Dr. Murray-Clay Group Meeting Notes
Week 1

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Written up by: John McCann

1 Parker Model of Solar Winds

1.1 Momentum Equation

For an isothermal, stationary wind we consider the Parker Model which states that

$$u\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} + \frac{GM_*}{r^2} = 0. \tag{1}$$

Where u is the velocity of the fluid, ρ is the density, p is the pressure, G is Newton's gravitational constant and M_* is the mass of the star.

1.2 Mass Conservation

We adopt an Eulerian specification of the flow field for the derivation of the mass conservation. Fixing some volume element in space we can calculate the total mass within the volume element as

$$M = \iiint_V \rho \,\mathrm{d}V. \tag{2}$$

We now think of how the total mass enclosed changes with time in terms of the mass leaving the volume element. This is the mass flux through an area element, integrated over the total surface area of the volume element. By convention we chose $\hat{n}$ to point outward, so what we have described is the net mass flow outwards, which is equal to negative the change in mass or negative net mass flow inwards.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\oiint_{\partial V} \vec{\Phi}_M \cdot \hat{n}\,\mathrm{d}S = -\oiint_{\partial V} (\rho\,\vec{u}) \cdot \hat{n}\,\mathrm{d}S. \tag{3}$$

Noting we have a closed surface integral of a flux, we invoke Gauss's theorem to rewrite the change of mass in time as

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\iiint_V \nabla \cdot (\rho\,\vec{u})\,\mathrm{d}V. \tag{4}$$

Differentiating (2) with respect to time and setting the RHS equal to the RHS of (4) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\,\mathrm{d}V = -\iiint_V \nabla \cdot (\rho\,\vec{u})\,\mathrm{d}V. \tag{5}$$

Since the volume element is fixed in space throughout time, we may commute the time derivate with the volume integral. Moreover the total time derivative of this integral is equal to the partial time derivative of the integral. Since these are the same integrals we may also add the integrands together by bringing the divergence term over to the LHS

$$\iiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\,\vec{u})\right] \mathrm{d}V = 0. \tag{6}$$

We have not put any restrictions on what V is, therefore this must be true for any arbitrary volume element. Thus we can deduce that the integrand must be zero

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\,\vec{u}) = 0. \tag{7}$$

This is our conservation of mass equation which can be rewritten to talk about the change of density in time as a function of the divergence of the mass flux.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\,\vec{u}). \tag{8}$$

1.3 Momentum Conservation

Begin the same as we did for mass conservation and write the total momentum in a volume element as

$$\vec{P} = \iiint_V \rho\,\vec{u}\,\mathrm{d}V. \tag{9}$$

We are interested in finding how the momentum in the volume element changes through time. To simplify the problem let's consider the flow of one component of the momentum, let's call this P_i for the i-th direction of momentum. So the i-th component of momentum in the volume element is

$$P_i = \iiint_V \rho\,u_i\,\mathrm{d}V. \tag{10}$$

This momentum can change with time due to several factors. We will consider: the flow of the fluid carrying momentum out or in of the volume element, internal fluid pressures on the volume element and external volumetric forces acting throughout the volume element.

The flow of the fluid will give rise to a momentum flux through the surface of the volume (N.B. the flow can carry some i component of the momentum in or out of the box through a fluid flow in the j-th direction). We can then write the change in momentum in the i-th direction as the momentum flux through the surface of the volume

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\oiint_{\partial V} \Phi_{P_i} \cdot \hat{n}\,\mathrm{d}S = -\oiint_{\partial V} (\rho\, u_i)\,\vec{u} \cdot \hat{n}\,\mathrm{d}S. \tag{11}$$

How to think of the change in momentum due to internal pressure from the fluid is similar to the external forces ($\vec{F}_p = p\vec{A} = \dot{\vec{P}}$). At each face of the volume element there will be a pressure exerted by the surrounding fluid. Pressure that will change the momentum in the i-th direction across some area element is $-p\,\hat{i} \cdot \hat{n}\,\mathrm{d}S$. Thus the change in the i-th direction of the momentum will be

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\oiint_{\partial V} p\,\hat{i} \cdot \hat{n}\,\mathrm{d}S. \tag{12}$$

The change in momentum, in the i-th direction, from external forces ($\vec{F}_{ext} = \dot{\vec{P}}$) can be written as the volume integral of the volumetric force, in the i-th direction, acting on the fluid

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = \iiint_V \rho\, f_i\,\mathrm{d}V. \tag{13}$$

Applying Gauss's Theorem to (11) and (12) turns them into

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\iiint_V \nabla \cdot (\rho\, u_i\, \vec{u})\,\mathrm{d}V, \tag{14}$$

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\iiint_V \nabla \cdot (p\,\hat{i})\,\mathrm{d}V. \tag{15}$$

Collecting the RHS of (13), (14) and (15), we find the total change of momentum due to all three effects is

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = -\iiint_V \nabla \cdot (\rho\, u_i\, \vec{u})\,\mathrm{d}V - \iiint_V \nabla \cdot (p\,\hat{i})\,\mathrm{d}V + \iiint_V \rho\, f_i\,\mathrm{d}V. \tag{16}$$

Since these integrals all talk about the same volume element we can combine these integrals into one big integrand. We can also rewrite the LHS by taking a time derivative of (10), which again commutes with the volume integral and is equivalent to a partial derivative with respects to time. This tells use that

$$\iiint_V \frac{\partial(\rho\, u_i)}{\partial t} \mathrm{d}V = \iiint_V \left[-\nabla \cdot (\rho\, u_i\, \vec{u}) - \nabla \cdot (p\, \hat{i}) + \rho\, f_i \right] \mathrm{d}V. \tag{17}$$

With the same arguments from the mass conservation section, we say that

$$\frac{\partial(\rho\, u_i)}{\partial t} = -\nabla \cdot (\rho\, u_i\, \vec{u}) - \nabla \cdot (p\, \hat{i}) + \rho\, f_i. \tag{18}$$

We can further simply this equation by using the product rule, mass conservation and the fact that the divergence of a unit vector is zero

$$\frac{\partial(\rho\, u_i)}{\partial t} = u_i \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial t} = -u_i \nabla \cdot (\rho \vec{u}) + \rho \frac{\partial u_i}{\partial t}.$$

$$-\nabla \cdot (\rho\, \vec{u}\, u_i) = -u_i \nabla \cdot (\rho\, \vec{u}) - \nabla u_i \cdot \rho\, \vec{u}.$$

$$-\nabla \cdot (p\, \hat{i}) = -\nabla p \cdot \hat{i} - p \nabla \cdot \hat{i} = -\nabla p \cdot \hat{i}.$$

Thus

$$\rho \frac{\partial u_i}{\partial t} = -\nabla u_i \cdot \rho\, \vec{u} - \nabla p \cdot \hat{i} + \rho f_i. \tag{19}$$

One last note is that $-\nabla u_i \cdot \rho\, \vec{u} = -\rho\, [(\vec{u} \cdot \nabla) \vec{u}]_i$. Writing this out in components we see this more easily

$$-\nabla u_i \cdot \rho\, \vec{u} = -\rho\, \vec{u} \cdot \nabla u_i = -\rho \sum_j u_j \frac{\partial u_i}{\partial x_j} = -\rho \left(\sum_j u_j \frac{\partial}{\partial x_j} \right) u_i = -\rho (\vec{u} \cdot \nabla) u_i = -\rho\, [(\vec{u} \cdot \nabla) \vec{u}]_i .$$

With this knowledge we can now write an equation for momentum conservation. We do so by noting that (19) is purely depended on the i-th component, this allows us to simply slap these together with their respective unit vector and sum the orthogonal unit vectors to get a vector.

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = -\frac{1}{\rho} \nabla p + \vec{f}. \tag{20}$$

This is the momentum conservation which takes only into account internal pressure and external forces acting on a flow.

1.4 Parting Question

Assume a plant of mass M has an atmosphere in hydrostatic equilibrium. Using an isothermal model find the density as a function of distance. What is the pressure at infinity? What does this mean?