FRW Cosmology

1.1 ***Ignoring angular terms, write down the FRW metric. Hence express the comoving distance r_M as an function of time and redshift. What is the relation between scale factor $a(t)$ and redshift z? How is the Hubble parameter related to the scale factor a(t)? Suppose a certain radioactive decay emits a line at 1000 Å, with a characteristic decay time of 5 days. If we see it at $z = 3$, at what wavelength do we observe it, and whats the decay timescale?***

The FRW metric is given as

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2[\mathrm{d}r^2 + S_\kappa(r)^2\mathrm{d}\Omega^2]. \tag{1}$$

Where for a given radius of curvature, R,

$$S_\kappa(r) = \begin{cases} R\sin(r/R) & \kappa = 1 \\ r & \kappa = 0 \\ R\sinh(r/R) & \kappa = -1 \end{cases}$$

Suppose we wanted to know the instantaneous distance to an object at a given time, such that $\mathrm{d}t = 0$. Ignoring the angular dependence, i.e. a point object for which $\mathrm{d}\Omega = 0$, we can solve for r_M as

$$\int_0^{d_p} \mathrm{d}s = a(t)\int_0^{r_M} \mathrm{d}r,$$

$$d_p(t) = a(t)r_M(t). \tag{2}$$

Where $d_p(t)$ is the proper distance to an object, and r_M is the coming distance. Note that at the present time the comoving distance and proper distance are equal. Consider the rate of change of the proper time

$$\dot{d}_p = \dot{a}r = \frac{\dot{a}}{a}d_p. \tag{3}$$

Hubble is famously credited for determining what is now called Hubbles law, which states that

$$v_p(t_0) = H_0 d_p(t_0). \tag{4}$$

Relaxing this equation to now be a function of time we will let the Hubble constant now be the Hubble parameter. Therefore we see that the definition of the Hubble parameter in relation to the scale factor is

$$H(t) \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\dot{a}}{a}. \tag{5}$$

To enlighten what the comoving distance is, consider a photon which is known to have a null geodesic, or in terms of the metric $\mathrm{d}s = 0$. Then the FRW tells us, again ignoring angular dependence,

$$\int_{t_e}^{t_0} \frac{c}{a(t)} \mathrm{d}t = \int_0^{r_M} \mathrm{d}r.$$

Thus,

$$r_M = \int_{t_e}^{t_0} \frac{c}{a(t)} \mathrm{d}t. \tag{6}$$

This comoving distance is then, the distance of which the photon has traveled, as measured by the photon—or something that was theorectically comoving with the photon.

To express this in terms of redshift we will show that $\lambda_e/a(t_e) = \lambda_0/a(t_0)$. This is derived from (6), where we say the comoving distance of a photon coming to us is equal to the comoving distance of another photon one wavelength behind the original photon.

$$\int_{t_e}^{t_0} \frac{c}{a(t)} dt = \int_0^{r_M} \mathrm{d}r = \int_{t_e+\lambda_e/c}^{t_0+\lambda_0/c} \frac{c}{a(t)} dt. \tag{7}$$

If we multiple both sides by the comoving distance of the photon one wavelength behind from where it starts to where it is when the first photon is observed $\int_{t_e+\lambda_e/c}^{t_0} \frac{c}{a(t)} \mathrm{d}t$, we get

$$\int_{t_e}^{t_e+\lambda_e/c} \frac{c}{a(t)} dt = \int_{t_0}^{t_0+\lambda_0/c} \frac{c}{a(t)} dt. \tag{8}$$

Note these time intervals are extremely short in the context on which a(t) varies. This is on the order of 10^{-15} s, compared to some fraction of the age of the universe, order of 10^{17} s, on which the scale factor changes of order itself during directly measurable epochs. Thus we can pull $a(t)$ out of these equations to good approximation to get

$$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_0}{a(t_0)}. \tag{9}$$

Note that we can clearly see this is a time dilation effect if you call $\delta t_e = \lambda_e/c$ and $\delta t_0 = \lambda_0/c$. Then we see that

$$\delta t_0 = \frac{\delta t_e}{a(t_e)}. \tag{10}$$

Taking the limit as $\lambda \to 0$, then $\delta t \to 0$, and we see that $\mathrm{d}t_0 = \frac{\mathrm{d}t_e}{a(t_e)}$, therefore, we see that $a(t)$ dilates the time making intervals longer in the observing frame.

With the definition of redshift being

$$z \equiv \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1. \tag{11}$$

We can use (9) and (11) together to find the familiar equation

$$a(t) = \frac{1}{1+z}, \tag{12}$$

where $a(t_0) = 1$. Now we can rewrite the comoving distance in terms of redshift, now having a relationship between the scale factor and the redshift. Since $z = a(t)^{-1} - 1$, then

$\dot{z} = -\dot{a}/a^2 = -H/a \to \mathrm{d}t = -\frac{a}{H}\mathrm{d}z$. Making the substitution we have, and noting the minus sign flips the integral bounds where t_e is at redshift z and t_0 is at redshift 0,

$$r_M = \int_{t_e}^{t_0} \frac{c}{a(t)}\mathrm{d}t. = \int_0^z \frac{c}{H(z)}\mathrm{d}z. \tag{13}$$

Now consider the atomic line being emitted at a redshift of $z = 3$. From (9), with $\lambda_e = 1000\mathring{A}$, and $z = 3$

$$\lambda_0 = a(t_0)\lambda_e/a(t_e) = (1+z)\lambda_e = 4000\mathring{A}.$$

To get the decay timescale in the observing frame we recall from (10) that the scale factor, and hence the redshift, dilates the time to be longer in the observing from by a factor of $1+z$, therefore,

$$\Delta t_0 = (1+z)\Delta t_e = 20 \text{ days}.$$

1.2 *Give definitions for the angular diameter distance, and luminosity distance (you dont have to derive them, just explain what they are). How are they related to the comoving distance in a flat universe? Explain why they are sensitive to cosmology. Mention a few ways we've used these to constrain cosmology.*

The angular distance is given by

$$d_A \equiv \frac{\ell}{\delta\theta}. \tag{14}$$

For a fixed length object of length ℓ, from the FRW metric we find that $\ell = a(t_e)S_\kappa(r)\delta\theta$. Thus we can rewrite the angular distance as

$$d_A = \frac{S_\kappa(r)}{1+z_e}. \tag{15}$$

For flat space this reduces to $d_A = a(t_e)r_M = d_p(t_e)$, that is to say the angular distance measured now is what the proper distance was to the object when it emitted the light that is reaching us now!

For luminosity distance is defined as

$$d_L \equiv \sqrt{\frac{L}{4\pi f}}. \tag{16}$$

We'll want an expression for f to determine d_L better, which alters from out normal definition in two aspects. First geometrically the space may not be flat so the area of the sphere in curved space is given by $A_p(t_0) = 4\pi S_\kappa(r)^2$. Next the energy of each photon is also decreased by a factor of $a(t_e)$, which is seen by the fact that $E = hc/\lambda$, and $\lambda_0 = \lambda_e/a(t_e)$. Secondly there is a time dilation effect that slows down the photons arriving to us, since initially the proper distance between two pulses is $c\delta t_e$, but the distance is stretched as it travels such that $c\delta t_0 = c\delta t_e/a(t_e) \to \delta t_0 = \delta t_e/a(t_e)$. Thus our definition of flux, in a curved space, is altered by a factor of $1/a(t_e)^2$, or

$$f = \frac{L\, a(t_e)^2}{4\pi S_\kappa(r)^2}.$$

Substituting this into (16) we arrive at

$$d_L = S_\kappa(1 + z_e). \tag{17}$$

Looking at (15) and (17), we can derive a relationship between the two as

$$d_A(1 + z_e) = S_\kappa(r) = \frac{d_L}{1 + z_e}. \tag{18}$$

In a flat universe, $\kappa = 0$, $S_\kappa(r) = r$. Thus (18) gives the relationship between the comoving distance and they two distance measures. Moreover, in a flat universe we also have $S_\kappa = d_p(t_0)$.

For low redshift, or observations of events that have occurred close to our current time (remember looking out into the universe is looking back in time), we can Taylor expand the scale factor. To second order we write the scale factor as

$$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2. \tag{19}$$

Where we have created a parameter q_0 called the deceleration parameter

$$q_0 \equiv -\frac{\ddot{a}}{aH^2}\bigg|_{t=t_0}. \tag{20}$$

Thus the sign of q_0 depends on the acceleration of the universe, positive for a negatively accelerating universe and positive for a positively accelerating universe. With some algebraic gymnastics we can define a comoving distance in terms of H_0 and q_0.

$$a(t)^{-1} \approx 1 - H_0(t - t_0) + \frac{1 + q_0}{2} H_0^2 (t - t_0)^2.$$

$$r = \int \frac{c}{a(t)} \mathrm{d}t \approx c(t_0 - t_e) + \frac{cH_0}{2}(t_0 - t_e)^2.$$

$$z = a(t_e)^{-1} - 1 \approx H_0(t_0 - t_e) + \frac{1 + q_0}{2} H_0^2 (t_0 - t_e)^2,$$

$$(t_0 - t_e) \approx H_0^{-1}\left[z - \frac{1 + q_0}{2} z^2\right].$$

With all of this we can now write the comoving coordinate in terms of H_0 and q_0 as we have set out to do.

$$r \approx \frac{c}{H_0}\left[z - \frac{1 + q_0}{2} z^2\right] + \frac{cH_0}{2} z^2 H_0^2 = \frac{c}{H_0} z \left[1 - \frac{1 + q_0}{2} z\right]. \tag{21}$$

In a flat universe the luminosity distance and angular distance can both be written in terms of the comoving coordinate, thus since our comoving coordinate depends on H_0 and q_0, we see there is a clear dependence on the cosmology, note the acceleration. Following the approximation $z << 1$ we see that

$$d_A \approx r(1-z) \approx \frac{c}{H_0} z \left[1 - \frac{3+q_0}{2} z\right]. \tag{22}$$

$$d_L = r(1+z) \approx \frac{c}{H_0} z \left[1 + \frac{1-q_0}{2} z\right]. \tag{23}$$

We have used the angular distance to make measurements of the CMB. Using CMB Doppler peaks we get an idea of the length of the ruler and then can infer the redshift of the surface of last scattering. Unfortunately there is no good standard ruler in cosmology to make other meaningful measurements, at least that are known or reasonably performable with current limits.

Luminosity distance has found uses in cosmology a little closer to us by measuring type IA supernovas. By making multiple measurements of these standard candles at various redshifts we have been able to infer that the universe is accelerating. Noting that an accelerating universe, with $q_0 < 0$, gives us smaller fluxes than a decelerating universe would. Which was found to be the case with $q_0 = -.55$.

1.3 *Write down the two Friedmann equations in a flat universe. Show that in an vacuum-dominated universe, $\rho_\Lambda \approx$ const, the scale factor increases exponentially with time.*

The first Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2}. \tag{24}$$

The second Friedmann equation, or the acceleration equation, is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P). \tag{25}$$

For completion the fluid equation relating P and ϵ is

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0. \tag{26}$$

In a flat universe simply set $\kappa = 0$. In a universe that is flat and vacuum-dominated such that the dominate energy density is $\epsilon_\Lambda = constant$, solving (24)

$$\int_1^{a(t)} \mathrm{d}\log a = \sqrt{\frac{8\pi G\epsilon_\Lambda}{3c^2}} \int_{t_0}^{t} \mathrm{d}t,$$

$$a(t) = e^{H_0(t-t_0)}, \tag{27}$$

where

$$H_0 = \sqrt{\frac{8\pi G\epsilon_\Lambda}{3c^2}}.$$

This definition is clearly merited if we look at $\left.\frac{\dot{a}}{a}\right|_{t=t_0}$ for the scale factor as defined in (27), which should be H_0.

1.4 *People talk a lot about the search for w in dark energy—how is 'w' defined? Find the relation between scale factor and time in a matter dominated and a radiation dominated universe. Write down the approximate value of the Hubble constant (including correct units), and hence give a rough estimate of the age of the universe.*

Given a component of the universe, labeled by 'w', which satisfies $P_w = w\epsilon_w$, the fluid equation tells us how the energy density scales with the scale factor. Plugging in this definition of P into (26)

$$\begin{aligned}\dot{\epsilon} + 3\tfrac{\dot{a}}{a}\epsilon(1+w) &= 0,\\ -3(1+w)\mathrm{d}\log a &= \mathrm{d}\log\epsilon.\end{aligned}$$

Therefore we see that the energy density evolves as

$$\epsilon_w(a) = \epsilon_{w,0}a^{-3(1+w)}. \tag{28}$$

Now using this in the first Friedmann equation, (24), where only one component of the universe is dominating at the given time

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \sqrt{\frac{8\pi G\epsilon_0}{3c^2}}a^{\frac{-(1+3w)}{2}}.$$

Guess a solution to the scale factor such that it is a power-law in time, such that $a \propto t^p$, thus $p - 1 = -\frac{1+3w}{2}p$. Solving for p we get that

$$p = \frac{2}{3(1+w)} \to a \propto t^{\frac{2}{3(1+w)}},$$

or

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}. \tag{29}$$

We can backtrack to solve for t_0 by noticing that from the first Friedmann equation and this result,

$$p\,\frac{t^{p-1}}{t_0^p} = \sqrt{\frac{8\pi G\epsilon_0}{3c^2}}\left(\frac{t}{t_0}\right)^{p-1} \to t_0 = \frac{p}{\sqrt{\frac{8\pi G\epsilon_0}{3c^2}}}.$$

Simplifying this gives us

$$t_0 = \frac{1}{1+w}\sqrt{\frac{c^2}{6\pi G\epsilon_0}}, \tag{30}$$

this is the age of the universe in a single component cosmology.

Lastly, before getting to the actual questions, we can talk about the Hubble constant in this single component universes by

$$H_0 \equiv \left(\frac{\dot{a}}{a}\right)\Big|_{t=t_0} = \frac{2}{3(1+w)}t_0^{-1}. \tag{31}$$

We can rewrite the scale factor in terms of the Hubble constant instead of the age of universe, giving us

$$a(t) = \left(\frac{3(1+w)}{2}H_0 t\right)^{\frac{2}{3(1+w)}} \tag{32}$$

For matter $w = 0$ and for radiation $w = 1/3$, thus we get the the scale factor for a matter domination universe is

$$a_m(t) = \left(\frac{t}{t_0}\right)^{2/3}, \tag{33}$$

and for radiation dominated universe

$$a_r(t) = \left(\frac{t}{t_0}\right)^{1/2}. \tag{34}$$

From the Benchmark model referred to in Ryden we have $H_0 = 70 \pm 7$ km s^{-1} Mpc^{-1}. This corresponds to an age of the universe of 13.5 ± 1.3 Gyrs with the appropriate cosmology. Roughly a Hubble time is $H_0^{-1} \approx 14$ Gyr, and for the Matter+Lambda model the numerical factor is .964 (This is very close to actual since radiation domination was so brief). If we wish to only consider a single component and the age in these universes we get

$$t_{m,0} = \frac{2}{3H_0} \approx 9.3Gyr, \tag{35}$$

and for radiation

$$t_{r,0} = \frac{1}{2H_0} \approx 7Gyr. \tag{36}$$

1.5 *Derive the critical density ρ_c in terms of the Hubble constant H_0. Given the present value of the Hubble constant, express ρ_c in physical units (g cm^{-3}). If I tell you that Ω_b = 0.04 and that the universe is pure hydrogen, what is the number density of protons today?*

The critical energy density ϵ_c is the energy density such that the universe is flat. If at a given time, $\epsilon(t) > \epsilon_c$, then the universe is positively curved, $\kappa = 1$. Likewise, if $\epsilon(t) < \epsilon_c$, then the universe is negatively curved, $\kappa = -1$. Thus to determine for ϵ_c, set $\epsilon(t) = \epsilon_c$, then $\kappa = 0$ and the first Friedmann equation, (24), tells us

$$\epsilon_c \equiv \frac{3c^2H^2}{8\pi G}. \tag{37}$$

Really this is the definition of the critical energy density so we are actually defining it, not solving for it. For the critical energy density today, we see it will be given in terms of the Hubble constant and thus we can solve for it—given knowledge of H_0.

$$\epsilon_{c,0} = \frac{3c^2H_0^2}{8\pi G} = 8.3\text{x}10^{-9}\text{ergs cm}^{-3}. \tag{38}$$

In the massive non-relativistic limit, $\epsilon \approx \rho c^2$, thus in some situation, e.g. matter dominated universes, we could say that

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} = 9.2\text{x}10^{-30}\text{g cm}^{-3}. \tag{39}$$

It is often convent to talk about the density parameter, Ω, of the energy density which a component makes up of the universe. This is defined as

$$\Omega_i(t) \equiv \frac{\epsilon_i(t)}{\epsilon_c(t)}. \tag{40}$$

Thus if we are given a component which massive and non-relativistic, which baryons are a good approximation of, we have

$$\rho_b = \Omega_b\, \rho_c.$$

Thus given $\Omega_{b,0} = .04$ we can find the number density of baryons today as

$$n_{b,0} = \frac{\rho_{b,0}}{\mu} = \frac{\Omega_{b,0}\, \rho_{c,0}}{\mu}.$$

If the universe was purely Hydrogen then $n_{b,0} = n_{p,0}$ and $\mu = m_p$, thus

$$n_{p,0} = \frac{\Omega_{b,0}\, \rho_{c,0}}{m_p} = 2.2\text{x}10^-7\text{cm}^{-3}.$$

1.6 *How does energy density in CDM scale with redshift? How does energy density in radiation scale with redshift? How does the temperature of the CMB scale with redshift?*

From (28) we determined the dependence of a component's energy density based on its energy/pressure relationship. For dark matter, similar to matter in that $w = 0$, we get

$$\epsilon_{DM}(t) = \epsilon_{DM,0} a^{-3} = \epsilon_{r,0}(1+z)^3. \tag{41}$$

For radiation, $w = 1/3$, we find that

$$\epsilon_r(t) = \epsilon_{r,0} a^{-4} = \epsilon_{r,0}(1+z)^4. \tag{42}$$

From statisitical mechanics we also know that a blackbody has an energy density of $\epsilon_{bb} = \sigma_{SB}\frac{c}{4}T^4$. Thus for a blackbody radiating we can get a temperature dependence on redshift as follows

$$\sigma_{SB}\frac{c}{4}T^4 = \sigma_{SB}\frac{c}{4}T_0^4(1+z)^4,$$

$$T = T_0(1+z). \tag{43}$$

Thermal History, Big Bang Nucleosynthesis

2.1 ***What are modern estimates for the CMB temperature T_γ today? What are modern estimates for Ω_m, Ω_Λ, h. Hence estimate the redshift z_{eq} of matter radiation equality (derive it, dont just state it). Give the characteristic age and temperature of the universe during primordial nucleosynthesis and recombination.***

From the Benchmark model provided by Ryden, based on current measurements available at the time of publication, we present the following table.

Modern Density Parameters			
photons			$\Omega_{\gamma,0}$ =5.0x10^{-5}
neutrinos			$\Omega_{\nu,0}$=3.4x10^{-5}
total radiation			$\Omega_{r,0}$=8.4x10^{-5}
baryonic matter			$\Omega_{b,0}$=0.04
nonbaryonic dark matter			$\Omega_{dm,0}$=0.26
total matter			$\Omega_{m,0}$ =0.30
cosmological constant			$\Omega_\Lambda \approx 0.70$
curvature			$\lvert\Omega_k\rvert \approx 0^*$
Epoch Equalities			
radiation-matter			$a_{rm} = 2.8$x10^{-4}
matter-lambda			$a_{m\Lambda} = 0.75$
Cosmology Parameters			
Hubble constant			$H_0 = 70$ km s^{-1} Mpc^{-1}
deceleration parameter			$q_0 = -0.55$
CMB temperature			$T_{\gamma,0} = 2.725$ K
baryon-to-photon ratio			$\eta = 5.5$x10^{-10}
Helium fraction			$Y = 0.24$
Events			
neutron freezeout	$T = 9$x10^9 K	$E = 1.29$ MeV	$t \approx 1$ s
nucleosynthesis	$T = 7.6$x10^8 K	$E = 66$ keV	$t \approx 200$ s
recombination	$T = 9730$ K	$E = .323$ eV	$t \approx 47$ kyr

*Current constrains $\lvert\Omega_k\rvert \leq 0.2$.

We then see that since h is the ignorance factor by saying $H_0 = 100h$ km s^{-1} Mpc^{-1}, that $h = 0.7$.

A simple way to find the scale factor of epoch equalities is to set the energy densities of the two epochs equal to each other, as follows.

$$\epsilon_X(t_{XY}) = \epsilon_Y(t_{XY}).$$

Then the ratio is equal to one, trivially. However, less trivially, the ratio is also equal to the density parameters now scaled by the scale factor depending on how each component scales. The scaling dependence can be given by (28), thus

$$\frac{\epsilon_X(t_{XY})}{\epsilon_Y(t_{XY})} = \frac{\epsilon_{X,0}}{\epsilon_{Y,0}} a(t_{XY})^{3(w_Y - w_x)} = \frac{\Omega_{X,0}}{\Omega_{Y,0}} a(t_{XY})^{3(w_Y - w_X)}.$$

Thus setting this equal to one and solving for $a(t_{XY})$ we get

$$a(t_{XY}) = \left(\frac{\Omega_{Y,0}}{\Omega_{X,0}}\right)^{\frac{1}{3(w_Y - w_X)}}. \tag{44}$$

Now we can see that for radiation-matter equality

$$a(t_{rm}) = \left(\frac{\Omega_{r,0}}{\Omega_{m,0}}\right)^{\frac{1}{3(1/3-0)}} = \frac{8.4\text{x}10^{-5}}{.3} = 2.8\text{x}10^{-4}. \tag{45}$$

Thus the redshift is $z_{rm} = 3570.43$.

2.2 *What was the temperature of the universe at recombination, and how did the characteristic energy, $k_B T$, compare with the ionization potential of hydrogen, $E_{ionize} = 13.6\,\text{eV}$. What redshift did recombination take place?*

To determine when recombination happened we need to define what is meant. Let's say recombination happens when half of the electrons have combined with the protons. Now we can use statistical mechanics to determine when this occurred since the protons and electrons will be in thermal equilibrium, thanks to the electromagnetic force. A speicies X will follow a Maxwell-Boltzmann distribution for some temperature T as follows

$$n_X = g_X \left(\frac{m_X kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(\frac{-m_X c^2}{kT}\right). \tag{46}$$

Taking the ratio of combined hydrogen to free protons and electrons we get the Saha equation

$$\frac{n_H}{n_e n_p} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{E_{ionize}}{kT}\right). \tag{47}$$

Defining the ionization fraction $X \equiv \frac{n_p}{n_p + n_H} = \frac{n_p}{n_b ary}$ and the baryon to photon ratio $\eta \equiv \frac{n_b}{n_\gamma}$, we can rewrite the Saha equation and solve for X. This results in

$$\frac{1-X}{X^2} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \eta \left(\frac{kT}{m_e c^2}\right)^{3/2} \exp\left(\frac{E_{ionize}}{kT}\right). \tag{48}$$

Setting $X = 0.5$, we find that $kT_{rec} = .323$ eV, or $T_{rec} = 3740$ K. Note that $kT_{rec} \approx \frac{1}{42} E_{ionize} << E_{ionize}$, thus a large fraction of the photons are no longer photoionizing combined hydrogen. Since radiation took place during a matter dominated epoch, we can use the handy relation from (43) to relate the temperature with a redshift to find

$$z_{rec} = \frac{T_{rec}}{T_{\gamma,0}} - 1 = 1371.48. \tag{49}$$

2.3 *Explain the concept of 'freeze-out'. In particular, if the weak interaction cross section $\sigma \propto E^2$, show that the neutron abundance 'freezes-out' falls as the temperature of the universe falls. At what redshift does the temperature of the ionized plasma differ from that of the CMB, and how is it that the two temperatures are still coupled after recombination?*

The concept of 'freeze-out', whence-fore dubbed freeze-out, is when a reaction can no longer efficiently occur to keep two or more populations in thermal equilibrium. In a cosmological context this happens due to the expansion of the universe and we declare a freeze-out when $\Gamma_{reaction} \leq H$. Typical the reaction rate has a dependence on a number density, cross section for interaction and the speed of the particles. As an example Thomson scattering has

$$\Gamma_{Tomson}(z) = n_e(z)\sigma_T c = (1+z)^3 n_{e,0}\sigma_T c. \tag{50}$$

Thus consider the low energy photons decoupling from the electrons,

$$\Gamma_{Tomson}(z) = (1+z)^3 X(z) n_{b,0}\sigma_T c \leq H_0\sqrt{\Omega_{m,0}}(1+z)^{3/2} = H. \tag{51}$$

Find the exact redshift for the freeze-out simply take the equality and solve for z. The result is $z_{Tdec} \approx 1100$.

Now let's consider the neutron freeze-out that occurs just prior to BBN, safely in the radiation dominated epoch. Given that $\sigma_{weak} \propto E^2$, and that $E = \epsilon_r V = \epsilon_{r,0} a^{-4} V_0 a^3 \propto a^{-1}$, therefore, $\sigma_{weak} \propto a^{-2}$. Further from the first Friedmann equation we know that $H = H_0\sqrt{\Omega_{r,0}}a^{-2}$.

Now the reaction rate of both: $n + \nu_e \leftrightharpoons p + e^+$ and $n + e^+ \leftrightharpoons p + \bar{\nu_e}$ are dependent on the weak cross section since they involve neutrinos. Thus we calculate when these reaction freeze out by

$$\Gamma = n_n \sigma_{weak} c \propto a^{-5},$$

and

$$H = H_0\sqrt{\Omega_{r,0}}a^{-2} \propto a^{-2}.$$

Thus look at the ratio of the reaction rate to the Hubble expansion rate, and recall that $a^{-1} \propto T$,

$$\frac{\Gamma}{H} \propto a^{-3} \propto T^3.$$

Thus as $T \rightarrow 0$, then so does the neutron abundance freeze-out as T^3. Thus to freeze-out more neutrons we want to do this ASAP.

The rewritten Saha equation, (48), to a decent approximation, will continue to hold true until the recombination rate is smaller than the expansion rate, that is $\Gamma_{recombination} \leq H$. Thus we will find that we have a freeze-out population of free electrons which will never combine with protons. Given a thermally average cross-section for recombination of $\langle\sigma v\rangle/c = 4.7\text{x}10^{-24}(k_b T/1eV)^{-1/2}$, we can find the freeze-out abundance as

$$\Gamma = n_e(z)\frac{\langle\sigma v\rangle}{c}c = H_0\sqrt{\Omega_{0,m}}a^{-3/2} = H \tag{52}$$

Solving for this numerically we find that $z_{ls} \approx 1000$. This leaves a freeze-out population of electrons of $X \approx 6\text{x}10^{-4}$. Note that recombination happens later for smaller η, or having more photons. This is because the freeze out happens because the photons follow a Planck distribution and if there are simply more, then there are more in the high energy tail and can keep hydrogen from being neutral.

This population of electrons can now stay coupled to the CMB loosely through Compton scattering. They will also loosely stay in contact with rest of the baryons, thus keeping the baryons and CMB coupled well beyond recombination, as you might have expected. To determine when the CMB and baryons actually decouple will we need to look when the Compton scattering reaction rate freezes-out.

We could look at the Klein-Nishina formula to get the cross section of Compton scattering, but that is beyond I will show here. Therefore I will just quote the result from that calculation and say that $z_{Cdec} \approx 150$.

2.4 *Given the temperature T_{freeze} of the neutrons, how can one crudely estimate the abundance of Helium in the universe? Name the 4 light elements produced in Big Bang Nucleosynthesis. How are other elements in the universe made? How would the Helium abundance in the universe change if the weak interaction was higher? If the baryon density was higher? If there were extra relativistic species in the universe. Explain how the abundance of light elements was used to constrain the number of neutrino species, $N_\nu = 3$*

We can write a Saha-like equation for the ratio of neutrons to protons. This becomes

$$\frac{n_n}{n_p} = \exp\left(\frac{-(m_n - m_p)c^2}{kT}\right). \tag{53}$$

Given a T_{freeze}, we can calculation this fraction f. Then you could estimate the abundance of Helium by assuming 100% efficiency in forming Helium. Defining $Y \equiv \rho_{He}/\rho_{bary}$, we find that

$$Y_{max} = \frac{2\rho_n}{\rho_n + \rho_p} = \frac{2f\rho_p}{f\rho_p + \rho_p} = \frac{2f}{1+f}. \tag{54}$$

From this calculation we find that $f = 0.2$, thus we get a $Y_{max} = 1/3$. This is definitely an upper bound since it assumes the neutrons don't decay during BBN, however they do start decaying before they can become bound since the universe first needs to cool down enough to for Deuterium! This doesn't happen until around $t \approx 200$ s, and a significant amount of neutrons have decayed. Thus the new ratio of $f = 0.15$, which places$Y_{max} \approx 0.27$.

The four 'elements' produced during the BNN are isotopes of Hydrogen, Helium, Lithium and Beryllium. During BBN both Deuterium and Tritium are produce, and while Tritium has a decay time of 18 years, this is effectively stable during BBN. Both ^{3}He and ^{4}He are made, and are stable. Both ^{6}Li and ^{7}Li are made, and are both stable. Both ^{7}Be and ^{8}Be are made, but ^{8}Be is extremely unstable and immediately decays back into two ^{4}He atoms. In the end ^{7}Be is also converted into ^{7}Li via electron caputre. This provided a major road black preventing elements of $A \geq 8$ from being formed during BBN. Elements heavy than these are made in stars where there is no race against time, and eventually $A \geq 8$ can be achieved by overcoming the unstable elements with $A = 5$ and $A = 8$.

The abundance of Helium produced during the BBN depends on a multiple parameters. Significant ones are the weak interaction strength, baryon-to-photon ratio, and the number of relativistic species during the BBN. First, if the weak interaction was weaker than what has been calculated, then the freeze-out of neutrons would happen sooner leading to a higher number of frozen-out neutrons. This in turn provides more neutrons for the BBN, which would increase the yield of Helium. The logical precession is seen as

$$\sigma\downarrow \Longrightarrow \Gamma\downarrow \Longrightarrow t_{freeze}\downarrow \text{ for } \Gamma(t) = H(t) \Longrightarrow T_{freeze}\uparrow \Longrightarrow \left(\frac{n_n}{n_p}\right)_{freeze}\uparrow \Longrightarrow n_{He}\uparrow$$

Next the baryon-to-photon ratio would also alter the abundance of Helium. Since the ratio of neutrons to Deuterium has η dependence, then changing this variable will alter the temperature of BBN, since we define BBN when the ratio of neutrons to Deuterium is 1. The Saha-like equation for this ratio is

$$\frac{n_D}{n_n} = \frac{12(1-f)\zeta(3)}{\sqrt{\pi}}\eta\left(\frac{kT}{m_n c^2}\right)^{3/2} \exp\left(\frac{(m_p + m_n - m_D)c^2}{kT}\right). \tag{55}$$

At the temperatures of BBN the ratio of Deuterium to neutrons is dominated by the exponential term. Thus if η goes up, then so does T_{nuc}. Thus BBN starts earlier and we have a more complete fusion of ^{4}He. However, if η goes up note the reaction rate for $p+n \leftrightharpoons D+\gamma$ also goes up since there is more baryons. Thus we have a later time for the freeze-out of D, and thus we have a lower abidance for D. Logical process of these occurrences are

$$\eta\uparrow \Longrightarrow T_{nuc}\uparrow \text{ for } \frac{n_D}{n_n} \approx 1 \Longrightarrow t_{nuc}\downarrow \Longrightarrow n_{He}\uparrow$$

$$\eta\uparrow \Longrightarrow \Gamma\uparrow \Longrightarrow t_{freeze}\uparrow \text{ for } \Gamma(t) = H(t) \Longrightarrow T_{freeze}\downarrow \Longrightarrow \left(\frac{n_D}{n_n}\right)_{freeze}\downarrow \Longrightarrow n_D\downarrow$$

Last parameter we will consider is the number of relativistic species. In the radiation dominated era, which BBN took place in, the number of relativistic species has a significant effect on the energy density. The exact relation is given by

$$\epsilon_r = \frac{g_*}{2}\frac{c}{4}\sigma_{SB}T^4 \tag{56}$$

Thus if the number of relativistic species goes up, i.e. $g_*\uparrow$, then we see that the energy density also goes up. This leads to the Hubble parameter increasing. Thus if we have a larger Hubble parameter the reaction will freeze-out earlier at higher temperatures. With higher freeze-out abundances as a result we will have more time and more material to fusion ^{4}He, and thus will have a higher abidance of such.

$$g_*\uparrow \Longrightarrow \epsilon_r\uparrow \Longrightarrow H\uparrow \Longrightarrow T_{freeze}\uparrow \Longrightarrow \left(\frac{n_n}{n_p}\right)_{freeze}\uparrow \Longrightarrow n_{He}\uparrow$$

Assuming we understand the weak interaction and baryon-to-photon ratio well, which they are fairly confident in, we could use the abidance of ^{4}He to then constrain the last parameter, the number of relativistic species. By the time of BBN it is widely believed we understand all of the components of the universe that were relativistic, rest mass energies << 66 keV (known particles that satisfy this are the photons, gluons [but suffer from confinement, thus do not contribute], neutrinos and perhaps gravitons [unconfirmed]). Thus we could place a constrain on g_* from ^{4}He, and therefore a bound on the number of neutrino families—if they are the only known constituents of the relativistic contribution. Doing this we have constrained the number to be three families, which is in agreement with what particle physicist have found.

2.5 *Explain why the neutrino temperature is different from the CMB temperature. Is it higher or lower? Extra credit if you can estimate its numerical value. What characteristic scale does the 'first Doppler peak' in the CMB power spectrum correspond to?*

The neutrinos decouple when the weak force became minuscule, around when the neutrons froze out on the order of 1 second. This was around 1 MeV, when the average photon could still pair produce. Thus the positron and electron are still relativistic when the neutrinos decouple. Now when the positron and electrons go non-relativistic, i.e. photons can no longer pair produce, the electrons and positron still annihilate, effectively dumping their energy into the photons without receiving any back from pair production. However, since the neutrinos decoupled from the electrons already, they get none of this energy. Therefore, while in thermal equilibrium with photons before now, the photons will become hotter than the neutrinos.

Here is a more detailed calculation of how to estimate the relic neutrino background temperature. The comoving specific entropy is given by

$$s = g_* \sigma_{SB} \frac{c}{4} T^3. \tag{57}$$

While two or more populations are in thermal equilibrium, they have the same specific entropy. Thus the comoving specific entropy of the neutrinos before decoupling is what the entropy will remain the same after the neutrinos are decoupled (as long as they still remain in thermal equilibrium with themselves). Designate two times, t_{before}, for just after the neutrinos decouple but before the electron/positrons go non-relativistic, and t_{after}, for after the electrons/positrons go non-relativistic. Now the neutrinos specific entropy is also equal to the photon's specific entropy at the time of decoupling which is

$$s_\gamma = s_\nu = g_{*before}\, \sigma_{SB} \frac{c}{4} T_\nu^3. \tag{58}$$

Now consider the photon's specific entropy after the electron/positrons go non-relativistic. Since the photons remain in thermal equilibrium with themselves, these two times will be equal

$$s_\gamma = g_{*before}\, \sigma_{SB} \frac{c}{4} T_{\gamma,before}^3 = g_{*after}\, \sigma_{SB} \frac{c}{4} T_{\gamma,after}^3. \tag{59}$$

From (58) we can see that $T_\nu = T_{\gamma,before}$ and lets just call $T_{\gamma,after} = T_\gamma$ now. Then from (59) we find that

$$T_\nu = \left(\frac{g_{*after}}{g_{*before}}\right)^{1/3} T_\gamma. \tag{60}$$

Now simply count the degeneracy factor for the conditions before and after to find what the neutrino temperature is in terms of the photon's. Recall

$$g_* = \sum_{Bosons} g_i + \frac{7}{8}\sum_{Fermions} g_i \tag{61}$$

Now the photons, electron, positrons have g=2, so $g_{*before} = 11/2$. And when only the photons are relativistic then $g_{*after} = 2$. Thus we find that

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma. \tag{62}$$

Or that today $T_{\nu,0} \approx 1.95$ K.

The location of the first Doppler peak corresponds to the size of Hubble length at the time of recombination. This is because this is the largest baryon acoustic oscillation amplitude possible, corresponding to a baryon-photon plasma that has just reached the center of the potential well, right before pressure pushes it back out. Modes that have been oscillating longer will have lower amplitudes due to Hubble friction.

Structure Formation

3.1 *Explain the difference between comoving and proper coordinates. If a filament is 5 Mpc in comoving coordinates, how big is it in proper coordinates?*

The exact definitions of comoving and proper coordinates were discussed in section on FRW. Coming coordinates are fixed in time, while the proper coordinate gives a physical distance between two points, and change in time. The relationship is given by (2) or restated in terms of redshift here

$$r = (1+z)d_p. \tag{63}$$

Thus if we have a filament that is 5 Mpc in coming coordinates it's proper distance is

$$d_p = \frac{5}{1+z}\ \text{Mpc}. \tag{64}$$

3.2 *Consider a uniform, initially static medium with density ρ and temperature T. Estimate the Jeans mass in terms of ρ, T and fundamental constants. Is it larger or smaller if the expansion of the universe is taken into account?*

The Jeans Length is given by

$$\lambda_J = 2\pi c_s t_{dyn}. \tag{65}$$

The dynamic time scale for gravitational collapse is

$$t_{dyn} = \sqrt{\frac{1}{4\pi G\bar{\rho}}}. \tag{66}$$

The speed of sound of a component with parameter 'w' is

$$c_s = c\sqrt{\frac{\mathrm{d}P}{\mathrm{d}\epsilon}} = \sqrt{w}c. \tag{67}$$

For massive baryonic matter we know that $w \approx kT/(\mu c^2)$. Thus, taking all this together we get the Jeans length for a static medium with density ρ and temperature T

$$\lambda_J = 2\pi\sqrt{\frac{kT}{\mu c^2}}c\sqrt{\frac{1}{4\pi G\bar{\rho}}} = \sqrt{\frac{\pi kT}{G\mu\rho}}. \tag{68}$$

Thus defining the Jeans mass to be the mass inclosed in a sphere of radius Jeans length we get a Jeans mass of

$$M_J = \frac{4}{3}\pi\rho\left(\sqrt{\frac{\pi kT}{G\mu\rho}}\right)^3 = \sqrt{\frac{16\pi^5k^3T^3}{9G^3\mu^3\rho}} \propto \frac{T^{3/2}}{\rho^{1/2}}. \tag{69}$$

Taking the expansion of the universe into account we have to reconsider what the dynamic time scale is. Recall that $\delta(r,t) = (\epsilon(r,t) - \bar{\epsilon}(t))/\bar{\epsilon}(t)$. The dynamic timescale in an initially static, homogeneous matter only universe can be derived from $\ddot{\delta} = 4\pi G\bar{\rho}\delta$. However, in an expanding universe there is an additional term called the Hubble drag which makes the perturbation equation as follows

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta \tag{70}$$

Therefore we can see that this frictional term will act to damp the growth of density perturbations, thus increasing the dynamical time for collapse. Therefore will a longer dynamical time, we have a longer Jeans length, which directly leads to a larger Jeans mass.

Note the a better description of how perturbations will grow involves a fluid dynamic approach to consider other physics involved in the problem. We have only considered large scale perturbations, a complete description also considers smaller scales, or larger k values is give

$$\ddot{\delta} + 2H\dot{\delta} = (\frac{3}{2}\Omega_m H^2 - \frac{k^2c_s^2}{a^2})\delta \tag{71}$$

3.3 *How does the growth factor D depend on the scale factor a(t) in the following 4 cases: superhorizon and subhorizon in both matter-dominated and radiation dominated eras (assume a flat universe). Give brief physical arguments why the radiation-dominated case is either faster or slower than the matter-dominated case for sup/super horizon perturbations. Given these scalings, explain with order-of-magnitude arguments why despite the fact that $\Delta T/T \sim 10^{-5}$ at recombination, structure formation was nonetheless able to go non-linear today.*

From (71) we can solve for the density perturbation in various regimes. Sometimes we write the density perturbation in Fourier transformed components as follows

$$\delta = D(t)\sum_k \delta_k \exp{(-i\vec{k}\cdot\vec{r})}. \tag{72}$$

To solve (71) lets assume that δ is a power-law in t, or $\delta \propto D\, t^p$. Then (71) becomes

$$p(p-1)D\, t^{p-2} + 2HpD\, t^{p-1} = (\frac{3}{2}\Omega_m H^2 - \frac{k^2 c_s^2}{a^2})pD\, t^p \tag{73}$$

In radiation and matter dominated periods the Hubble parameter goes as t^{-1}. Thus we can solve for p from these equation for cases when we can ignore the sound speed term, the sub horizon terms. We find that D grows as $t^{2/3} \propto a(t)$ in matter dominated period and as $\log(t) \propto \log(a(t))$ in radiation dominated periods. We quote the results for superhorizons and summarize in the follow table

$D \propto a(t)$	Sub-Horizon	Super-Horizon
Radiation Dominated	constant	$a(t)^2$
Matter Dominated	$a(t)$	$a(t)$

We can think of the radiation dominated results as a consequence of the speed of light being finite and the definition of the horizon. For sub-horizon scales the photons can easily stream out of any potential wells and then smooth out the density perturbations. However on super-horizon scales the photons, by definition, cannot transverse this scale. Thus the matter and photons combine together to create a density perturbation.

Despite the $\Delta T/T \sim 10^{-5}$ observed in the CMB, leading to a density pertubation of $\delta \sim 10^{-2}$ today, we find that $\delta \sim 1$ (enabling non-linear growth). This is because the dark matter decoupled from radiation during the radiation-matter equality epoch and not at recombination. Thus the dark matter density perturbations have been growing much longer, and when the baryons decoupled from the photons they quickly feel into the dark matter potential wells giving them a larger perturbation to start growing from ($\delta_{DM} = 10^{-3}$). Note the dark matter perturbation is not seen in the CMB since the baryonic plasma's sound speed was so large, preventing it from falling in to the CDM wells.

3.4 *Write down the perturbed Poisson equation, and show that in a flat universe, linear potential fluctuations are time-independent.*

The perturbed Poisson equation is

$$\nabla^2\phi = 4\pi G a^2 \bar{\rho}\delta. \tag{74}$$

Consider large scale structure, or superhorizon scales. In the radiation dominated epoch $\delta \propto a^2$ and $\bar{\rho} \propto a^{-4}$. While for matter dominated epoch $\delta \propto a$ and $\bar{\rho} \propto a^{-3}$. Thus in both epochs the combination $\delta\bar{\rho} \propto a^{-2}$. Thus taking this into account in (75), we find

$$\nabla^2\phi \propto \text{constant}. \tag{75}$$

This seems to be circular logic since the requirement that $\nabla^2\phi \propto$ constant for superhorizon perturbations, is what got us the growth rate factors.

3.5 ***Sketch the power spectrum $P(k)$ vs. k, and explain the reason for the asymptotic slopes at the smallest and largest wavenumbers (you should derive the latter). What characteristic scale does the peak in the power spectrum P(k) correspond to? How do mass fluctuations $\sigma(M)$ depend on $P(k)$? How do they depend on the growth factor? Explain why this implies that the CDM is a bottom-up hierarchical cosmology, with the smallest structures collapsing first.***

The power spectrum has asymptotic behavior of $\propto k$ for $k << 1$ and $\propto k^{-3}$ for $k >> 1$. We can derive this by considering when perturbations collapse into the horizon. During radiation dominated era, perturbations do not grow, thus perturbations smaller than the horizon are frozen until matter-radiation equality. Thus consider two cases: the perturbation crosses within the horizon before matter-radiation equality, and the perturbation crosses within the horizon after matter-radiation equality. Then we can say the density perturbation for $z_{cross} > z_{rm}$ are frozen in until z_{rm} and lose out on a perturbation growth by a factor of the ratio of the scale factors. While for perturbations with $z_{cross} < z_{rm}$ will have never experience a freezing. Thus since this takes place during the radiation dominated period $\delta \propto a(t)^2$. Then

$$\delta \propto \begin{cases} 1 & z_{cross} < z_{rm} \\ \left(\frac{1+z_{rm}}{1+z_{cross}}\right)^2 & z_{cross} > z_{rm} \end{cases} \tag{76}$$

Now we want this in terms of k, so find the relation between k and z_{cross}. We expect the perturbation will have a horizon crossing when $\lambda_{proper} \sim d_H$. Thus

$$\lambda_{proper}(t_{cross}) = a(t_{cross})\frac{2\pi}{k} = \frac{2\pi}{k(1+z_{cross})}.$$

$$d_H(t_{cross}) = \frac{c}{H} = \frac{c}{\sqrt{\Omega_r}}a(t_{cross})^2 = \frac{c}{\sqrt{\Omega_r}(1+z_{cross})^2}$$

Thus we find that $k \propto (1+z_{cross})$. Now we are told that $P(k) = A\,k\,\delta^2$ (Harrison-Zel'dovich), thus

$$P(k) \propto \begin{cases} k & k\,d_H << 1 \text{ or } (z_{cross} < z_{rm}) \\ k^{-3} & k\,d_H >> 1 \text{ or } (z_{cross} > z_{rm}) \end{cases}. \tag{77}$$

From this derivation it is easy to see that the peak in the power spectrum corresponds to the event horizon at time of matter-radiation equality.

The mass fluctuation $\sigma(M)$ is define as

$$\sigma(M)^2 = \int P(k)|\hat{W}(k)|\mathrm{d}^3k \propto P(k)\,k^3. \tag{78}$$

Since $P(k) = \langle|\delta_{\vec{k}}|^2\rangle$, then we know that $P(k) \propto \delta^2 \propto D^2$. But also we assume the power spectrum is invariant, thus $P(k) \propto k^n$. Thus

$$\sigma(M) \propto D\,k^{(3+n)/2}. \tag{79}$$

Thus we have that $\sigma \propto k^{(3+n)/2}$. For $n > -3$, which we just showed, we have that as you go to smaller scales, larger k, you have larger mass fluctuations, $\sigma(M)$. Therefore you will have collapse acting faster where there is larger fluctuations, therefore we will collapse first at small scales and then larger scales. This shows that CDM is hierarchal.

3.6 $P(k) \propto k$ *is often referred to as a scale invariant power spectrum. Explain what this means. What is the relation between perturbations in matter* δ_m *and perturbations in radiation* δ_r *if the perturbations are adiabatic?*

A scale invariant power spectrum predicted by most inflationary scenarios, (n=1), means that the mass fluctuations has the same amplitude at all scales when the enter the horizon in the radiation dominated epoch. Let's show what the condition for a invariant power spectrum is, and why we chose $n = 1$ or $P(k) \propto k$.

$$\sigma(M) \propto Dk^{(3+n)/2} \propto a^2 M^{-(3+n)/6}. \tag{80}$$

Since $k \propto \lambda^{-1} \propto M^{-1/3}$ and $D \propto a^2$, since these perturbations are happening during the radiation dominated era, outside the horizon. Now the M or the horizon mass is equal to

$$M \propto \frac{\rho}{H^3} \propto \frac{a^{-3}}{(a^{-2})^3} \propto a^3. \tag{81}$$

Thus we find $a \propto M^{1/3}$, telling us that

$$\sigma(M) \propto M^{(1-n)/6} \implies \sigma(M) = \text{ constant, for n} = 1. \tag{82}$$

For adiabatic perturbations, we assume that the entropy is constant. Therefore the perturbations between the matter and radiation are in communication and are related somehow. Recall the comoving specific entropy $s \propto T^3$. Then the specific entropy per baryon, s_m is

$$s_b \propto \frac{T^3}{\rho_m} \propto \frac{\rho_r^{3/4}}{\rho_m}. \tag{83}$$

Therefore, since the fluctuations are constant so that, (mathematical note: $\frac{\delta s_b}{s_b} \approx \mathrm{d}\log s_b$ in the limit $\delta \to 0$)

$$\frac{\delta s_b}{s_b} = \frac{3}{4}\frac{\delta\rho_r}{\rho_r} - \frac{\delta\rho_m}{\rho_m} = 0. \tag{84}$$

Thus,

$$\delta_m = \frac{3}{4}\delta_r. \tag{85}$$

3.7 *Define the correlation function* $\zeta(r)$*, and define the power spectrum* $P(k)$ *(if you cant give a strict mathematical explanation, at least give a qualitative explanation). What is the relation between them?*

The galaxy correlation function is the excess probability of finding a galaxy at a radius r of a known one. Mathematically that is

$$dN = N_0[1 + \zeta(r)]dV. \tag{86}$$

Thus is galaxies are uncorrelated at a radius R, $\zeta(R) = 0$, and we would get $\frac{dN}{dV} = N_0$, as expected. We can calculate the correlation function by

$$\zeta(r) = \left\langle \frac{\rho(r)\rho(0)}{\bar{\rho}^2} \right\rangle - 1 = \langle \delta(r)\delta(0) \rangle \,. \tag{87}$$

Recall from earlier that $P(k) = \langle |\delta_{\vec{k}}|^2 \rangle$, thus looking at these two expression we see that they are simply the Fourier transformation of each other.

3.8 *Top-hat model. Write down the relation between kinetic and potential energy for virialized objects. Hence, what is the relation between the turnaround radius and the virial radius? How does the velocity dispersion σ scale with mass for virialized objects; hence, how does the virial temperature scale with mass? What is the linear overdensity at which an object collapses into a halo? What is the corresponding non-linear overdensity?*

From classical mechanics, relating the potential energy, kinetic energy and second time derivative of the moment of inertia together, we can derive what is called the virial theorem.

$$\ddot{I} = 2U + 4K. \tag{88}$$

We say a system is virialized if $\ddot{I} = 0$, or $K = -\frac{1}{2}U$. From this relationship we can derive another relationship between the turnaround radius and virial radius. We define the turnaround radius at which the kinetic energy is zero, thus

$$E = U = -\frac{GM}{r_{ta}}.$$

And for the virial radius do the same equation for something that has been virialized, and then set these energies equal to each other (this is saying that we want objects within a radius turnaround to become virialized).

$$E = \frac{GM}{2r_{vir}}.$$

Thus,

$$2r_{vir} = r_{ta}. \tag{89}$$

From the virial theorem, given a velocity dispersion, σ,

$$\sigma^2 \approx \frac{GM}{r_{vir}} \propto M^{2/3} a(t). \tag{90}$$

Where we used the fact that $r_{vir} = r_{vid,p} a^{-1} = (\frac{3\pi M}{4})^{1/3} a^{-1}$.

Thus we find that $\sigma \propto M^{1/3}\sqrt{a}$. The virial temperature is just proportional to the disperision velocity, thus $T \propto M^{2/3} a$.

The magic number to remember is that for objects to collapses into halos the linear over density is $\delta_{crit} \sim 1.69$ and for non-linear case $\Delta_{crit} \sim 18\pi^2 \sim 178$.

3.9 *Sketch the Press-Schechter mass function. Which end evolves most sharply with redshift? Explain the concept of bias—why are massive halos strongly clustered?*

The Press-Schechter mass function talks about the number of collapsed halos as a function of the halo mass and redshift. The result is the number of objects between M and $M + \mathrm{d}M$ is

$$N(M)\mathrm{d}M = \frac{1}{\sqrt{\pi}}(1+\frac{n}{2})\frac{\bar{\rho}}{M^2}\left(\frac{M}{M_{crit}}\right)^{(3+n)/6}\exp\left(-\left(\frac{M}{M_{crit}}\right)^{(3+n)/3}\right). \tag{91}$$

For small scale objects, $n = 1$, we that masses above M_{crit}, will evolve more rapidly. Also note below M_{crit}, $N(M) \approx AM^{-2} \propto M^{-2}$.

The concept of bias comes from the fact that we tend to find structure around other structure. Thus we it seems that there might be some underlaying bias causing this. The underlaying bias is that if you have collapse you are biased towards having formed from a higher underlaying long wavelength density perturbation. Thus since it is long wavelength your neighbor is also on this higher density perturbation that average, so your neighbor is also likely to collapse and form structure. If everyone was trying to build a tower 10km above sea level, than those who built it in the Himalayas would have an easier time than those who started at sea level.

Inflation

4.1 *Briefly explain some of the problems in cosmology inflation was designed to solve. Why doesnt inflation reduce the number density of photons to undetectable levels? Give the condition for inflation, either in terms of the scale factor a or the pressure and density (P, ρ).*

Cosmology has a fair deal of problems, making the subject mostly bullshit. Without invoking inflation, there is the problem of uniformity of the CMB (Horizon Problem), the lack of magnetic monopoles (Monopole Problem) and the observation of an fairly flat universe (Flatness Problem).

The Horizon Problem, is the issue that the CMB is in complete thermal equilibrium with itself. This might not be an issue if the entire Horizon had even been in casual contact. However it turns out at only a 2° chunk of the CMB is in causal contact with itself. Thus there are on the order of 10^4 patches that are not in casual contact with each other, yet have $\delta T/T \sim 10^{-5}$. How could this be?

The Monopole Problem, is the issue that we don't see observe monopoles. Most accepted cosmologies involve a Big Bang that creates monopoles, and thus there should be an observable amount of monopoles we could detect today. The fact that Maxwell wrote $\nabla \cdot \vec{B} = 0$, and is still consider an acceptable law today, tells us this is a glaring problem. Additionally, the predict mass of the monopole is huge, on the order of $10^{12} TeV$. This would have quickly become non-relativistic and would actually have been a dominate part of the universe very early, effectively negating a radiation dominated period. This would cause a whole range of issues for cosmology as we understand it now.

The Flatness Problem, is a fine tuning problem. For the universe to be flat today, with current constraints requiring that $|\Omega_k| < 0.2$. Using the Benchmark model and following the evolution of curative back to the Planck time, right before relativity breaks down and we should stop applying our theories of cosmology, the bound on curvature would be $|\Omega_k| < 10^{-60}$. This means that the universe when created was pretty much perfectly flat. This seems like a huge fine tuning issue, since there is nothing in physics whats-so-ever that requires a flat universe.

Introducing the concept of inflation helps solve a lot of these issues. The basic idea is that there was a period when the universe went rapid exponential growth very early in the universe. One inflation model has inflation turning on at $approxt_{GUT} \approx 10^{-36}$. The universe is then briefly dominated by some component of the universe with $w < -1/3$. This causes and exponential growth in the scale factor. Inflation ends after N e-foldings of the scale factor, around 10^{-34} s, when the component of the universe loses all of it's energy.

This rapid growth allows for a few things to occur. First we can have a CMB that is casually connected before inflation and reaches thermal equilibrium, then after inflation the CMB is no longer in casual contact as we observe it today. Next the monopole problem would be resolved since the monopoles would be drastically diluted. The reason photons are not diluted to undetectable levels with the monopoles is that the component driving inflation annihilates and dumps its energy into the relativistic components of the universe. Since the monopoles were so massive they do not get any of the energy, while the photons do. Thus we can effectively dilute the monopoles while keep photons at detectable levels. Lastly the flatness problem is adverted since we could have a highly curved universe to start with, but after inflation the universe will be rapidly flattened, that is the radius of curvature exponentially grew instead of contracting.

The condition for inflation is that the comoving Hubble distance decreases with time, this means

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c}{aH}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c}{\dot{a}}\right) = -\frac{c\ddot{a}}{\dot{a}^2} < 0 \Longrightarrow \ddot{a} > 0. \tag{92}$$

From the acceleration equation we see that this corresponds to a $w < -1/3$. Thus something like the vacuum would be a good candidate.

4.2 *Name four testable predictions of inflation, and give brief explanation.*

4.2.1 Gaussian density fields, i.e., the phases of the Fourier components are uncorrelated with each other. This leads to the power spectrum being scale invariant with a power-law spectrum form.

4.2.2 Statistical significance of finding monopoles. Inflation predicts that if you find one monopole, you are highly unlikely to ever find another one. Thus if we find a handful of monopoles inflation will be in serious trouble.

4.2.3 Flat universe. Inflation predicts that the universe should be flat, since the initial condition are highly suppressed and we should expect a flat universe now. Somehow probing before inflation and finding a significantly curved universe would bode well for inflation, though lack of one is not incriminating.

4.2.4 Background gravitational waves. Gravitational waves come from flocculation in the metric, and their amplitude scales with the energy of inflation. We are looking to find this through indirect detections from the B-mode polarization of the CMB.