Normalization of Power Spectra

Omer Blaes and Ken Henisey

Department of Physics, University of California, Santa Barbara, Santa Barbara CA 93106

ABSTRACT

A description of our thoughts on the normalization of the Zeus power spectra.

1. Definition of Power Spectra

First off, we chose to normalize our power spectra to the mean squared amplitude of the original data. Notice that we will assume a real data set so $H(f) = H(-f)$. Thus, we will simplify notation by taking $|H(f)|^2 + |H(-f)|^2 \to |H(f)|^2$. So,

$$\frac{1}{T}\int_0^T |h(t)|^2\, dt = \int_0^\infty P(f)\, df. \tag{1}$$

Because we are dealing with a discretized set of data with N elements, we must pad our data with zeroes to fit into an array with length $N_{\rm ft} = 2^m$, for m satisfying $2^{m-1} < N \leq 2^m$. It is convenient to define the following terms imitating the perscriptions of Numerical Recipies,

$$\Delta \equiv \frac{T}{N} \tag{2}$$

$$t_n \equiv n\Delta \tag{3}$$

$$f_k \equiv \frac{k}{N_{\rm ft}\Delta}. \tag{4}$$

Similarly, after performing a discrete Fourier transform we find,

$$h(t_n) \equiv h_n \tag{5}$$

$$H(f_k) \equiv H_k\Delta, \tag{6}$$

Where H_k represents the discrete output from the FFT algorithm. We can write the mean squared amplitude from (1) using Parceval's theorem,

$$\frac{1}{T}\int_0^T |h(t)|^2\, dt = \frac{1}{T}\int_0^\infty |H(f)|^2\, df \tag{7}$$

So we can identify,

$$P(f_k) = \frac{|H(f_k)|^2}{T} = \frac{\Delta}{N}|H_k|^2 \tag{8}$$

2. Power Binning

Now suppose we group our power spectrum data into M bins such that $0 = N_0 < N_1 < N_2 < \cdots < N_{M-1} < N_M = \frac{N_{\rm ft}}{2} + 1$. Then,

$$\Pi_i \equiv \frac{1}{N_i - N_{i-1}} \sum_{k=N_{i-1}}^{N_i - 1} P(f_k). \tag{9}$$

Furthermore, each Π_i is associated with a frequency range $[\frac{N_{i-1}}{N_{\rm ft}\Delta}, \frac{N_i}{N_{\rm ft}\Delta})$. Thus we have,

$$\int_0^\infty P(f)\, df \approx \sum_{k=0}^{\frac{N_{\rm ft}}{2}} \frac{P(f_k)}{N_{\rm ft}\Delta} \tag{10}$$

$$= \sum_{i=1}^{M} \sum_{k=N_{i-1}}^{N_i - 1} \frac{P(f_k)}{N_i - N_{i-1}} \frac{N_i - N_{i-1}}{N_{\rm ft}\Delta} \tag{11}$$

$$= \sum_{i=1}^{M} \Pi_i \frac{N_i - N_{i-1}}{N_{\rm ft}\Delta}. \tag{12}$$

But notice, if Π_i is representative of the $i^{\rm th}$ frequency interval above, which has width $\frac{N_i - N_{i-1}}{N_{\rm ft}\Delta}$ in frequency space, we can directly write $P(f_k) \to \Pi_k$ and $df \to \frac{N_i - N_{i-1}}{N_{\rm ft}\Delta}$. So,

$$\int_0^\infty P(f)\, df \approx \sum_{i=1}^{M} \Pi_i \frac{N_i - N_{i-1}}{N_{\rm ft}\Delta}. \tag{13}$$

Therefore, we must identify Π_i with the $i^{\rm th}$ interval in order to satisfy the above integral normalization condition, even though $\sum_i \Pi_i \neq \sum_k P(f_k)$.