Superconductivity

(1) Condensation into state: not occupied/occupied pairs of electrons of opps. momentum (time-reversed)

(2) All pairs have same free parameter \( \delta \)

\[
\Psi = (\rho | \nu \phi e^{i\delta} c_{\downarrow}^* c_{\uparrow}^* ) / |\Psi| \\
= (\lambda | \nu \phi e^{i\delta} c_{\downarrow}^* c_{\uparrow}^* ) / |\Psi|
\]

(3) \( S \propto \text{prop. to } \nabla \phi \) as any quantum state

(4) Energy Gap \( \Delta \) (~2k\( T \)) For breaking of pair \( \rightarrow \) robust state

(5) For tunnel junction:

\[ I = I_0 \sin \delta \]

\( (I_0 \propto A) \) thickness barrier

\[ V = \frac{\Phi_0}{2\pi} \frac{\delta}{\Phi} = \frac{\hbar}{2e} \frac{\delta}{\Phi} \]

Later equation like Faraday's law

\[ V = \frac{\Phi}{\Phi} \text{ } \Phi = \left( \frac{\hbar}{2e} \delta \right) \]

Can many \( \delta \) with B field (Flux)

(6) In a loop, \( \delta \) is 2\( \pi \) periodic, so see device behavior periodic in B field, with applied flux

\[ B \cdot A = n(\Phi_0) = n(\hbar/2e) \]

In tunnel junction. See S current

No \( I \) \& \( V \) \leq 2\( \Delta \) because Far e's to tunnel, need to break CP's & \( \angle 2\Delta \) \& }
Tunnel Junctions
Tunnel Junctions

AI/AlOx/AL: Good quality junctions

Dist prop to $T_C$

Low sub-gap leakage ($\sim 10^{-3}$ at $2\Delta$ rise)
Additional Discussion

(1) Understand phase parameter in S. C.
   (BCS,n) explicit focus of J
   "spin model"

(2) Josephson Effect
   Proper 2nd order calculation of effect.
   → Understand where comes from (Not in textbooks)
   → Why S. C. has no resistance
   → An actual calculation!

(3) Exact Calc. of Josephson Effect
   (Arbitrary coupling) via Andreev Bound states
   • S. C. Gap protects from Dissipation
   • Understand subgap current

All with simple (as possible) Mathematics.
(learn in 1st year of Grad. School).
understanding of the junction physics is thus needed so that nonideal behavior can be more readily identified, understood, and eliminated. Although we will not discuss specific imperfections of junctions in this paper, we want to describe a clear and precise model of the Josephson junction that can give an intuitive understanding of the Josephson effect. This is especially needed since textbooks do not typically derive the Josephson effect from a microscopic viewpoint. As standard calculations use only perturbation theory, we will also need to introduce an exact description of the Josephson effect via the mesoscopic theory of quasiparticle bound-states.

The outline of the paper is as follows. We first describe in Sec. 2 the nonlinear Josephson inductance. In Sec. 3 we discuss the three types of qubit circuits, and show how these circuits use this nonlinearity in unique manners. We then give a brief derivation of the BCS theory in Sec. 4, highlighting the appearance of the macroscopic phase parameter. The Josephson equations are derived in Sec. 5 using standard first and second order perturbation theory that describe quasiparticle and Cooper-pair tunneling. An exact calculation of the Josephson effect then follows in Sec. 6 using the quasiparticle bound-state theory. Section 7 expands upon this theory and describes quasiparticle excitations as transitions from the ground to excited bound states from nonadiabatic changes in the bias. Although quasiparticle current is typically calculated only for a constant DC voltage, the advantage to this approach is seen in Sec. 8, where we qualitatively describe quasiparticle tunneling with AC voltage excitations, as appropriate for the qubit state. This section describes how the Josephson qubit is typically insensitive to quasiparticle damping, even to the extent that a phase qubit can be constructed from microbridge junctions.

2 The Nonlinear Josephson Inductance

A Josephson tunnel junction is formed by separating two superconducting electrodes with an insulator thin enough so that electrons can quantum-mechanically tunnel through the barrier, as illustrated in Fig. 1. The Josephson effect describes the supercurrent $I_J$ that flows through the junction according to the classical equations

$$ I_J = I_0 \sin \delta $$

(2.1a)

$$ V = \frac{\Phi_0}{2\pi} \frac{d\delta}{dt} $$

(2.1b)

where $\Phi_0 = \hbar/2e$ is the superconducting flux quantum, $I_0$ is the critical-current parameter of the junction, and $\delta = \phi_L - \phi_R$ and $V$ are respectively the superconducting phase difference and voltage across the junction. The
Fig. 1. Schematic diagram of a Josephson junction connected to a bias voltage $V$. The Josephson current is given by $I_J = I_0 \sin \delta$, where $\delta = \phi_L - \phi_R$ is the difference in the superconducting phase across the junction.

dynamical behavior of these two equations can be understood by first differentiating Eq. 2.1a and replacing $d\delta/dt$ with $V$ according to Eq. 2.1b

$$\frac{dI_J}{dt} = I_0 \cos \delta \frac{2\pi}{\Phi_0} V.$$ \hspace{1cm} (2.2)

With $dI_J/dt$ proportional to $V$, this equation describes an inductor. By defining a Josephson inductance $L_J$ according to the conventional definition $V = L_J dI_J/dt$, one finds

$$L_J = \frac{\Phi_0}{2\pi I_0 \cos \delta}.$$ \hspace{1cm} (2.3a)

The $1/\cos \delta$ term reveals that this inductance is nonlinear. It becomes large as $\delta \to \pi/2$, and is negative for $\pi/2 < \delta < 3\pi/2$. The inductance at zero bias is $L_{J0} = \Phi_0/2\pi I_0$.

An inductance describes an energy-conserving circuit element. The energy stored in the junction is given by

$$U_J = \int I_J V dt$$

$$= \int I_0 \sin \delta \frac{\Phi_0}{2\pi} d\delta dt$$

$$= \frac{I_0 \Phi_0}{2\pi} \int \sin \delta \, d\delta$$

$$= -\frac{I_0 \Phi_0}{2\pi} \cos \delta.$$ \hspace{1cm} (2.4c)

This calculation of energy can be generalized for other nondissipative circuit elements. For example, a similar calculation for a current bias gives $U_{\text{bias}} = -(I\Phi_0/2\pi)\delta$. Conversely, if a circuit element has an energy $U(\delta)$, then the current-phase relationship of the element, analogous to Eq. 2.1a, is

$$I_J(\delta) = \frac{2\pi}{\Phi_0} \frac{\partial U(\delta)}{\partial \delta}.$$ \hspace{1cm} (2.5)

Useful relations used later
A generalized Josephson inductance can be also be found from the second derivative of \(U\),
\[
\frac{1}{L_j} = \left(\frac{2\pi}{\Phi_0}\right)^2 \frac{\partial^2 U(\delta)}{\partial \delta^2}.
\]  \(2.6\)

The classical and quantum behavior of a particular circuit is described by a Hamiltonian, which of course depends on the exact circuit configuration. The procedure for writing down a Hamiltonian for an arbitrary circuit has been described in detail in a prior publication [11]. The general form of the Hamiltonian for the Josephson effect is \(H_j = U_j\).

3 Phase, Flux, and Charge Qubits

A Josephson qubit can be understood as a nonlinear resonator formed from the Josephson inductance and its junction capacitance. Nonlinearity is crucial because the system has many energy levels, but the operating space of the qubit must be restricted to only the two lowest states. The system is effectively a two-state system [12] only if the frequency \(\omega_{10}\) that drives transitions between the qubit states \(0 \leftrightarrow 1\) is different from the frequency \(\omega_{21}\) for transitions \(1 \leftrightarrow 2\).

We review here three different ways that these nonlinear resonators can be made, and which are named as phase, flux, or charge qubits.

The circuit for the phase-qubit circuit is drawn in Fig. 2(a). Its Hamiltonian is
\[
H = \frac{1}{2C} \frac{\partial^2}{\partial \delta^2} - \frac{I_0 \Phi_0}{2\pi} \cos \delta - \frac{I \Phi_0}{2\pi} \delta^n,
\]  \(3.1\)

where \(C\) is the capacitance of the tunnel junction. A similar circuit is drawn for the flux-qubit circuit in Fig. 2(b), and its Hamiltonian is
\[
H = \frac{1}{2C} \frac{\partial^2}{\partial \delta^2} - \frac{I_0 \Phi_0}{2\pi} \cos \delta + \frac{1}{2L} (\Phi - \frac{\Phi_0}{2\pi} \delta)^2.
\]  \(3.2\)

The charge qubit has a Hamiltonian similar to that in Eq. 3.1, and is described elsewhere in this publication. Here we have explicitly used notation appropriate for a quantum description, with operators charge \(Q\) and phase difference \(\delta\) that obey a commutation relationship \([\delta, Q] = 2\pi\). Note that the phase and flux qubit Hamiltonians are equivalent for \(L \to \infty\) and \(I = \Phi / L\), which corresponds to a current bias created from an inductor with infinite impedance.

The commutation relationship between \(\delta\) and \(Q\) imply that these quantities must be described by a wavefunction. The characteristic widths of this wavefunction are controlled by the energy scales of the system, the charging energy of the junction \(E_C = \varepsilon^2 / 2C\) and the Josephson energy \(E_J = I_0 \Phi_0 / 2\pi\). When the energy of the junction dominates, \(E_J \gg E_C\).
LC Microwave Resonator
Most all qubits formed by considering 2 states (|0⟩ + |1⟩) labeled of more complex “atom”. To understand Josephson, first look at LC osc. Like:

\[ H = \frac{\hat{p}^2}{2m} + \frac{\hat{y}^2}{2L} \]

\[ [\hat{y}, \hat{p}] = i\hbar \]

Solve in standard way for Harm Osc.:

\[ a^+|n⟩ = \sqrt{n+1}|n+1⟩ \]

\[ a|n⟩ = \sqrt{n-1}|n-1⟩ \]

\[ a^+a + \frac{1}{2} = \frac{1}{\sqrt{LC}} \]

Problem(s) for qubit:

1. All energy level separations same, can't define |0⟩, |1⟩ and keep in that subspace.

2. Response (to driving) is linear / classical

Response to drive is 6th state moves in a classical trajectory.

Need non-linearity for qubit states.
Circuit H / energies

(1) (Paper by Devoret describes mathem. to determ. H, depends on wiring).

(2) Not spend time, but can understand basic energy terms:

\[ \frac{1}{C} \rightarrow \frac{1}{\epsilon_C} \]
\[ \frac{1}{L} \rightarrow \frac{1}{\Phi^2} \]

(4) biases:
\[ \phi \]
\[ \phi_0 \]
\[ \phi_0 \]
\[ \frac{1}{C_g} \]
\[ \left( \phi - C_g \phi_0 \right)^2 \rightarrow \frac{1}{\epsilon_C} \text{ offset to change (new equilibrium value)} \]
\[ \frac{1}{2L} \]

(5) J-junction
\[ E = -E_J \cos \delta \] in \( \delta \)-space

(6) J-effect is transp 1 pair
\[ J = \frac{\hat{\phi}}{2} \]
Feynman (6)
\[ \left[ \frac{\hat{\phi}}{2}, \phi \right] = i \hat{\phi} \]
\[ \left[ \phi, \phi \right] = 2i \] dimless units
\[ \hat{\phi} = FT \phi \text{ potential; } e^{-i \hat{\phi}} + e^{i \hat{\phi}} \text{; chg. } \phi \text{ by } 2 \]
\[ E = -\frac{E_J}{2} \left[ \phi \phi^* + 1 + \phi^2 \right] \] \( \hat{\phi} \)-space
Charge Qubit

Made by evap Al thru a shadow mask stencil made from e-beam resist.
Charge Qubit

\[ H = \frac{e^2}{2(C+C_0)}(\hat{\phi} - \frac{C_0V}{E})^2 - \frac{I_0\Phi_0}{2\pi} \cos \delta \]

\[ = E_C(\hat{\phi} - \hat{\phi}_b)^2 - \frac{E_J^2}{2}\left[ 1\hat{\phi}\hat{\phi}_b + 1\hat{\phi}_b\hat{\phi} + 1\hat{\phi}\hat{\phi}_b + 1\hat{\phi}_b\hat{\phi} \right] \]

To solve:

1. \( E_C \gg E_J \): more classical cond. better basis (discrete values)
2. \( E_J \gg E_C \): is better basis

Near Resonance, basis \( \{ \hat{\phi} = 0, \hat{\phi} = z \} \):

\[ H = 2E_C(1 - \hat{\phi}_b)\frac{\hat{\phi}}{2} - \frac{E_J}{2} \hat{\phi}_b \]

\[ E = \pm \sqrt{C_0^2 + 10E_C^2} \]

\[ \hat{\phi}_b = 1 \]

\[ 10 = (1\hat{\phi} = 0 + 1\hat{\phi} = z) / \sqrt{2} \]

\[ |1\rangle = (\cos \theta - i \sin \theta) \]

\[ \text{Avoided Crossing} \]
Charge Noise Problem:

All fine, except nature not so kind!

Example of limitations to thy, from exp'l observation!

8b random, sample-to-sample (have to try)

\[ 8b \text{ Fluct. over time by } 10^{-2} \text{ to } 10^{-3} \text{ (charge noise).} \]

\[ \text{D Fluct.'s } E_{10}(t), \text{ random } Z \text{ oper., } \not\sim \text{ noise.} \]

\[ \text{ns coherence time.} \]

Big problem! (I orig. thought s.c. gubits never would work)

1. Solution (Saday):

   Operate at \( g_{0} = 1 \), where \( \frac{dE_{10}}{dg_{0}} = 0 \).

   Insens. to noise (1st-order, 2nd-order significant)

2. Also, make band flatter

   \( E_{f} \) big; \( E_{f} > E_{c} \) useful limit!

   This is params for more modern design
   (can't really use this \( \hat{g} \) basis well)

   since \( \Psi_{r} \) is super of many \( |g = 0\rangle, \]
   \( |g = 2\rangle, |g = 4\rangle, \ldots \), states.)
Solution for $E_j > E_c$ ($\delta$ basis)

1. Charge quantizes. Periodic $\Psi(\delta + 2\pi) = e^{i2\pi \delta / \delta}$ from $-\pi \to \pi$

2. Solve S.E. in $\cos \delta$ potential

   (H.O.-like states, but non-linear)

3. See non-linear from non-quad cos potential.

4. Make $E_j > 2SE_c$ so $\Psi$ is small at 0, $\pi$, then small change with charge $g\delta$.

5. Possible to get exp. small charge noise, but reasonable non-lin $1 < 2/\omega$

(Transmon circuit from Yale)

(If $g$ does not affect the charge qubit, why still call it that?)
Flux Qubits
FLUX qubits

(1) $I_0 \frac{\partial}{\partial t}\Phi_B$

(2) $H = E_c \hat{\delta}^2 + \left( \Phi_0 - \frac{\hat{\delta} \Phi_0}{2\pi} \right)^2 - E_j \cos \frac{\hat{\delta}}{2 L}$

Parab. + cosine pot

(3) $=$

(4) Idea: Make quartic potential, tunneling between 2 diff states

(5) Why interesting?

- $E_{10}$ low, because determ by tunneling;
- $E_{21}$ high, so very nonlinear potential; forget about other states!

(6) How get highly nonlinear (quartic) potential?

- Cancel $-L_{10} \rightarrow$ bias at $\Phi_0/2$ with $L$
- Careful design with $L \gg L_{10}$

- Make $C$ small so tunnels fast (with $N^2$)

DX short (L) across junction, $\hat{\delta}$ is continuous, $\hat{\delta}$ not periodic
In basis of \( \Psi_L, \Psi_R \):

Tunneling mixes states; avoided crossing \( E_s \), determ by param's hard \( \Rightarrow \) (expon. depend).

\[
H = \frac{\alpha \Phi_0^2}{2L} (\Phi_0 / E_s - \frac{1}{2}) \sum_i \frac{E_s}{2} \mathcal{T}_i
\]

\( (\Psi_L, \Psi_R) \) basis

\( \alpha, E_s \) calculated; depends on parameters

Nature: Flux noise small; \( \sim 10^{-5} \Phi_0 \), but gives decoher in ms if operate away from deg. pt. So operate at \( \Phi_0 = \frac{1}{2} \Phi_0 \).

Real device: Hard to match \( L_5 \); \( L_5 = \frac{\Phi_0}{2\pi I_0} \)

Solve with device design

\( \sim 2I_0 \times I_0 \)

\( \sim 2I_0 \times I_0 \)

\( \times I_0 \)

2 junctions; \( \frac{1}{2} \) induct; \( \simeq L_5 \).

Good because \( I_0 \)'s all scale with oxid. changes.

In practice, solve q.m. of entire system, \( \Phi \) qubit mode dominates

\( 2I_0 \Rightarrow \simeq 1.6I_0 \) for best design.

(Don't make \( C \)'s too small or will see charge noise effects!)
Phase Qubits

(old NIST design)
Phase Qubits

1. Current bias $I$

2. $H = \frac{\hat{B}^2}{2C} - \frac{I_0 \Phi_0}{2\pi} \cos \frac{\hat{\phi}}{2} - \frac{I \Phi_0}{2\pi} \hat{\phi}$

   [Equation for cosine potential tilted by bias current]

3. $E$ levels in non-linear potential is $\propto$ cubic.

4. $\omega_p = \sqrt{\frac{1}{L \cdot j C}} = \sqrt{\frac{1}{L_0 \cdot j C_0}}$

   $I = I_0 \sin \Phi_0$

   $\sin^2 \Phi_0 = (I/I_0)^2$

   Slow with $I$

5. $\Delta \omega \approx \frac{4 \sqrt{2}}{3} \frac{I_0 \Phi_0}{2\pi} \left(1 + \frac{I}{I_0}\right)^{3/2}$

   Fast with $I$

6. No $\Phi$, $\Phi$ noise, so no degeneracy fine (except more complicated)

   $\omega_p$, $\Delta \omega$ changes with $I \to$ tunable.
Qubit: Nonlinear LC resonator

\[ I = I_0 \sin \delta \]
\[ V = \frac{\Phi_0}{2\pi} \frac{\dot{\delta}}{} \]
\[ \dot{I}_j = I_0 \cos \delta \frac{\dot{\delta}}{} \]
\[ \frac{L_J}{(1/L_J)}V \]
\[ L_J = \frac{\Phi_0}{2\pi I_0 \cos \delta} \]

**nonlinear inductor**

\[ \langle V \rangle = 0 \]

\[ \Delta U = \frac{4\sqrt{2} I_0 \Phi_0}{3} \left[ 1 - \frac{I}{I_0} \right]^{3/2} \]

\[ \omega_p = \left( \frac{2\sqrt{2\pi} I_0}{\Phi_0 C} \right)^{1/2} \left[ 1 - \frac{I}{I_0} \right]^{1/4} \]

\[ \gamma_{10} \approx \frac{1}{RC} \text{ Lifetime of state } |1> \]

1: Tunable well (with $I$)
2: Transitions non-degenerate
3: Tunneling from top wells
4: Lifetime from $R$
(7) Small Non-linearity (~5%) 

- Use ~ 5-10ns microwave pulses 
- Practically, ok since  
  a) Hard to accurate & cheap shape faster pulses. 
  b) Rot. Wave Approx breaks down faster pulses  
     (Logic errors when) 

(8) Qubit control: \( I + \Delta I \) in \( |0\rangle, |1\rangle \) of eigen states 

\[
H = \begin{pmatrix} 0 & 0 \\ 0 & E_{10} \end{pmatrix} = \frac{\Phi_0}{2\pi} \Delta I \begin{pmatrix} \langle 0|1 \rangle \langle 10 | & \langle 0|1 \rangle \langle 01 | \\ \langle 1|0 \rangle \langle 10 | & \langle 1|0 \rangle \langle 01 | \end{pmatrix} 
\]

\[
\begin{array}{c}
\text{Std. H.O. M.E.'s not zero} \\
\text{H.O. coupling non-linear}
\end{array}
\]

(9) LF (\( \Delta I \)) gives \( I_x, I_z \) rotations 
HF (\( \Delta I \)) gives \( I_x, I_y \) rotations 

(10) Practical: \( I^0 \) with large \( R \) does not exist!  
(Only model only) 

\[
\sum \Rightarrow \sum \quad \text{Transf I bias} \\
\text{Like } \Phi \text{ qubit; but not deg.}
\]

- Note: Now sens. to \( \Phi \) noise 
- (100-200 ms decoherence)  
  (can be made smaller!) 

\[
\text{cubic well}
\]
Experiments

Go through my standard $\phi$-qubit descr.

1) Init state  
2) Rotate  
3) Meas.  
4) Fabrication  
   Our / std s.c uses special materials  
   $\text{Al, } T_c \gg 1K \text{ ok}$  
   Fabrication of small junction (e-beam)  
5) Meas. Setup in OR  
   Low $T$  
   Filtering  
   Generation of microwave pulses  
6) Sequence to take data  
   0 or 1  
   10,000x for prob.  
7) Spectr.  
8) Rabi's ...
Josephson-Junction Qubit

(1) State Preparation
Wait \( t > 1/\gamma_{10} \) for decay to \( |0> \)
(20 mK; \( kT < \hbar \omega_{10} \); 6 GHz)

(2) Qubit logic with current bias
\[ I = I_{dc} + \delta I_{dc}(t) + I_{\muwc}(t)\cos\omega_{10}t + I_{\muws}(t)\sin\omega_{10}t \]

(3) State Measurement: \( \Delta U(I_{dc} + \delta I_p) \)
Fast single shot – high fidelity

3 ns Gaussian pulse
\( |1> : \) tunnel
\( |0> : \) no tunnel
IC Fabrication

Qubit

X, Y

Z

(Old NIST design)

100 μm

Al junction process
& optical lithography

Al₂O₃ substrate

Al SiNx Al
Spectroscopy

\[ \Delta U / \hbar \omega_{0} \]

\[ \omega_{10}(t) \]

\[ P_{1} = \text{grayscale} \]

Bias current \( I \) (au)

GQubit frequency [GHz]

\[ 8 \]

\[ 7.5 \]

\[ 7 \]

\[ 6.5 \]

\[ 6 \]

\[ 5 \]

\[ 4 \]

\[ 3 \]

\[ 2 \]

\[ 1 \]

\[ 0 \]

\[ I_{\mu \omega} \]

\[ I_{p} \]

\[ \text{meas.} \]

\[ \text{saturate} \]
Qubit Fidelity Tests

- Large Visibility: \( T_1 = 110 \text{ ns}, T_\phi \sim 85 \text{ ns} \)

- Probability 1 state

- ~90% visibility

- (slightly detuned)

- (no detuning)
Qubit Taxonomy

**Charge**
- Potential (in $\delta$)
- Area ($\mu m^2$): 0.01
- $E_J = I_0 \Phi_0 / 2\pi e^2 / 2C$
- Non-linearity: 50\% - 5\%
- Impedance $1/\omega_r C$: 30 k\$
- Noise: $Q_N$ - 1 ns
- (more complex coupling)

**Flux**
- Area ($\mu m^2$): 0.1-1
- $E_J = I_0 \Phi_0 / 2\pi e^2 / 2C$
- Non-linearity: >100\%
- Impedance $1/\omega_r C$: 1 k\$
- Noise: $\Phi_N$ - 10 ns

**Phase**
- Area ($\mu m^2$): 10-100
- $E_J = I_0 \Phi_0 / 2\pi e^2 / 2C$
- Non-linearity: ~5\%
- Impedance $1/\omega_r C$: 50 \$
- Noise: $\Phi_N$ - 200+ ns

Experimentally Important:
- Small C
- $L \approx L_{J0}$
- $I \rightarrow I_0$