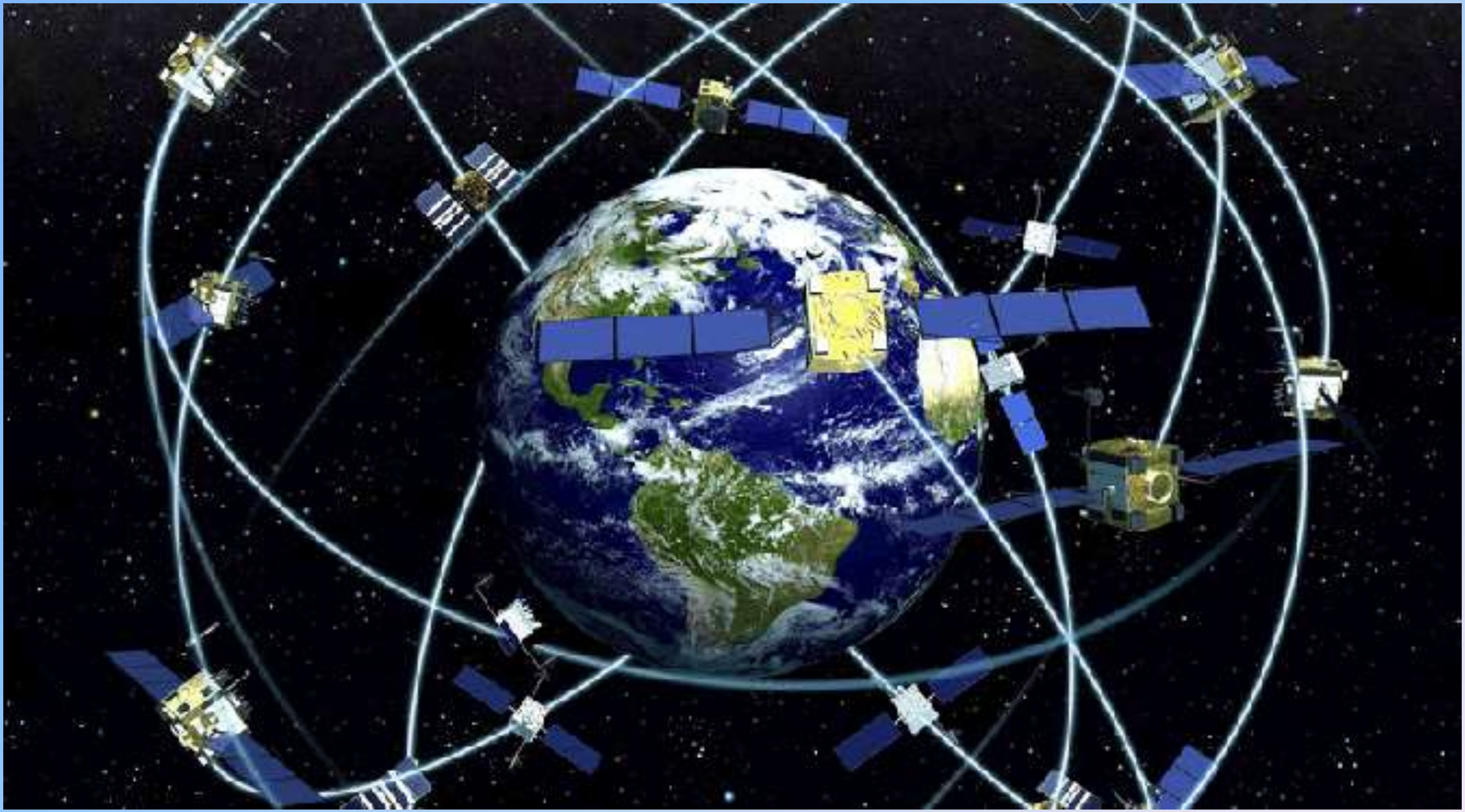
Planck satellite orbiting L2

animation:
European Space Agency

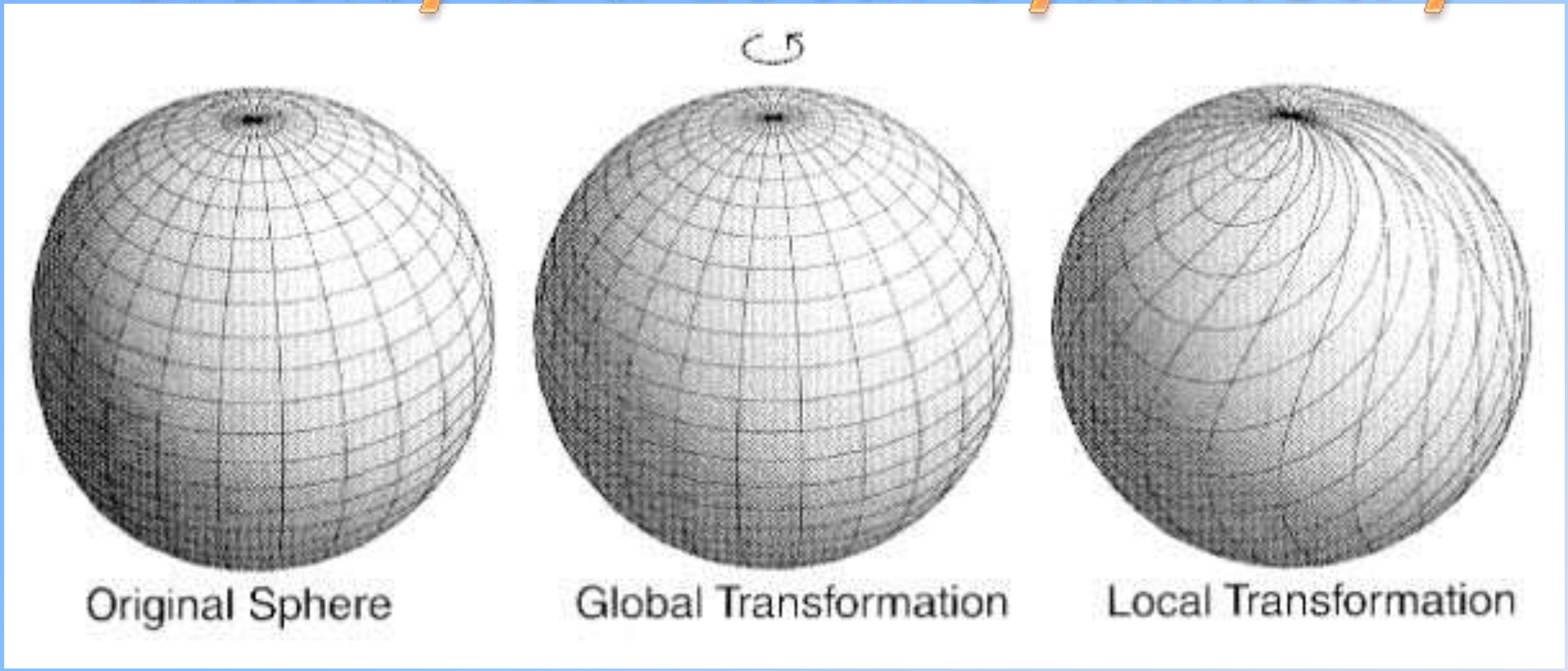
Clocks actually tick at different rates in a gravitational field!



GPS satellites must account for the different rates of time on the ground and at their altitude.



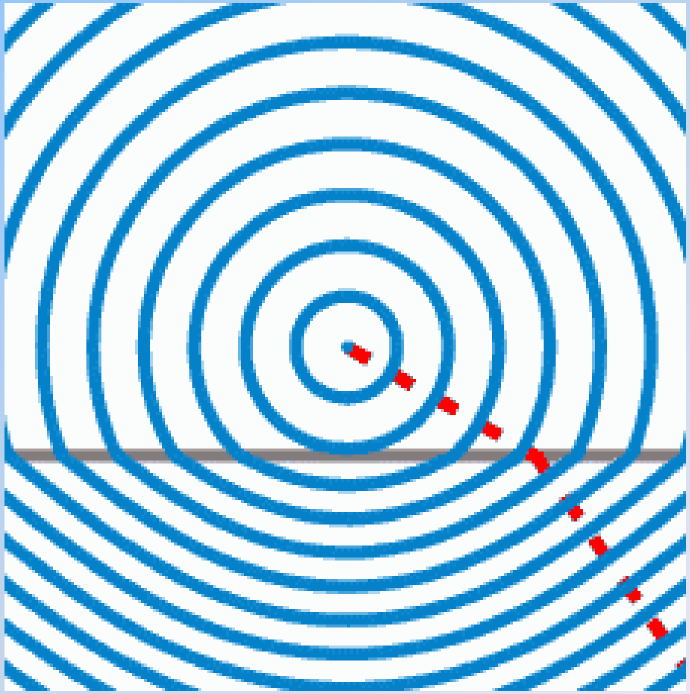
Gravity is a Local Symmetry



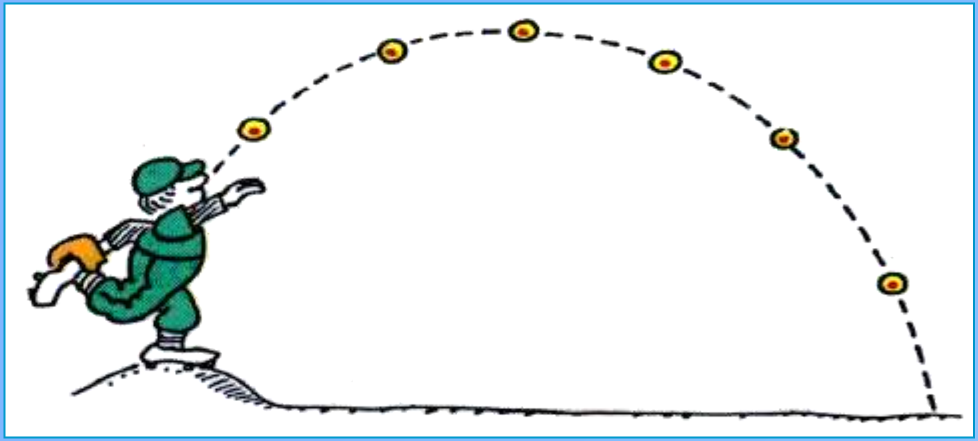
- A global symmetry does not depend on spacetime.

* A local symmetry depends on spacetime.

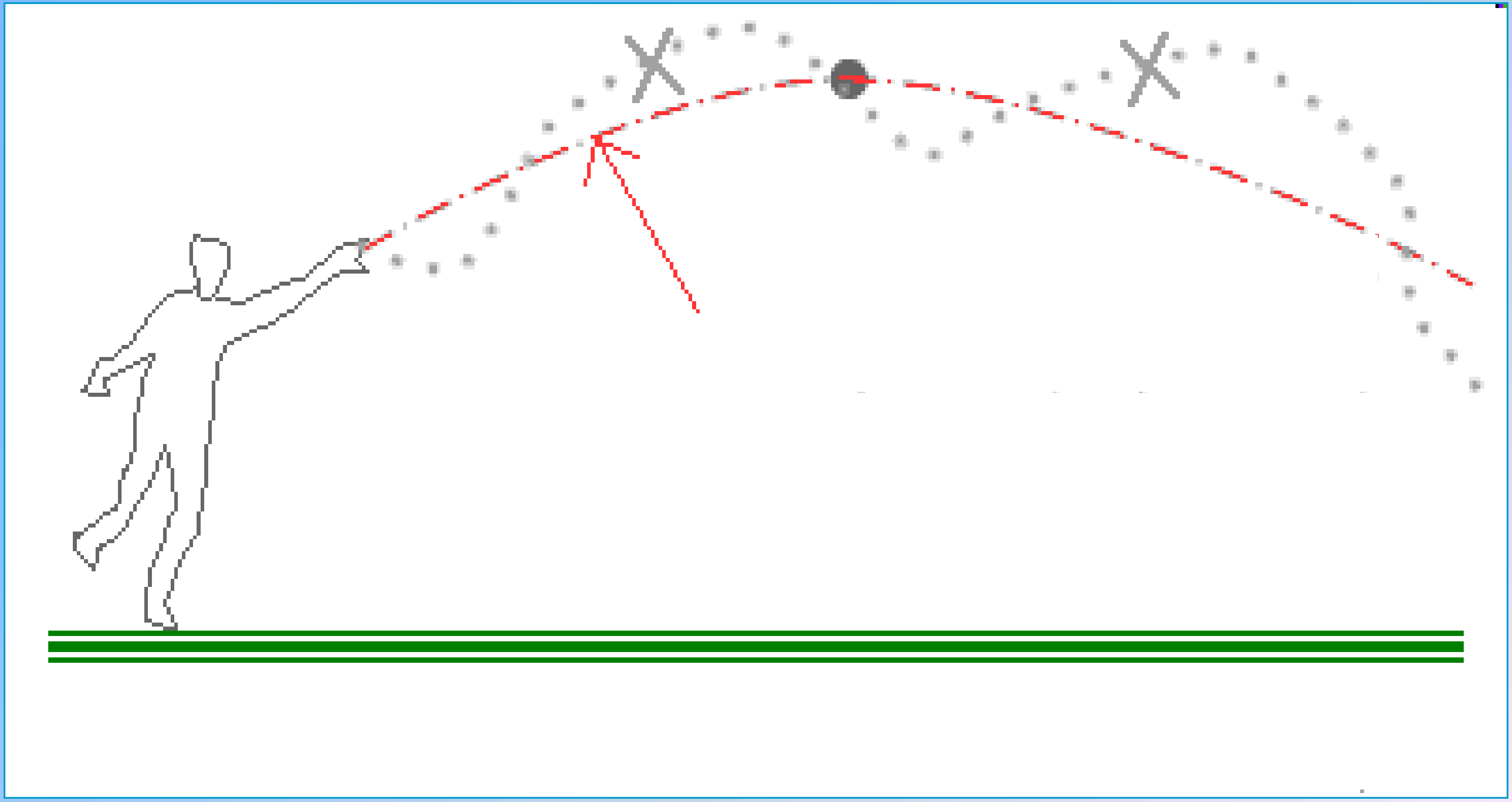
How does Humpty Dumpty know to fall straight down?



And how does light “know” ahead of time what path it will take?



The shortest distance in space is the path of maximum proper time.



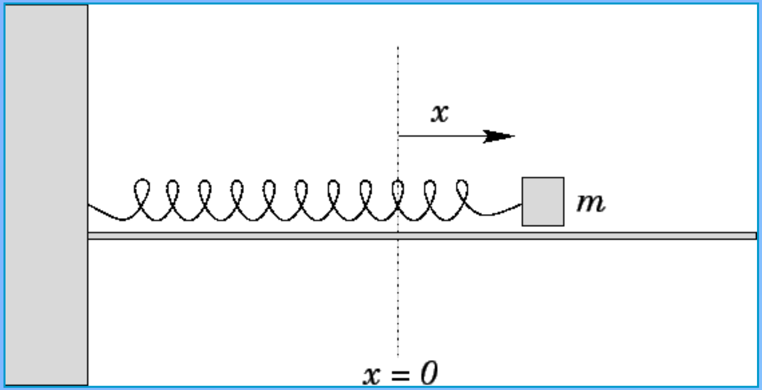
$$S = \int L(x, \dot{x}, t)\,dt = \int_{t_1}^{t_2} (T - V)\,dt$$

S is the Action.
L is called the Lagrangian.
L = T – V
T = kinetic energy
V = potential energy

- **The Principle of Stationary Action:**

  The path of a particle is the one that yields a stationary value of the action.

Example of a simple mass on a spring:



$$T = \frac{m}{2}\dot{x}^2$$

$$V = \frac{k}{2}x^2$$

m = mass
k = spring constant,
or ‘stiffness’

Action:

$$S = \int L(x, \dot{x}, t)dt = \int_{t_1}^{t_2} \left(\frac{m}{2}^{2} \dot{x}^2 - \frac{k}{2}x^2\right)dt$$

If S is constant (stationary) then the derivative of S must be zero.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

Substitute in the values for T and V…

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

$$L = \frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

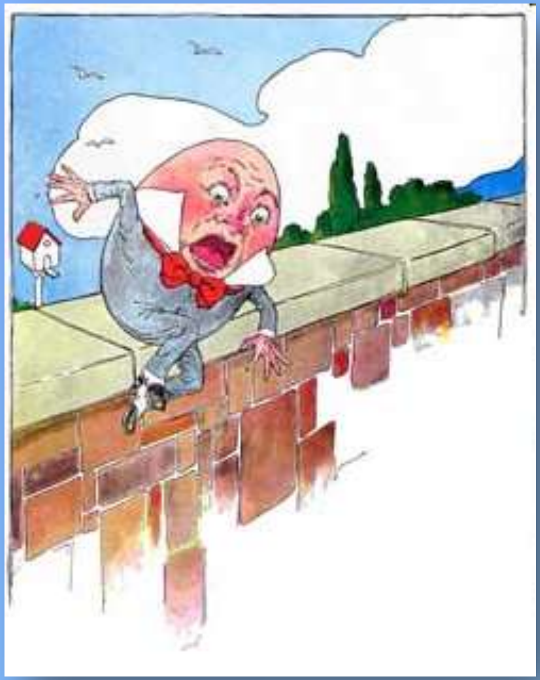
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = m\ddot{x} - kx = 0$$

ma ***F***

F = ma **which is just Newton's second law!**



Humpty Dumpty will always follow a geodesic in spacetime!

That is, he will always follow a path such that the difference between his kinetic and potential energies is stable to small perturbations.

we looked at these previously…

classically,

$$E = \frac{mv^2}{2}$$

$$p = mv$$

combining these:

$$p^2 = m^2 v^2$$

$$E = \frac{p^2}{2m}$$

we get the dispersion relation between energy and momentum in classical physics

Einstein's guess:

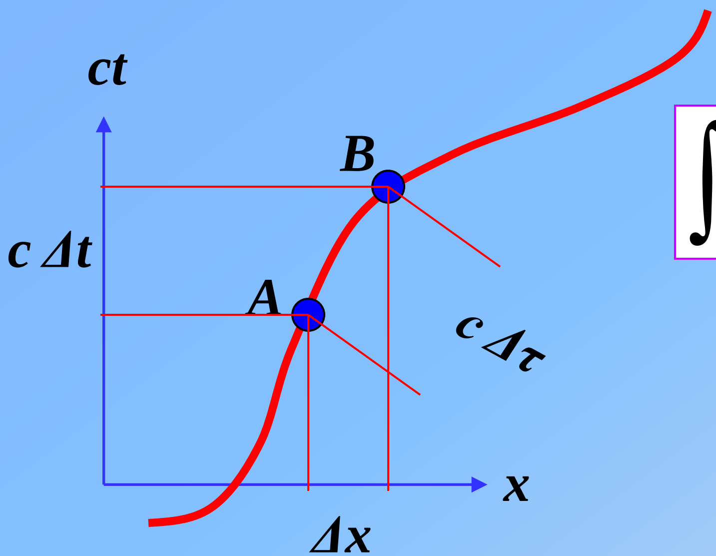
$$\boxed{E^2 - c^2 p^2}$$

is the invariant quantity

$$\longrightarrow \boxed{E^2 - c^2 p^2 = (m_0 c^2)^2}$$

(Energy – momentum) = rest mass

We defined the invariant interval in Special Relativity as the proper time, from Einstein's derivation:



$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$$
$$\Delta \tau^2 \equiv - \Delta s^2 / c^2$$

$$\int ds^2 = \int dx^2 + dy^2 + dz^2 - c^2 dt^2$$

$$\Delta s^2 = -c^2 \Delta \tau^2$$

$$E^2 - c^2 p^2 = (m_0 c^2)^2$$

$$\int ds^2 = \int dx^2 + dy^2 + dz^2 - c^2 dt^2 = c^2 \Delta \tau^2 = \left(\int_{t_1}^{t_2} L\,dt\right)^2$$

The invariant quantity is the Lagrangian –

General Covariance: An accelerating observer and a non-accelerating observer can interpret the different physical realities that each perceives as being due to a gravitational field.

Progression of symmetry from obvious to subtle:

1. Rotation of coordinate axes in space ~ invariance of the length of a line
2. Relative motion of inertial observers at slow speeds ~ Galilean invariance

*** Discovery: Speed of light is a property of Nature, same for all observers ***

3. Relative motion of inertial observers at high speeds (Lorentz Boosts)
   ~ Lorentz invariance

*** Discovery: Gravity is Lorentz invariant → Gravity is not a force but a curvature of spacetime ***

4. General covariance ~ Dynamical symmetry between accelerating observers

Putting it all together...

1. **Symmetry groups are defined by the operations that leave an object invariant.**

2. **Noether proved that symmetries of the Lagrangian ($\mathcal{L}$) lead to conserved quantities in Nature – i.e., conservation laws in physics.**

3. **Thus the symmetry operations which transform $\mathcal{L} \rightarrow \mathcal{L}'$ so that energy and momentum (and other quantities) are conserved must be also identifiable as belonging to certain symmetry groups.**

4. **Our task: Find the symmetry group which leaves the Action invariant under translations, rotations, Lorentz boosts, and general covariance! If we can find this group, then all objects belonging to it – i.e., such that they remain structurally invariant under the symmetry operations, then we can find hidden relationships and conservation laws which explain a wider range of phenomena.**

Noether's profound realization:

Symmetries of the Lagrangian

Lead to

Conserved Quantities

in Nature

conservation of momentum ⟷ *invariance of the laws of physics to translations, rotations and boosts in space*

conservation of energy ⟷ *invariance to the laws of physics to translations, rotations and boosts in time*

Term which describes the shape of spacetime

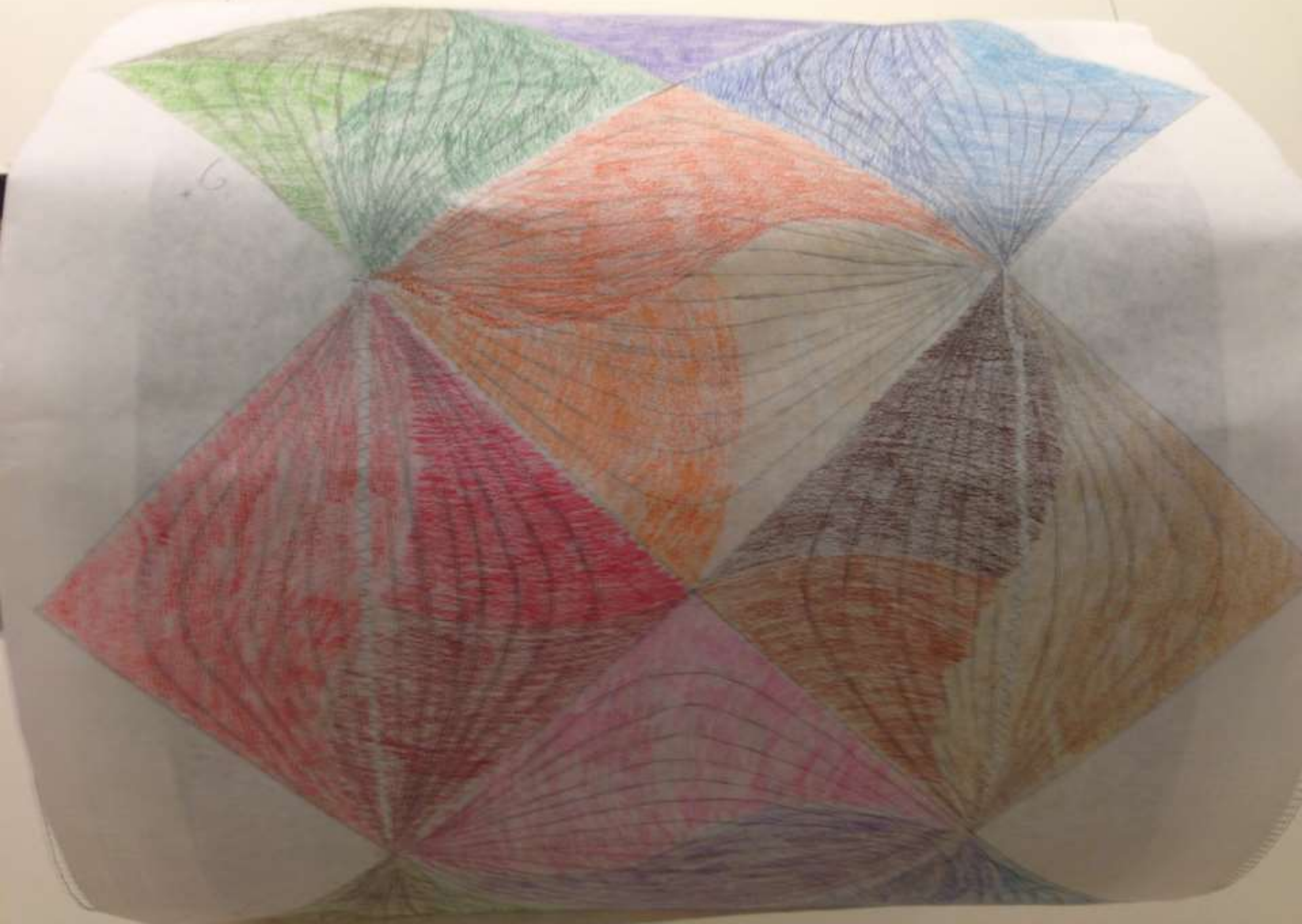
Term which describes all the energy and momentum contained within a small volume of spacetime

The double sub script indicates that these are tensor quantities.

This description of gravity as a curvature of spacetime due to the presence of matter and energy has led to our understanding of many interesting phenomena in the Universe, such as…

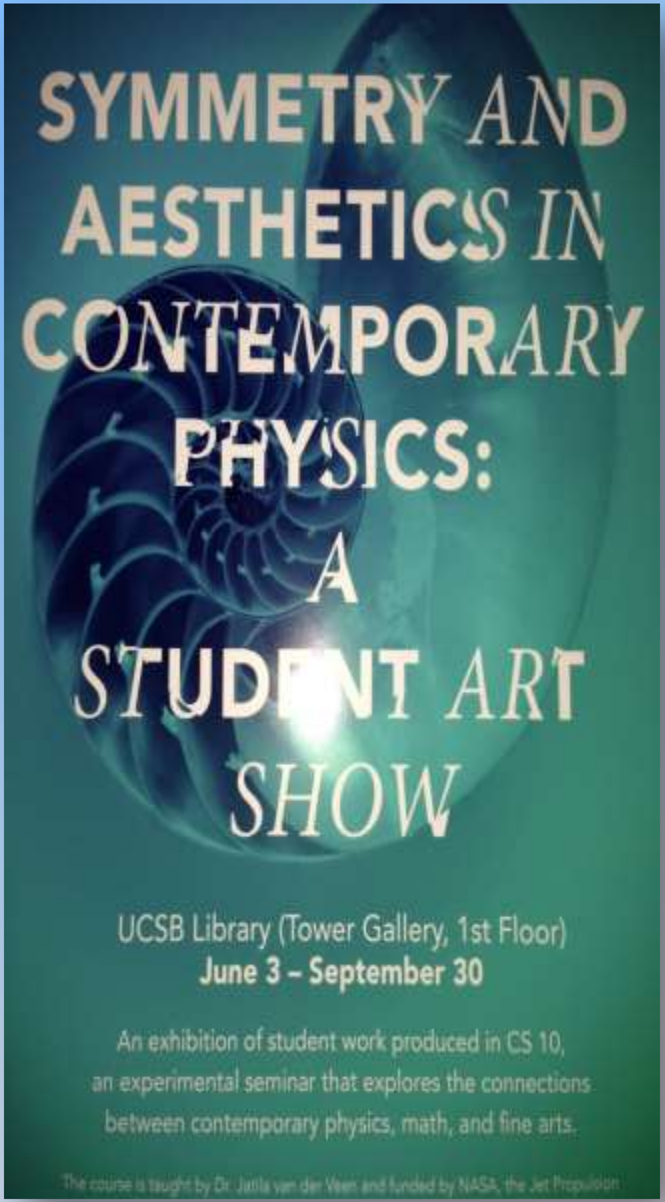
Time for a break!

ARTISTIC EXPLORATIONS OF CURVED SPACETIME




Axiom

And now for yours…


SYMMETRY AND AESTHETICS IN CONTEMPORARY PHYSICS: A STUDENT ART SHOW
UCSB Library (Tower Gallery, 1st Floor)
June 3 – September 30
An exhibition of student work produced in CS 10, an experimental seminar that explores the connections between contemporary physics, math, and fine arts.
The course is taught by Dr. Jatila van der Veen and funded by NASA, the Jet Propulsion

Discussion of final projects:

1. **Do you have an idea of what you want to do? Take some time now to brain storm.**

2. **Going over final presentation process:**
   - **By next week I need a general description of what you plan to do, and any equipment you will need.**
   - **By May 27th I need a short write up that the Library people will print out to hang with your work of art.**
   - **By next week I need you to sign the release forms.**
   - **June 3rd: We will meet in Rm. 1312 in the library – more instructions to follow!**

EXTRA SLIDES

We did not discuss these in class, but I'm including them for optional reading about General Relativity. The slides are based on a discussion of an article by physicist John Baez, which I had assigned in 2012. I removed this article from the reader because it seemed to be too difficult for most people who sign up for this class, but this year I think we have a number of students who are quite advanced, so I put it up on GauchoSpace for anyone who wishes to read it.

The Meaning of Einstein's Equation
John C. Baez and Emory F. Bunny
January 4, 2006

Historic Perspective:

After the invention of special relativity, Einstein tried for a number of years to invent a Lorentz-invariant theory of gravity, without success. His eventual breakthrough was to replace Minkowski spacetime with a curved spacetime, where the curvature was created by (and reacted back on) energy and momentum. (quoting Sean Carroll…)

After arriving at the amazing realization that mass and energy curve spacetime, and that gravity is not a force but the reaction of mass and energy to curved spacetime, Einstein had two major problems to solve: 1) He had to understand how to define curved spaces mathematically; and 2) He had to figure out how to prove that the Laws of Physics remain invariant to rotations, translations, and Lorentz boosts in curved spacetime. To do this, he had to find the symmetry rule that allows observers to understand physics in each other's reference frames in curved spacetime, such that they will arrive at the same conclusions about the Laws of Physics, *even though they may observe different "realities" due to local gravity!*

<u>The Big Questions:</u>

What remains invariant under transformation of coordinate systems in curved spacetime, when there is no such thing as an inertial observer, no preferred reference frame, and it is not possible to even define relative motion unless two observers are close enough that they do not experience the curvature of spacetime?

And how can we define coordinate systems in curved spacetime?

And how is GR a theory of curved spacetime which satisfies the known Laws of Physics?

Next, we explore these ideas as described by Professor John Baez (the father of the famous folk singer) in his article The Meaning of Einstein's Equation.

Discussion of Baez

Baez' goal is to explain Einstein's Equation in simple terms, and point out how it illuminates our understanding of gravity and 4D spacetime.

THE EINSTEIN FIELD EQUATION

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

Term which describes the shape of spacetime

The double sub script indicates that these are tensor quantities.

Term which describes all the energy and momentum contained within a small volume of spacetime

On p. 93 Baez states: In Special Relativity it makes no sense to talk about absolute velocities; only relative velocities. In General Relativity it makes no sense to even talk about relative velocities, except if you measure them for particles at the same point of spacetime.

… in GR we take very seriously the notion that a vector is a little arrow sitting at a particular point in space-time…

What does he mean by that? What is a vector, and what sorts of physical quantities does it represent?

In the flat, 4D Lorentz-invariant reference frame (Minkowski spacetime) in which we envision ourselves 'at rest' in a moving (but still inertial) frame, at some velocity which is a large fraction of the speed of light, we have 'four vectors' (-t, x, y, z).

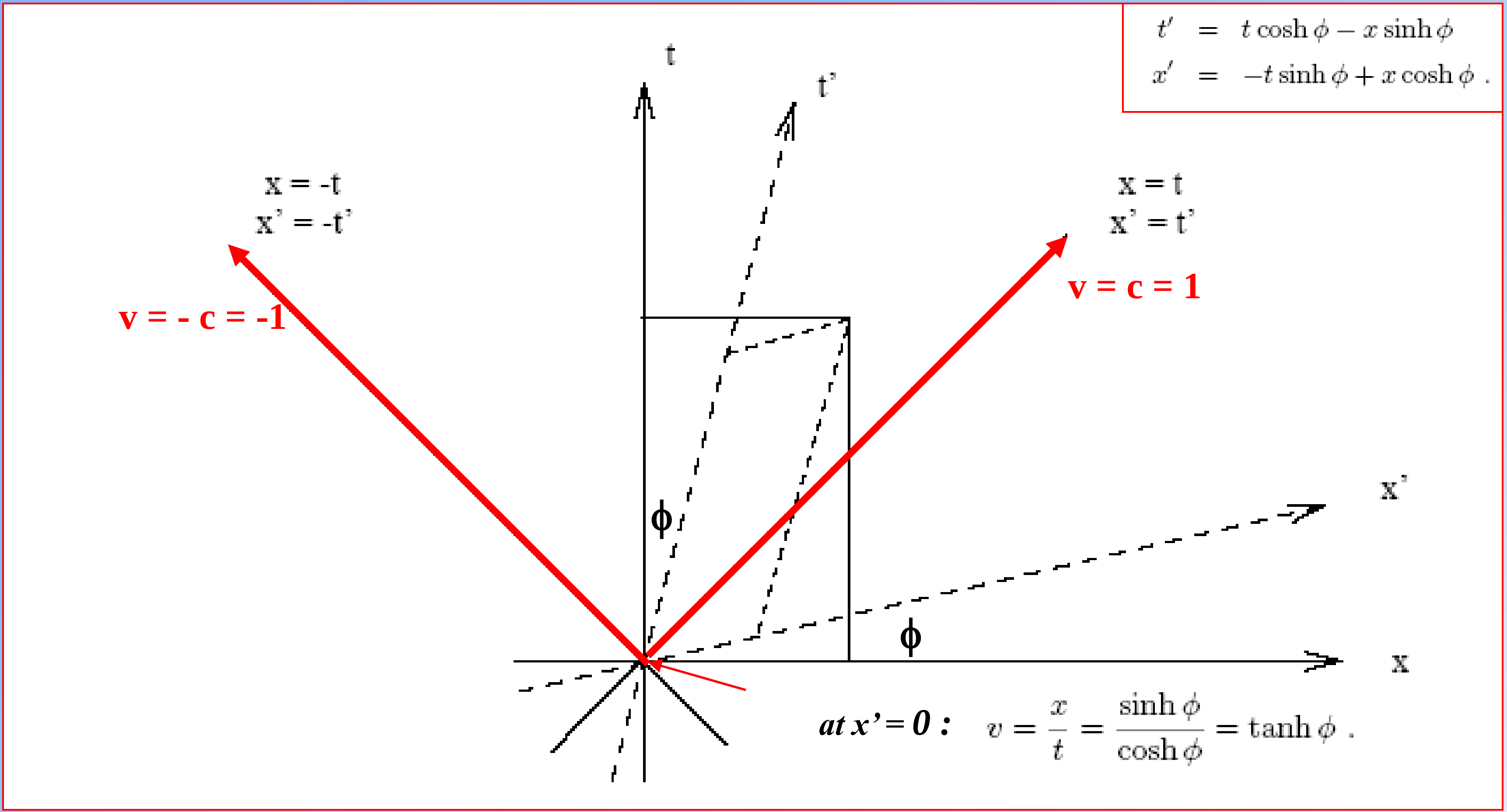
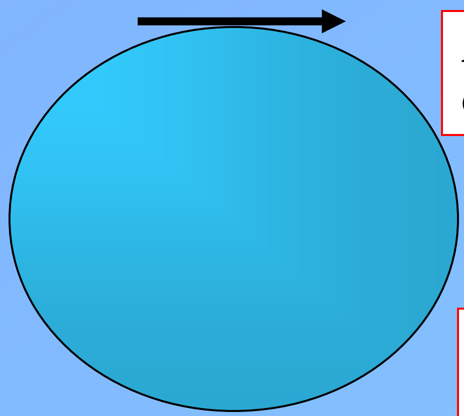


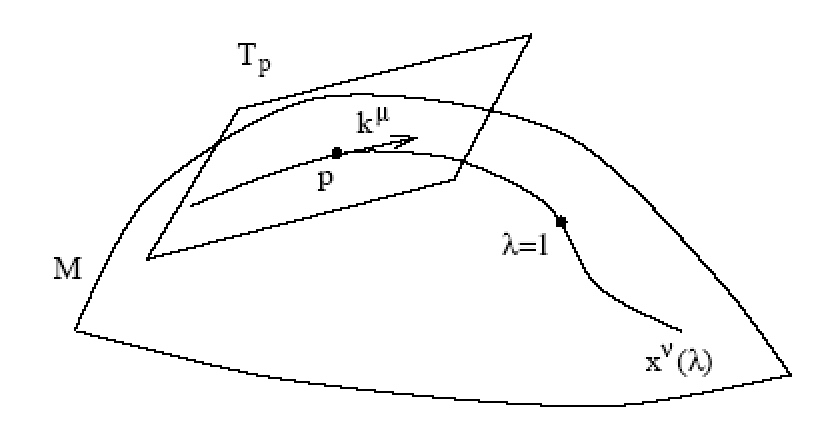
Diagram of a Lorentz boost taken from Sean Carroll's on-line notes on General Relativity, available at http://arxiv.org/PS_cache/gr-qc/pdf/9712/9712019v1.pdf.

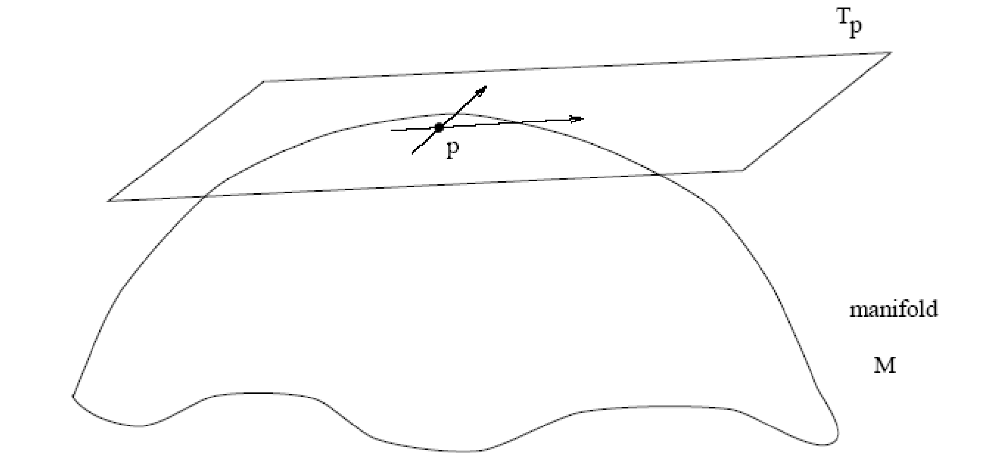
But the notion of vectors that have any extent beyond a single point makes no sense in curved spacetime, where all ‘rulers’ conform to the local curvature, like the bugs on Feynman’s hotplates! “Flat” vectors are thus envisioned as being embedded in a higher dimensional space.

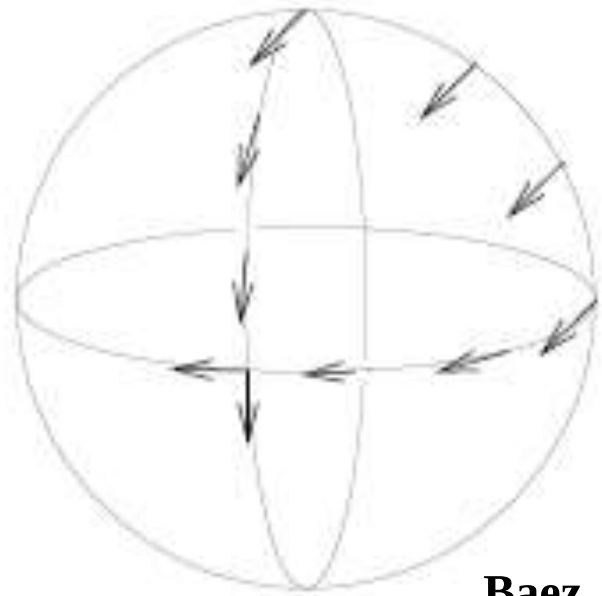


A tangent vector at one point on a 2D circle embedded in a 3D space.

Vectors in curved spacetime can only exist at one point; thus, there must be an infinite number of them which are collected at every point. So, since any two lines define a plane, in curved spacetime a bundle of vectors defines a plane which is tangent to the curved space at a point – a tangent plane.







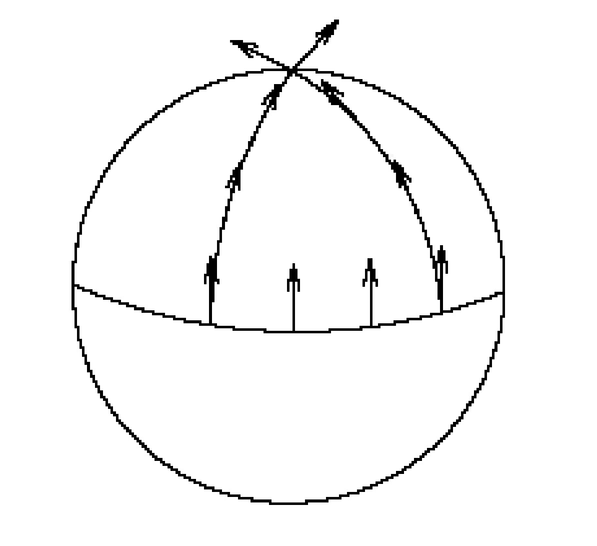
Baez, p. 3

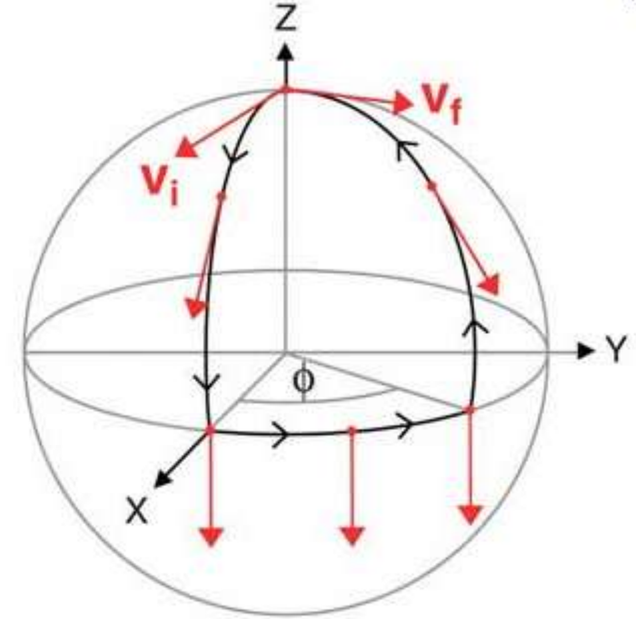
Baez uses the example of parallel transport to illustrate how, in curved spacetime, if you move a vector from one place to another, even if you keep it pointing in the same direction, without rotating it, the path along which you move it makes a difference. This is a conceptual break with our every day notions of flat space.

A couple of other illustrations
I found on line:

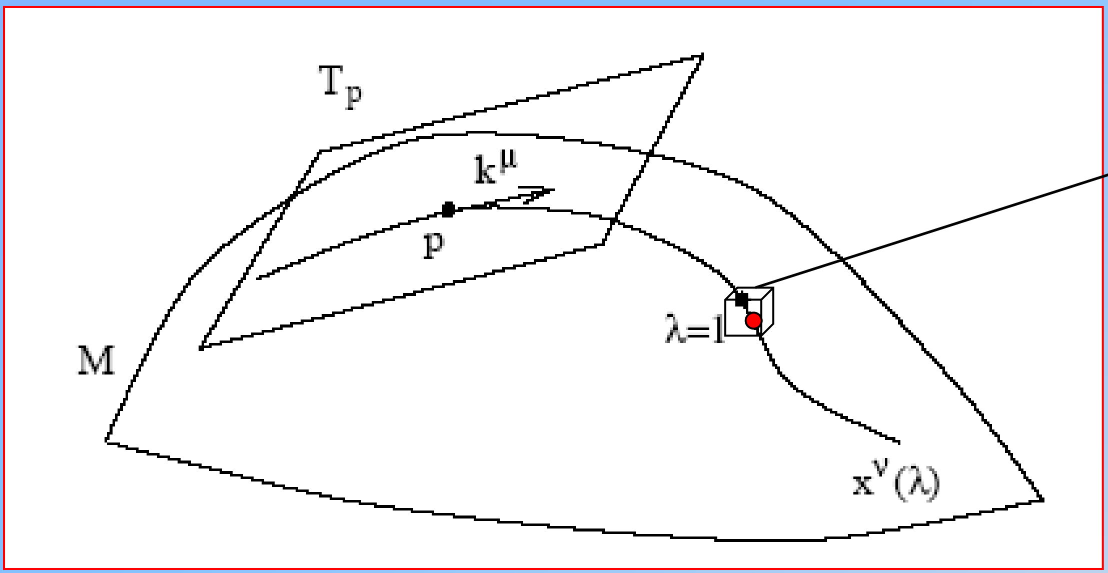
Left: two people who start out at the equator walking north, parallel to each other, will collide at the North Pole.

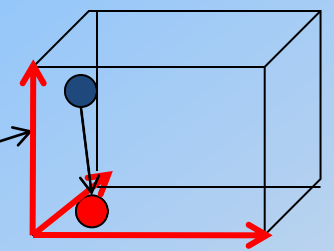
Right: Vectors that point east and west at the pole, if parallel transported to the equator, will point south.





If we are willing to put up with limited accuracy, we can still talk about the relative velocity of two particles in the limit where they are very close, since curvature effects will then be very small. In this approximate sense, we can talk about a 'local' inertial coordinate system. However, we must remember that this notion makes perfect sense only in the limit where the region of spacetime covered by the coordinate system goes to zero in size. **Baez, p. 3**



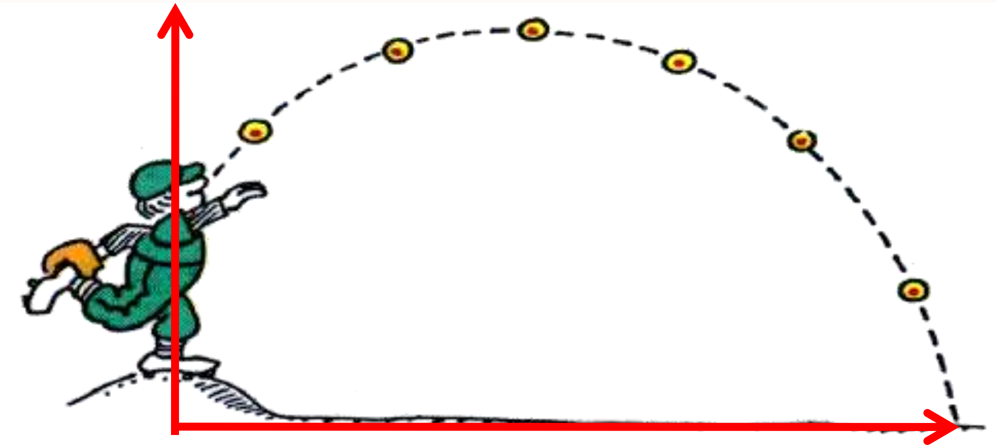


A local approximation of an inertial coordinate system in curved spacetime.
A geodesic is the path a test particle takes in freefall.

Einstein's equation can be expressed as a statement about the relative acceleration of very close test particles in free fall. Let us clarify these terms a bit. A 'test particle' is an idealized point particle with energy and momentum so small that its effects on spacetime curvature are negligible. A particle is said to be in 'free fall' when its motion is affected by no forces except gravity. In general relativity, a test particle in free fall will trace out a 'geodesic'. This means that its velocity vector is parallel transported along the curve it traces out in spacetime. A geodesic is the closest thing there is to a straight line in curved spacetime.

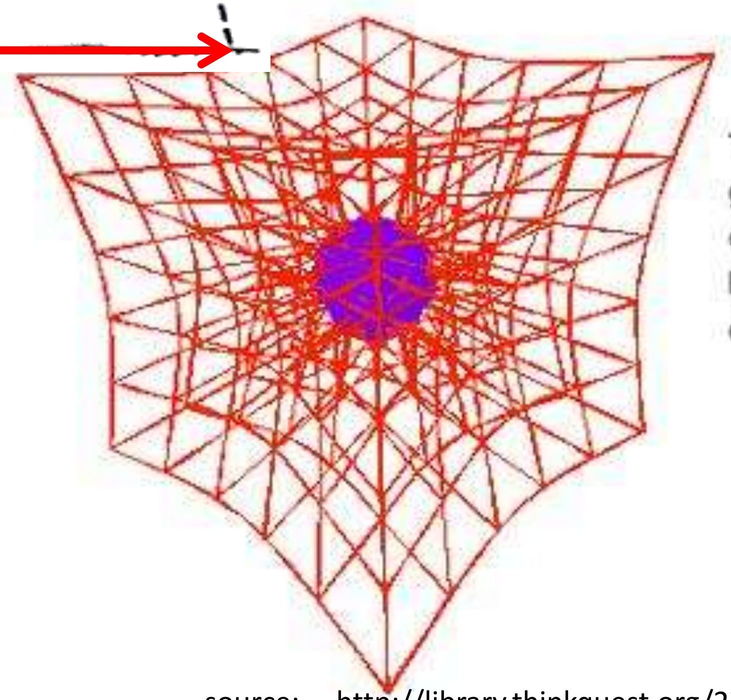
a cat in free fall along a geodesic in spacetime

Baez then reminds us that when we observe a projectile in a parabolic path in spacetime from our limited perspective, it looks quite curved, even though the curvature of space due to the Earth's gravitational field is imperceptible to us. This is because the projectile's path is quite curved in the time dimension, because one second in time = 300,000 km.



A test particle following a path in the x-y plane near the surface of the earth, from the human perspective …

… can be visualized as following a curve in 3D (one spatial axis must be suppressed in order to draw it) on a grid like this:

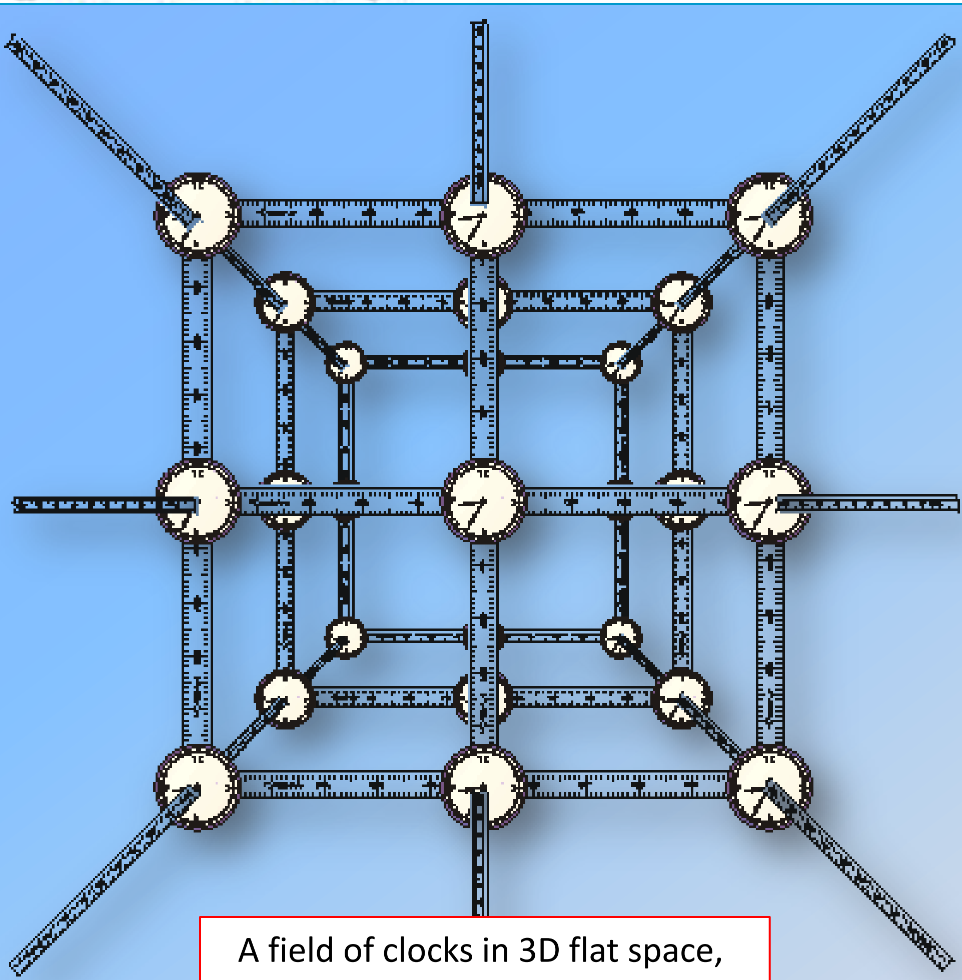


This three-dimensional grid gives a better idea of what curved space-time might look like than the two-dimensional analogies do.

source: http://library.thinkquest.org/27585/what/what7.html

I found this diagram on line. I can't draw the ball's trajectory on this grid, but you can imagine the ball going from (t_1, x_1, y_1) to (t_2, x_2, y_2) and then imagine projecting that curved path in (t, x, y) space onto the x-y plane, and it will come out looking like a parabola.

Now, in special relativity we can think of an inertial coordinate system, or 'inertial frame', as being defined by a field of clocks, all at rest relative to each other. In general relativity this makes no sense, since we can only unambiguously define the relative velocity of two clocks if they are at the same location. Thus the concept of inertial frame, so important in special relativity, is *banned* from general relativity! **Baez, p. 3**

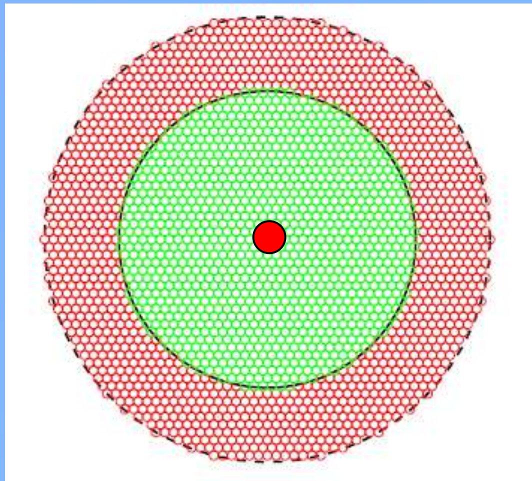


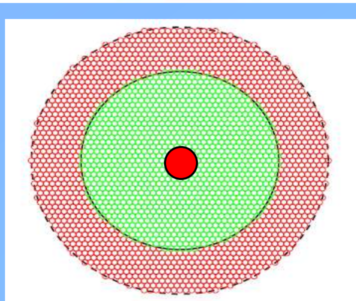
A field of clocks in 3D flat space, all at rest relative to each other

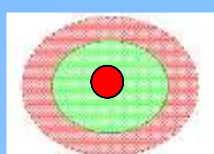
Why does this make no sense?

Because, as we read in Feynman and in Zee, clocks don't run at the same rates in regions where the gravitational field strength is different. Remember the Pound and Rebka experiment!

Curved spacetime is equivalent to a gravitational field, thus where ever you move your clock ('time ruler'), the curvature of spacetime is different, so the time ruler is stretched. Think of the bugs on a hotplate, with time-like meter sticks. Clocks tick at different rates due to the curvature of spacetime – i.e., the local gravitational field.







Over time, a round ball of test particles deforms in the presence of a gravitational field.

OK, so if we extend this idea of the stress tensor to spacetime, we include the time dimension. We define a little spherical volume of test particles which deforms over time, due to the effects of energy and momentum contained within its local volume of spacetime – that is, the curvature of spacetime, or the local gravitational field.

Baez states:

Let $V(t)$ be the volume of the ball after a proper time t has elapsed, as measured by the particle at the center of the ball. Then Einstein's equation says:

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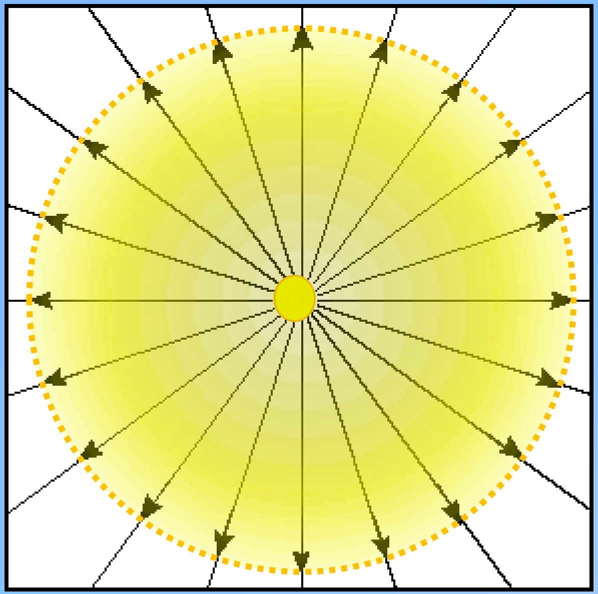

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Don’t forget that we’re extending the symmetry of Lorentz invariance, so we’re thinking in the Einsteinian paradigm that $E = mc^2$ in the rest frame of a particle, so density = energy density = ρ.

And - keep in mind that GR is a description of the behavior of spacetime on large, ideal scales, that gravity is not a force but a curvature of spacetime, and that on such large scales matter and radiation behave like *perfect fluids*. Perfect fluids are homogeneous and isotropic, characterized only by their pressure and density.

A perfect fluid is one which is isotropic in its rest frame – that is, it looks the same in any direction to a bug sitting on a particle at rest in the fluid. To really pick apart Baez's description here gets a bit messy, in that we have to consider energy and momentum 'flows' as four-vectors in spacetime. Rather, just consider the following:

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For the ideal fluid, the off-diagonal components are zero, so we get:

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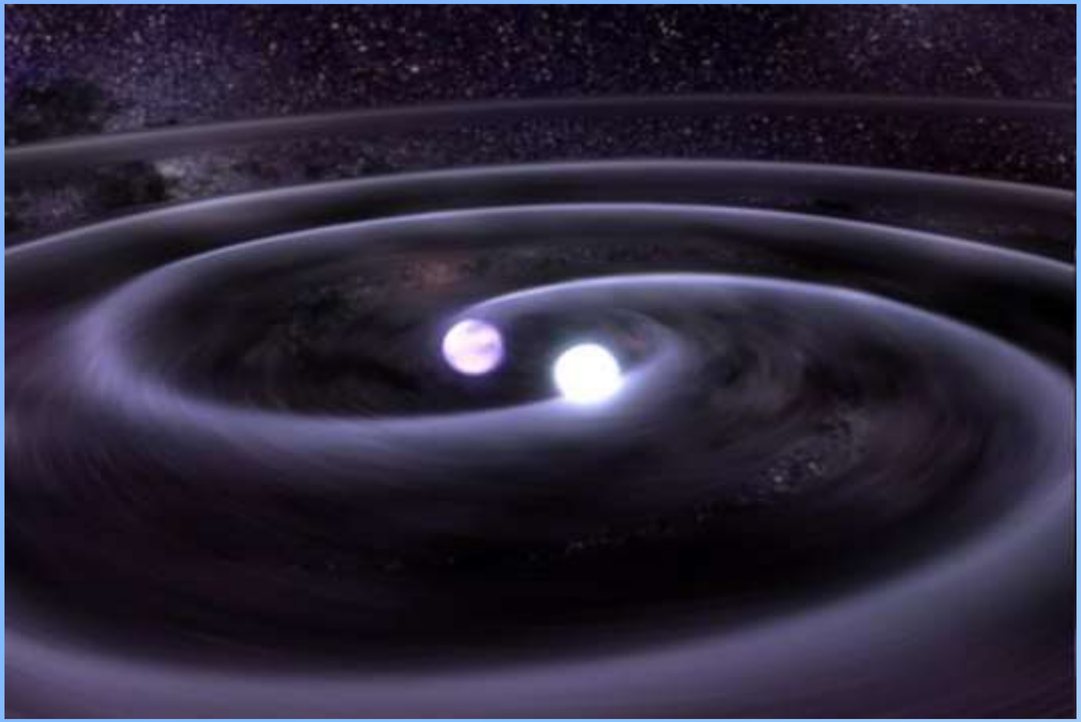
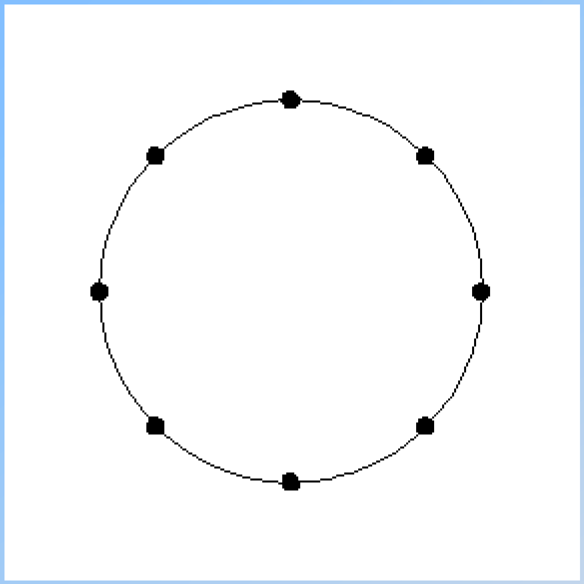
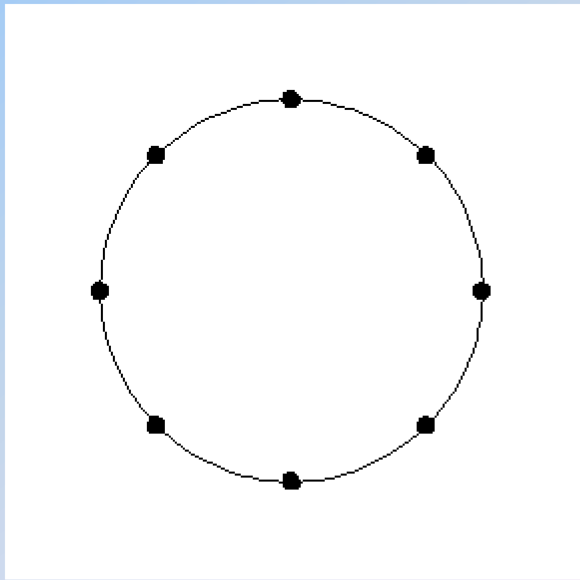
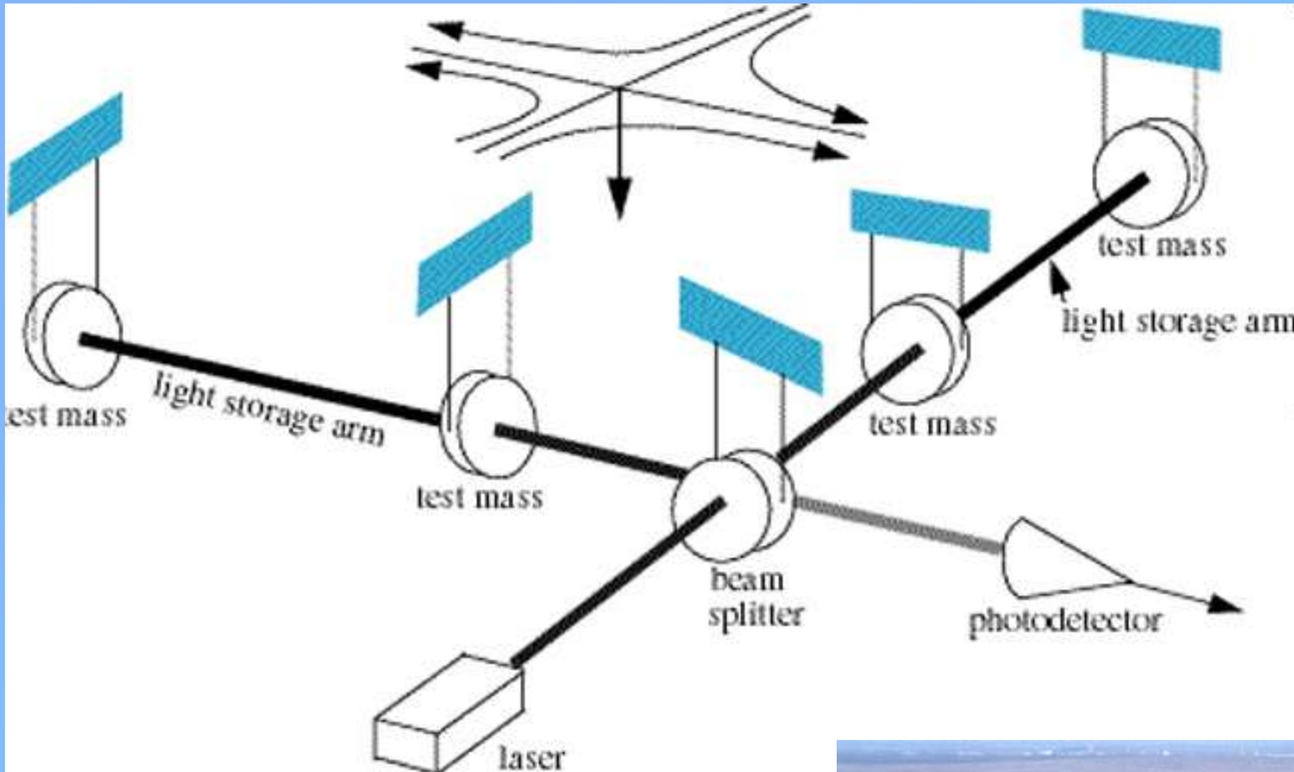


Gravity waves produced by two rotating massive objects





disturbance of test particles due to passage of gravity waves



and the design of instruments that can measure small deformations of spacetime:

the LIGO gravity wave detector

General Covariance: An accelerating observer and a non-accelerating observer can interpret the different physical realities that each perceives as being due to a gravitational field.

Progression of symmetry from obvious to subtle:

1. Rotation of coordinate axes in space ~ invariance of the length of a line
2. Relative motion of inertial observers at slow speeds ~ Galilean invariance

*** Discovery: Speed of light is a property of Nature, same for all observers ***

3. Relative motion of inertial observers at high speeds (Lorentz Boosts) ~ Lorentz invariance

*** Discovery: Gravity is Lorentz invariant → Gravity is not a force but a curvature of spacetime ***

4. General covariance ~ Dynamical symmetry between accelerating observers

So…

If spacetime is interdependent with the mass and energy contained within it – that is, there is no sense in which anyone is a perfectly inertial observer, and we can't tell that all our rulers and light beams are actually bent because we are part of the curvature of spacetime, HOW do we know that the laws of physics are still invariant to rotations and translations? How do we KNOW that the laws of physics are the same here and now as they were 10 billion years ago, in another part of the universe?

In the simple case of rotations in flat space, we proved that the length of a line is invariant to rotations of the coordinate axes. In Special Relativity we proved that inertial observers in relative motion will each measure the other's lengths as being contracted, relative to their own rest frame, by the same amount, and each other's times as being dilated, relative to their own rest frame, by the same amount.

But in General Relativity, there are no inertial observers; we can't define relative velocities except in teensy-weensy regions of spacetime where we can't notice the curvature. So what symmetry rule is there which allows us to prove that the Laws of Physics are still the same for all observers? Well, Einstein guessed that it had to do with the constancy of the speed of light, and the equivalence of mass and energy… which leads us to Section III of Zee: Into the Limelight.

Historic Perspective:

After the invention of special relativity, Einstein tried for a number of years to invent a Lorentz-invariant theory of gravity, without success. His eventual breakthrough was to replace Minkowski spacetime with a curved spacetime, where the curvature was created by (and reacted back on) energy and momentum. (quoting Sean Carroll…)

After arriving at the amazing realization that mass and energy curve spacetime, and that gravity is not a force but the reaction of mass and energy to curved spacetime, Einstein had two major problems to solve: 1) He had to understand how to define curved spaces mathematically; and 2) He had to figure out how to prove that the Laws of Physics remain invariant to rotations, translations, and Lorentz boosts in curved spacetime. To do this, he had to find the symmetry rule that allows observers to understand physics in each other's reference frames in curved spacetime, such that they will arrive at the same conclusions about the Laws of Physics.

The Big Questions:
What remains invariant under transformation of coordinate systems in curved spacetime, when there is no such thing as an inertial observer, no preferred reference frame, and it is not possible to even define relative motion unless two observers are close enough that they do not experience the curvature of spacetime? And how can we define coordinate systems in curved spacetime? And how is GR a theory of curved spacetime which satisfies the known Laws of Physics?

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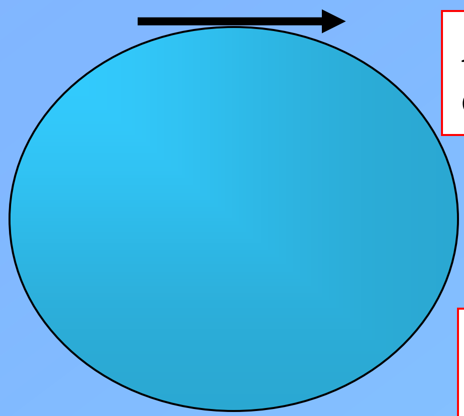

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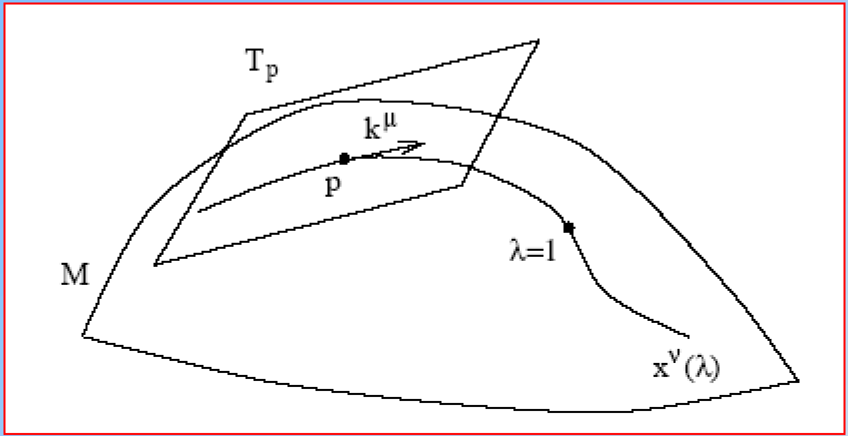

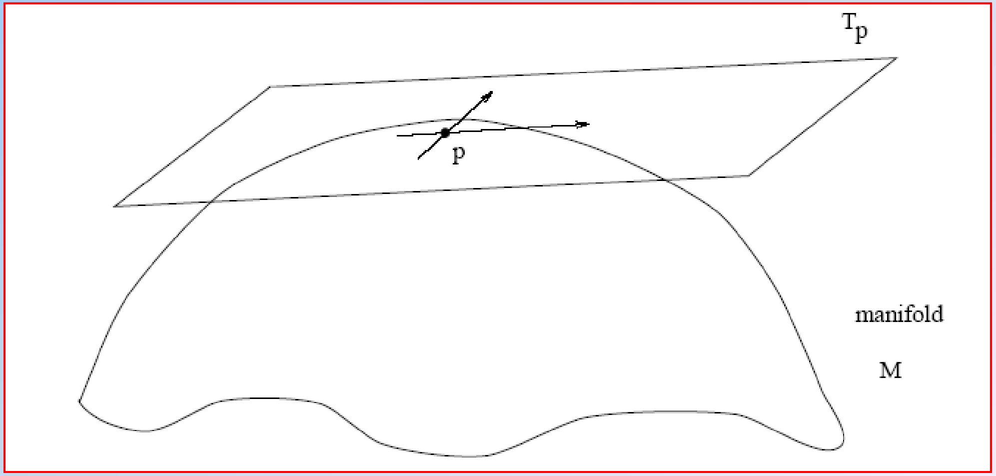

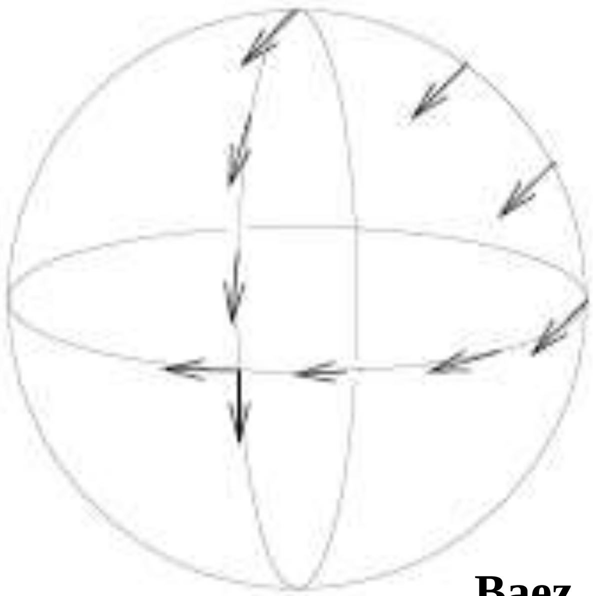

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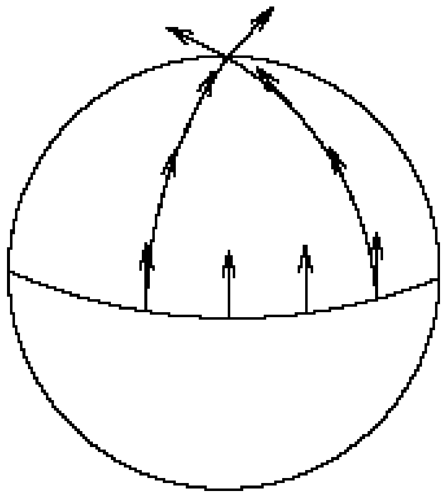

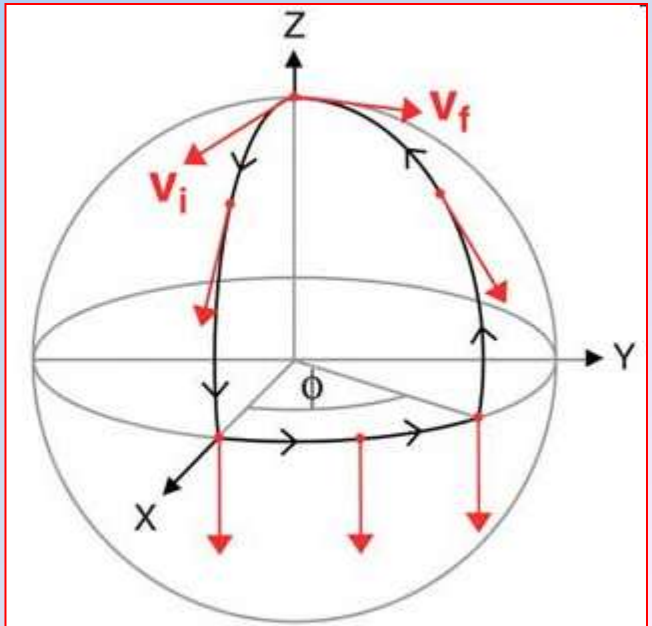

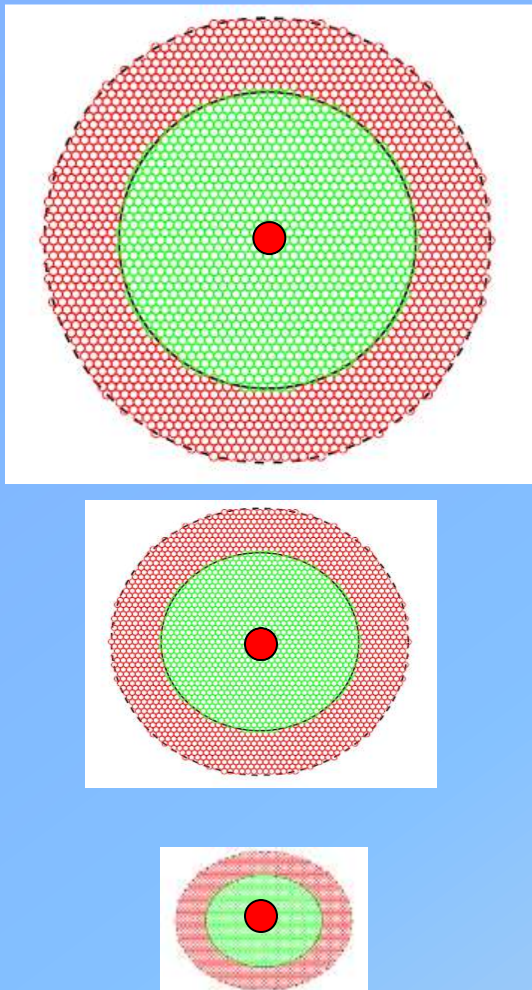

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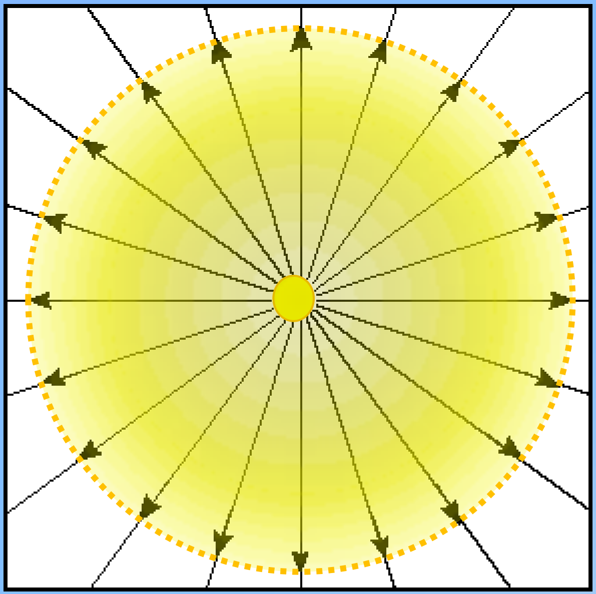

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Now, what is this "V-double-dot?" Well, think of x-double-dot as an acceleration. In our usual linear thinking, this is a free fall straight down. Now, combine this in your mind's eye with y and z, and you get a collapse!

Given a small ball of freely falling test particles initially at rest with respect to each other, the rate at which it begins to shrink is proportional to its volume times: the energy density at the center of the ball, plus the pressure in the x direction at that point, plus the pressure in the y direction, plus the pressure in the z direction.

This is just a short hand notation for all of the above.

THE EINSTEIN FIELD EQUATION

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

Term which describes the shape of spacetime. This, too is a tensor, and it describes the shape of spacetime in a local region.

Term which describes all the energy and momentum contained within a small volume of spacetime. This is the energy momentum tensor.

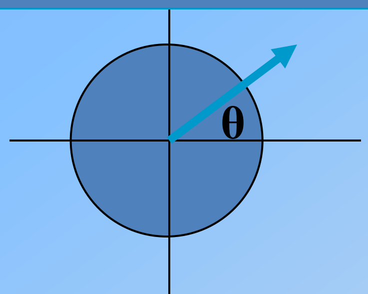
This description of gravity as a curvature of spacetime due to the presence of matter and energy has led to our understanding of many interesting phenomena in the Universe, such as…

OPTIONAL DISCUSSION : **Review of Groups…**

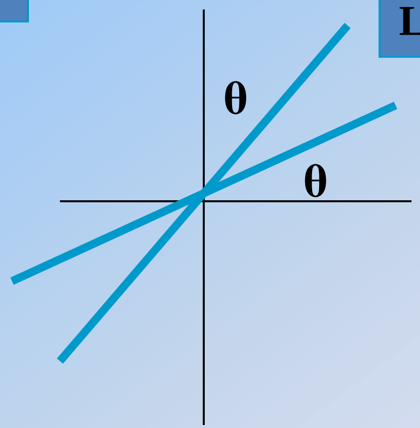
Recall last quarter, we found all the symmetry operations (rotations and reflections) and combinations of them that left an equilateral triangle unchanged… There were 6 independent operations (3 rotations and 3 reflections) and 36 combinations, which form the group "D(3)." (See lecture 4 from last quarter.)

The equilateral triangle has discrete symmetry, that is, you can't rotate it by any arbitrary angle – only rotations of 120º are symmetry operations. For conserved quantities in Nature, we look for CONTINUOUS symmetries – like those we looked at for circles and spheres.

We showed that rotations in the x-y plane are represented by little 2x2 matrices.



$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$



And Lorentz boosts in spacetime are represented by the 4x4 Lorentz transformation matrix

$$\begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-\gamma\beta}{c} & 0 & 0 & \gamma \end{pmatrix}$$

OPTIONAL DISCUSSION

p. 127 – 131: take a look at the group SO(3) – the group of rotations in 3D space.

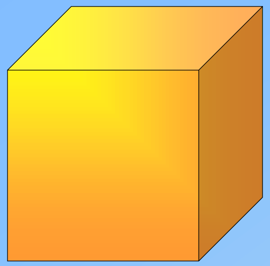
S = "special" → **Determinant of the matrix = 1**

O = "orthogonal" → **Matrix multiplied by its transpose = Identity**

In general, the order "n" tells the degrees of freedom. For Special Orthogonal groups, the degrees of freedom are given by

$$\frac{n(n+1)}{2}$$

For SO(3) *df* = $\frac{3(4)}{2} = 6$



6 directions, each one normal to one face of the cube

But how did he come up with the dimensions? SO(3) is represented by a 3 x 3 tensor. You can decompose a tensor of rank n into a symmetric, traceless component; the trace; and an antisymmetric component.

n x n = $\frac{n(n+1)}{2} - 1 + \frac{n(n-1)}{2} + 1$

Which is how he gets his decomposition of SO(3), a 3 x 3 = 9 dimensional matrix, into 5, 3, and 1 dimensional

OPTIONAL DISCUSSION

$$\begin{pmatrix} 0 & 2 & 3 & 1 \\ 2 & 0 & 1 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \end{pmatrix}$$

$T_{ij} = T_{ji}$

$i \neq j$

$T_{ij} = 0$

$i = j$

A symmetric traceless tensor

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$T_{ij} = T_{ji} = 0$

$i \neq j$

$T_{ij} \neq 0$

$i = j$

A tensor with only a trace

$$\begin{pmatrix} 0 & 2 & 3 & 1 \\ -2 & 0 & 1 & 3 \\ -3 & -1 & 0 & 2 \\ -1 & -3 & -2 & 0 \end{pmatrix}$$

$T_{ij} = -T_{ji}$

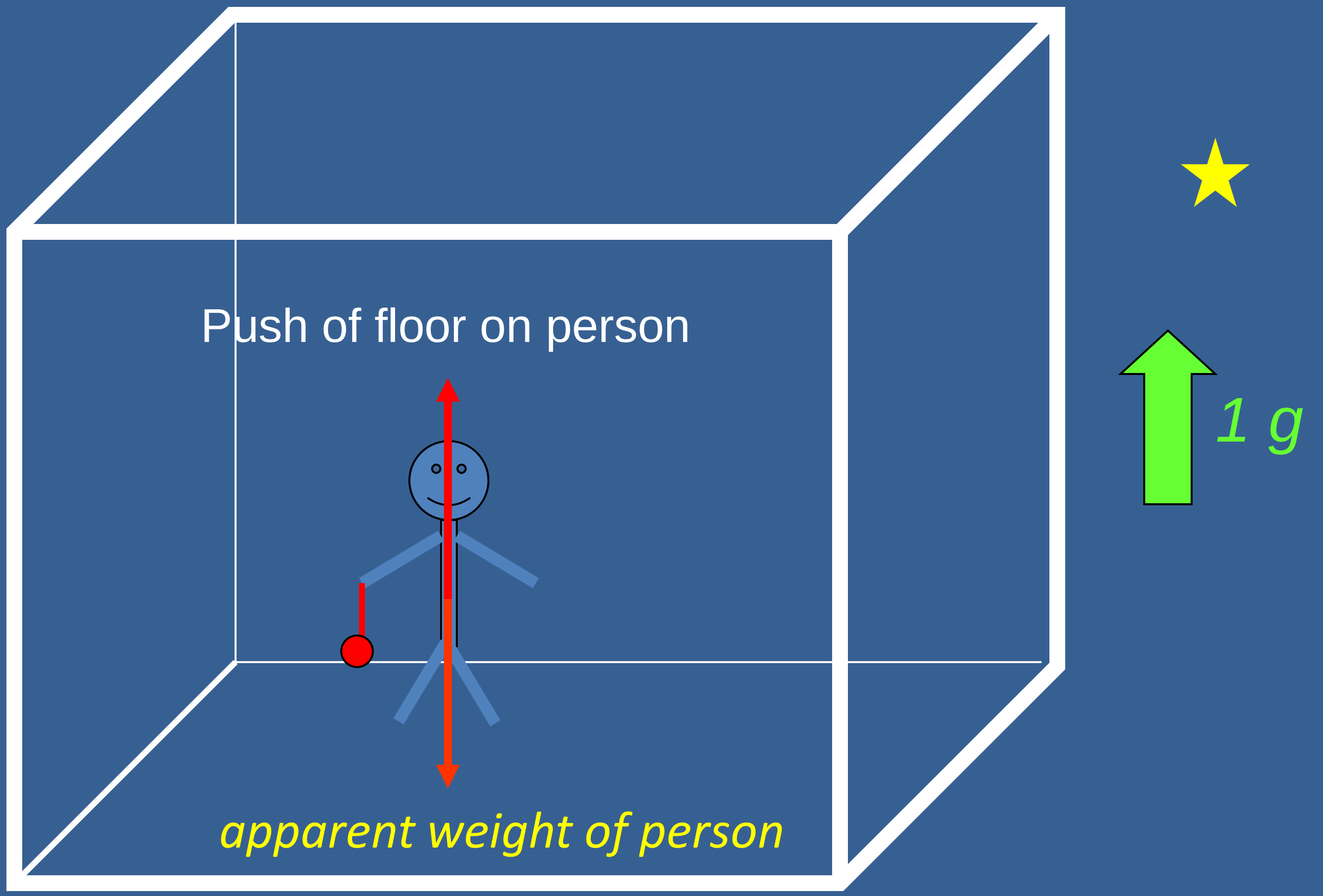
$i \neq j$

$T_{ij} = 0$

$i = j$

An antisymmetric tensor

In a small enough region of spacetime, such that the gravitational field strength does not vary, a person accelerating at 1g “in outer space” cannot distinguish this from standing still on the surface of the Earth, at sea level.



"traditional" Mercator projection of the world which reflects political dominance of the US and Europe.

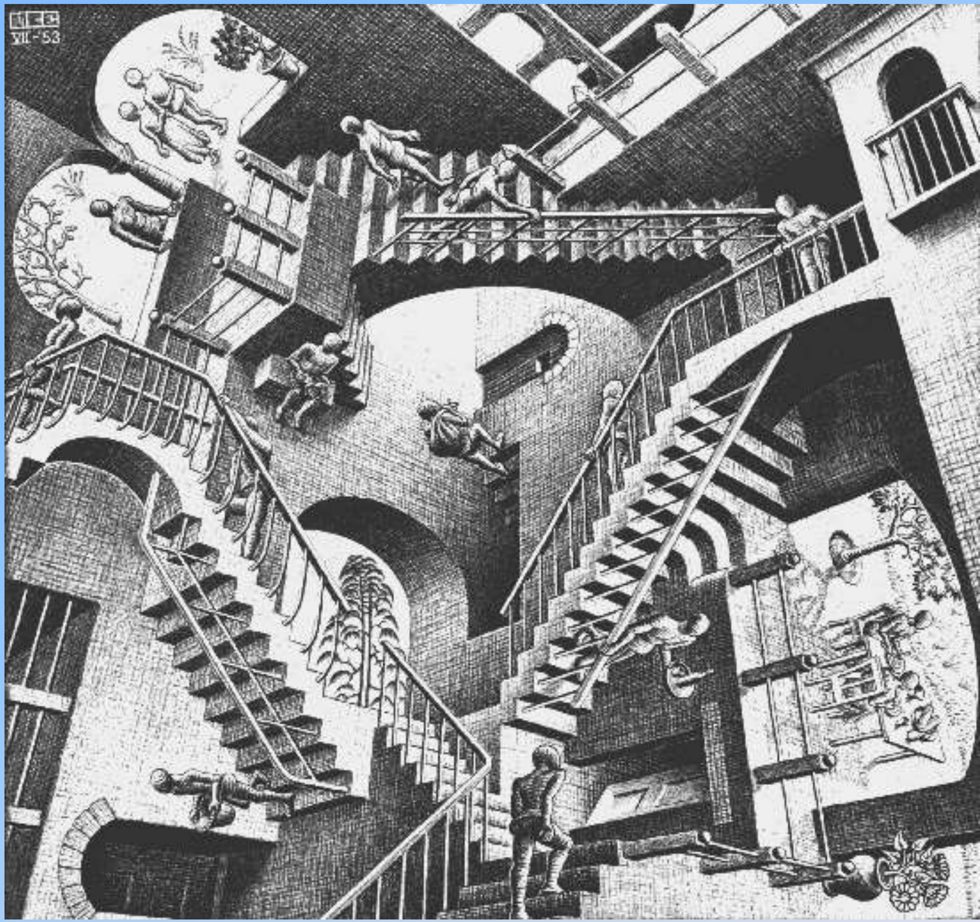
vs.

Peters projection, which is geopolitically more correct in showing the relative sizes of land

our perceptions under coordinate transformations

on p. 84 Zee uses the analogy of the Mercator vs. the Peters projections

Escher is well known for his explorations of gravity and higher dimensions through art.



http://escherdroste.math.leidenuniv.nl/index.php?menu=escher