Effective Potential

Consider a three-body system with $m_3 << m_2 \leq m_1$, from this point narratored with m_1 as a star, m_2 as a planet and m_3 as a small satellite. We shall use a rotating non-inertial coordinate system, which rotates about the barycenter but with the origin centered on the planet. Oriented such that the barycenter falls along the x–axis, in the $x > 0$ half, and the axis of rotation is parallel to the z–axis.

Rewrite, $m_1 \equiv M_*$ as the mass of the star, and $m_2 \equiv M_P$ as the mass of the planet. Define $\vec{a}$ as the vector from center of the planet to the center of the star, $\vec{\ell}$ as the vector from the center of the planet to the barycenter, and $\vec{r}_\perp \equiv \vec{\rho}$, as projection of the vector from the center of the planet to a given point into the plane normal to the axis of rotation (such given point denoted as $\vec{r}$).

To be succinct, $\vec{r}_\perp = |\vec{r}|\sin(\theta)\left(\sin(\theta)\hat{r} + \cos(\theta)\hat{\theta}\right) = \rho\hat{\rho}$, where the angle is the usual spherical coordinate definition and ρ is the standard cylindrical coordinate, as used by physicist. We chose to define this last vector, since it is the relevant distance for determining the centrifugal potential, along with ℓ and Ω. The effective potential per unit mass, u, for a tertiary object in a planet-star system is

$$u_{eff}(\vec{r}) = -\frac{GM_P}{|\vec{r}|} - \frac{GM_*}{|\vec{a}-\vec{r}|} - \frac{1}{2}\Omega^2|\vec{r}_\perp - \vec{\ell}|^2. \tag{1}$$

Respectively the terms are Newton's gravitational potential from the planet (thus defining G as Newton's gravitational constant), the gravitational potential from the star and the centrifugal potential from an object moving about the barycenter with angular frequency Ω.

Since we can describe a scalar field potential to the entire space, there is a corresponding curl-free force. This is given as $F_{eff}(\vec{r}) = -\nabla U_{eff}(\vec{r})$. Assuming $|\vec{a}|$ is constant, we limit ourselves to circular orbits, and carrying out the calculation for the force per unit mass, f, we find

$$\begin{aligned} f_{eff}(\vec{r}) = \quad & -\frac{GM_P}{|\vec{r}|^2}\hat{r} \quad -\frac{GM_*}{|\vec{a}-\vec{r}|^2}\left[\frac{(|\vec{r}| - |\vec{a}|\cos(\phi)\sin(\theta))\hat{r} + |\vec{a}|\sin(\phi)\hat{\phi} - |\vec{a}|\cos(\phi)\cos(\theta)\hat{\theta}}{|\vec{a}-\vec{r}|}\right] \\ & +\Omega^2|\vec{r}_\perp - \vec{\ell}|\left[\frac{\left(|\vec{r}_\perp| - |\vec{\ell}|\cos(\phi)\right)\hat{\rho} + |\vec{\ell}|\sin(\phi)\hat{\phi}}{|\vec{r}_\perp - \vec{\ell}|}\right]. \end{aligned}$$

Note that the parts of the second and third term enclosed in square brackets, are unit vectors. Moreover, they are the unit normalized vectors of $\vec{r} - \vec{a}$ and $\vec{r}_\perp - \vec{\ell}$, respectively. Physically, the first one points from the star to the point r, and the second one points from the barycenter to the projection of the point in the plane normal to the rotation axis. Let's define unit vectors such that they point opposite of the terms enclosed in square brackets, physically said they are directed from a given point, or it's projection, towards the object of interest. We will call these $\hat{r}_{ra}$ and $\hat{r}_{r_\perp\ell}$.

$$\hat{r}_{ra} = -\frac{(|\vec{r}| - |\vec{a}|\cos(\phi)\sin(\theta))\hat{r} + |\vec{a}|\sin(\phi)\hat{\phi} - |\vec{a}|\cos(\phi)\cos(\theta)\hat{\theta}}{|\vec{a}-\vec{r}|}, \tag{2}$$

$$\hat{r}_{r_\perp\ell} = -\frac{(|\vec{r}_\perp| - |\vec{\ell}|\cos(\phi))\hat{\rho} + |\vec{\ell}|\sin(\phi)\hat{\phi}}{|\vec{r}_\perp - \vec{\ell}|}, \tag{3}$$

$$f_{eff}(\vec{r}) = -\frac{GM_P}{|\vec{r}|^2}\hat{r} + \frac{GM_*}{|\vec{r}-\vec{a}|^2}\hat{r}_{ra} - \Omega^2|\vec{r}_\perp - \vec{\ell}|\hat{r}_{r_\perp \ell}. \tag{4}$$

For a Keplerian system $\Omega^2|\vec{a}|^3 = G(M_* + M_P)$, thus

$$f_{eff}(\vec{r}) = -\frac{GM_P}{|\vec{r}|^2}\hat{r} + \frac{GM_*}{|\vec{r}-\vec{a}|^2}\hat{r}_{ra} - \frac{G(M_*+M_P)|\vec{r}_\perp - \vec{\ell}|}{|\vec{a}|^3}\hat{r}_{r_\perp \ell}. \tag{5}$$

$$u_{eff}(|\vec{r}|) = -\frac{GM_P}{|\vec{r}|} - \frac{GM_*}{|\vec{r}-\vec{a}|} + \frac{G(M_*+M_P)|\vec{r}_\perp - \vec{\ell}|^2}{2|\vec{a}|^3}. \tag{6}$$

Up until now we have been exact for circular orbits, let's consider realistic systems with $M_P << M_*$ fields of interest being $\vec{r} << \vec{a}$. That means $M_* + M_P \approx M_*$ or the limit that the ratio of the planets mass to the stars is zero. Then we get that

$$\lim_{M_P/M_* \to 0} \vec{\ell} = \vec{a}. \tag{7}$$

Furthermore, let's use $\vec{r} << \vec{a}$, such that $\theta \approx \pi/2$. Then $\hat{r}_{ra} \approx \hat{r}_{\ell a}$. This regime leads to the equation simplifying to

$$f_{eff}(\vec{r}) = -\frac{GM_P}{|\vec{r}|^2}\hat{r} + \frac{GM_*}{|\vec{r}-\vec{a}|^2}\hat{r}_{ra} - \frac{GM_*|\vec{r}-\vec{a}|}{|\vec{a}|^3}\hat{r}_{ra}. \tag{8}$$

$$u_{eff}(|\vec{r}|) = -\frac{GM_P}{|\vec{r}|} - \frac{GM_*}{|\vec{r}-\vec{a}|} + \frac{GM_*|\vec{r}-\vec{a}|^2}{2|\vec{a}|^3}. \tag{9}$$

Let's now group the star's gravitational potential/force and the centrifugal potential/force into one term and Taylor expand.

$$\begin{aligned}
\frac{GM_*}{|\vec{r}-\vec{a}|^2}\hat{r}_{ra} - \frac{GM_*|\vec{r}-\vec{a}|}{|\vec{a}|^3}\hat{r}_{ra} &= \frac{GM_*}{|\vec{a}|^3}\left(\frac{|\vec{a}|^3 - |\vec{r}-\vec{a}|^3}{|\vec{r}-\vec{a}|^2}\right)\hat{r}_{ra} \\
&= \frac{GM_*}{|\vec{a}|^2}\left(\frac{1 - |\frac{\vec{r}}{\vec{a}} - 1|^3}{|\frac{\vec{r}}{\vec{a}} - 1|^2}\right)\hat{r}_{ra} \\
&= \left(\frac{3GM_*|\vec{r}|}{|\vec{a}|^3}\cos(\phi) + \mathcal{O}\left(\left(\frac{|\vec{r}|}{|\vec{a}|}\right)^2\right)\right)\hat{r}_{ra}
\end{aligned}$$

$$\begin{aligned}
-\frac{GM_*}{|\vec{r}-\vec{a}|} + \frac{GM_*|\vec{r}-\vec{a}|^2}{2|\vec{a}|^3} &= -\frac{GM_*}{2|\vec{a}|^3}\left(\frac{2|\vec{a}|^3 - |\vec{r}-\vec{a}|^3}{|\vec{r}-\vec{a}|}\right) \\
&= -\frac{GM_*}{2|\vec{a}|}\left(\frac{2 - |\frac{\vec{r}}{\vec{a}} - 1|^3}{|\frac{\vec{r}}{\vec{a}} - 1|}\right) \\
&= \left(-\frac{GM_*}{2|\vec{a}|} - \frac{2GM_*|\vec{r}|}{|\vec{a}|^2}\cos(\phi) + \mathcal{O}\left(\left(\frac{|\vec{r}|}{|\vec{a}|}\right)^2\right)\right)
\end{aligned}$$

$$f_{eff}(\vec{r}) = -\frac{GM_P}{|\vec{r}|^2}\hat{r} + \frac{3GM_*|\vec{r}|}{|\vec{a}|^3}\cos(\phi)\hat{r}_{ra} + \mathcal{O}\left(\left(\frac{|\vec{r}|}{|\vec{a}|}\right)^2\right). \tag{10}$$

$$u_{eff}(|\vec{r}|) = -\frac{GM_P}{|\vec{r}|} - \frac{GM_*}{2|\vec{a}|} - \frac{2GM_*|\vec{r}|}{|\vec{a}|^2}\cos(\phi) + \mathcal{O}\left(\left(\frac{|\vec{r}|}{|\vec{a}|}\right)^2\right). \tag{11}$$

Note the potential is the potential from the planet, plus half the potential from the star and some funny "tidal" potential. The half potential from the star can be understood from the virial theorem, since the effective potential is really a kinetic energy term moved into the potential ($U_{eff} = \frac{1}{2}m\dot{\phi}^2 + U$), thus the kinetic energy plus the potential energy is half the potential energy. Assumptions: circular orbit ($|\vec{a}| = constant$), $M_P << M_*$, and $|\vec{r}| << |\vec{a}|$ to ignore higher order terms.

What is the centrifugal force? Consider a Lagrangian description! Given some mass fixed to a 2-dimensional plane, we can write $\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$. If the potential is independent of ϕ, then ϕ is a cyclic coordinate, meaning $\partial_{\dot{\phi}}\mathcal{L} = constant = L$. Therefore we can move the kinetic energy involving ϕ into the potential, creating an effective potential, since it is only a function of r, namely $-\frac{1}{2}mr^2\dot{\phi}^2 = -\frac{L^2}{2mr^2}$ (the negative since we are moving it from the kinetic to potential term and in Lagrangian formalisms they differ by a sign). By doing this we have reduced the problem to a 1-dimensional problem, namely r. Physically what has happened is we have shifted into a rotating reference frame co-rotating with the particle. Thus this effective potential term is what is known as the centrifugal potential, now we can get the centrifugal force by taking the negative gradient of this potential term, $-\nabla(-\frac{1}{2}mr^2\dot{\phi}^2) = mr\dot{\phi}^2\,\hat{r}$.

Unfortunately, the potential description does not yield the full suite of fictitious forces. This is because we thought only of stationary positions in our field; while the Coriolis force discusses the force felt from moving from point to point, and the Euler force talks about spinning up or down our frame itself. These two effects slightly complicate the picture and will be discussed in future notes.