Dr. Murray-Clay Group Meeting Notes
Meeting 4

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1 Ionization Equilibrium

From Murray-Clay et al. 2009, equation 7 states

$$\frac{F_{UV}\, e^{-\tau}}{h\nu_0} n_0 \sigma_{\nu_0} = n_+^2 \alpha + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 n_+ v \right). \tag{1}$$

Where n_0 is the number density of neutral hydrogen, σ_{ν_0} is the cross section for photoionization of hydrogen from light at a frequency of ν_0, F_{UV} is the flux in the uv, τ is the optical depth, α is the recombination coefficient, n_+ is the number density of ionized hydrogen, r is the distance to the center of the planet, and v is the velocity of the wind.

1.1 L.H.S.

The L.H.S. of this equation represents the number of ionizations per unit volume, per unit time. Picking apart the expression, let's look at

$$\frac{F_{UV}\, e^{-\tau}}{h\nu_0}. \tag{2}$$

The term F_{UV} is the flux produced from the star in the uv, which is the amount of energy emitted at ν_0 per area per time. To get the number of photons per area per time we divide F_{UV} by the energy of a photon at that frequency, namely, $h\nu_0$. Thus the term

$$\frac{F_{UV}}{h\nu_0} = \frac{\text{the number of photons at } \nu_0}{\text{area} \cdot \text{time}}.$$

However, in our model we will include some extinction from material between the star and the planet. Thus the flux that reach the planet is attenuated by a factor of $\exp(-\tau)$. Therefore, accounting for this extinction, the true number of photons to reach a position r per time per area is given by (2).

To find the number of ionizations per unit volume per time, i.e. (1), we will want to find the number of collisions per number of photon per length, in addition to our number of photons per area per time, i.e., (2).

The number of collisions per number of photons per length, is the number of collisions per time per number of photons times the amount of time elapsed divided by distanced traveled by a photon in that time. More compactly, almost incorrectly, said the collision rate divided by the speed of light.

$$n_0\sigma_{\nu_0} = \frac{\text{collision rate}}{\text{speed of light}}. \tag{3}$$

Thus putting (2) and (3) together we exactly get (1), and our work is done.

To illuminate what was just done, we can reimagine the picture. Using our physical intuition, unit analysis and cleverness

$$\text{total \# of collisions} = \text{collision rate} \cdot \text{\# of photons} \cdot \text{time},$$

$$\frac{\text{total \# of collisions}}{\text{time}} = \text{collision rate} \cdot \text{\# of photons},$$

$$\frac{\text{total \# of collisions}}{\text{time} \cdot \text{volume}} = \text{collision rate} \cdot \frac{\text{\# of photons}}{\text{volume}}. \tag{4}$$

Now we can also figure out how many photons there are in a given volume,

$$\frac{\text{\# of photons}}{\text{volume}} = \frac{\text{\# of photons}}{\text{area} \cdot \text{time}} \cdot time \cdot \frac{1}{\text{length}},$$

$$\frac{\text{\# of photons}}{\text{volume}} = \frac{F_{UV}e^{-\tau}}{h\nu_0} \cdot t \cdot \frac{1}{\text{ct}}.$$

Plugging this back into (4), and recalling what the collision rate expression is, we can see that

$$\frac{\text{total \# of collisions}}{\text{time} \cdot \text{volume}} = n_0\sigma c \cdot \frac{F_{UV}e^{-\tau}}{h\nu_0} \cdot t \cdot \frac{1}{\text{ct}}.$$

Thus we get that

$$\frac{\text{total \# of collisions}}{\text{time} \cdot \text{volume}} = n_0\sigma\frac{F_{UV}e^{-\tau}}{h\nu_0}.$$

Where we are using the word collision and ionization interchangeably.

1.2 R.H.S.

There are two terms on the R.H.S. The first being recombination and the second being advection. Both of these will lead to a loss of ions, since you lose one ion every time an electron recombines with an ion to form a neutral hydrogen, and advection is simply the outflow of sometime, physically leaving the system.

The R.H.S. is the continuity equation rewritten. Where the continuity equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot j = \sigma.$$

In this situation the rate of change of ion density with respect to time is the recombination rate term

$$\frac{\partial \rho}{\partial t} \to n_+^2 \, \alpha.$$

The divergences of the flux of ions is the amount of ions being transported out via advection, exactly as written.

$$\vec{\nabla} \cdot j \to \vec{\nabla} \cdot (n_+ \vec{v}) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \, n_+ \, \vec{v} \right).$$

These two terms together is equal to the number of recombinations per volume per time, which is what σ is defined as in the continuity equation.

Recombination. Following our previous argument on how to "derive" the ionization rate, we can use the same idea to talk about the recombination rate. The only difference is now we consider electrons colliding with ions, instead of photons colliding with neutral hydrogen. Reformulating (4) we replace photons with electrons and neutral hydrogen with ions to get

$$\frac{\text{total \# of collisions}}{\text{time} \cdot \text{volume}} = \text{collision rate} \cdot \frac{\text{\# of electrons}}{\text{volume}}.$$

We know that the collision rate will be the density of ions, times the cross sectional area, times the speed of the electrons. We will denote these as n_+, σ_n and v respectively. Where σ_n is the cross sectional area for an electron becoming bound in the *n-th* hydrogen energy level.

We want to consider that an electron can recombine into any of the hydrogen energy levels, and thus we will sum over all possible recombinations. Moreover, the cross sectional area will depend on velocity and we will have some distribution of velocity for a gas of electrons. There we will integrate over all possible velocities to find the recombination rate for a single energy state as follows

$$\alpha_n = \int \sigma_n(v) v f(v) dv.$$

Where $f(v)$ is the distribution of the electrons (Maxwellian). Note this distribution depends on the temperature of the gas by definition of temperature. Now to get the total recombination rate we will sum over all recombination coefficients at various energy levels, and denote this as α_A.

$$\alpha_A = \sum_{n=0}^{\infty} \alpha_n,$$

This is called the total recombination coefficient. We want to exclude recombinations into the ground state, since we will assume that the emitted photon will quickly be resorbed and ionize a neutral hydrogen. Resulting in no net change in the ionization rate. This is call the recombination rate B, often denoted as α_B,

$$\alpha_B = \sum_{n=1}^{\infty} \alpha_n.$$

Now we can say that the collision rate in this situation is as follows

$$\text{collision rate} = \alpha_B \, n_+.$$

Therefore all that is left to complete the recombination rate is to mutliple by the number density of electrons. Note that we are considering a neutral gas, such that $n_e = n_+$, thus we can write the recombination rate as

$$\frac{\text{total \# of collisions}}{\text{time} \cdot \text{volume}} = n_+^2 \alpha_B.$$

Where we use the word collision and recombination interchangeably. As referenced in the paper from Storey & Hummer 1995, the value for $\alpha_B = 2.7 \times 10^{-13} (T/10^4 K)^{-0.9}$.

Advection. Skipping motivation for advection, I will just not that we are using spherical coordinates and the equation as written in (1) simply expands $\nabla \cdot \vec{A}$, assuming that $\vec{A}$ is only a function of r. In our model we only consider a radial velocity field.

1.3 Rewriting the advection

The advection term in (1) is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 n_+ v \right),$$

note that this is equivalently

$$\vec{\nabla} \cdot (n_+ \vec{v}).$$

Neglecting the mass contribution from ionized elections in the gas, we can say that $\rho = n\mu$, where $n = n_0 + n_+$, the number density of neutral plus ionized hydrogens, and μ is the mean molecular mass. Note that

$$\mu = \frac{n_0 m_H + n_+ m_H + n_e m_e}{n_0 + n_+ + n_e} \approx \frac{n}{n + n_+} m_H = \frac{1}{1 + f_+} m_H.$$

Where we have said the mass of the electron is negligible, the number of electrons is equal to the number of ions, and $f_+ = \frac{n_+}{n}$.

Now we can rewrite the advection as follows

$$\begin{aligned}
\vec{\nabla} \cdot (n_+ \vec{v}) &= \vec{\nabla} \cdot \left(\frac{n_+}{\rho} \rho \vec{v} \right) \\
&= \frac{1}{\mu} \vec{\nabla} \cdot (f_+ \rho \vec{v}) \\
&= \frac{1}{\mu} \left[\vec{\nabla} f_+ \cdot \rho \vec{v} + f_+ (\nabla \cdot (\rho \vec{v})) \right] \\
&= nv \frac{\partial f_+}{\partial r}.
\end{aligned}$$

Which is what we wanted. Note that we used mass conservation to say that $\nabla \cdot (\rho \vec{v}) = 0$, and that there is only a radial component of the velocity.

1.4 Notes about collision rates

Consider hydrogen with a cross sectional area of size σ. Then a single photon passing through that area is going to interact with that atom, by the definition of cross sectional area. The photon will traverse a distance $c\,t$ in a time t, where c is the speed of that object. If there is a number density n of hydrogen in the volume swept out by the photon, where that volume is $\sigma\, c\, t$, then there will be $n\, \sigma\, c\, t$ interactions. Thus

$$\text{collision rate} = n\sigma c,$$

$$\text{time between collisions} = \frac{1}{n\sigma c},$$

$$\text{mean free path} = \frac{1}{n\sigma}.$$

1.5 Notes about optical depth

From wikipedia, "Optical depth is defined as the negative natural logarithm of the fraction of radiation (e.g., light) that is not scattered or absorbed on a path." Better said

$$I(\tau) = I_0 e^{-\tau}.$$

From radiative transfer the optical depth between points a and b is defined as

$$\tau \equiv \int_a^b \alpha(s) ds.$$

Where α is the absorvitiy coefficient, which we define as σn. Therefore in our situation of the star and planet

$$\tau(r,a) = \int_r^a n_0(s)\sigma_{\nu_0} ds.$$

Where a is the distance from the center of the planet to the star. Note that the cross section is independent of our location on the path, and thus can be pulled outside the integral. Further, we can say that $n_0(s)$ is essentially 0 at distances greater than a. If that is true then integrating out to infinity will not change the optical depth. Thus

$$\tau(r) = \sigma_{\nu_0} \int_r^\infty n_0(s) ds.$$

2 Critical point of Parker Wind

Picking up where we left off in Meeting 1 notes, the momentum equation for Parker winds is as follows

$$v\frac{\partial v}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{GM_*}{r^2} = 0. \tag{5}$$

2.1 Reformulating the equation in terms of velocities

Using the ideal gas law, defining the speed of sound as $c_s^2 = \frac{kT}{\mu}$, we can rewrite the pressure term as

$$\frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{1}{\rho}\left(c_s^2 \frac{\partial \rho}{\partial r}\right).$$

Where $P = nkT = c_s^2 \rho$. Now using mass conservation, we can say the mass passing through a sphere at a given radius is

$$\dot{M} = 4\pi r^2 \rho(r) v(r) = 4\pi F_m.$$

Where F_m is the flux of mass per steradian, which in our model is a constant since we state the planet is losing mass at a constant rate and the wind is in a steady state. Now we can say what the density is as a function of r

$$\rho(r) = \frac{F_m}{r^2\, v(r)}.$$

Therefore

$$\begin{aligned}
\frac{1}{\rho(r)}\frac{\partial \rho(r)}{\partial r} &= \frac{1}{\rho(r)}\frac{\partial F_m}{\partial r}\frac{1}{r^2\, v(r)} + \frac{F_m}{\rho(r)}\frac{\partial}{\partial r}\left(\frac{1}{r^2\, v(r)}\right), \\
&= \frac{F_m}{\rho(r)}\left(-\frac{2}{r^3\, v(r)} - \frac{1}{r^2\, v(r)^2}\frac{\partial v(r)}{\partial r}\right), \\
&= -\frac{2}{r} - \frac{1}{v(r)}\frac{\partial v(r)}{\partial r}.
\end{aligned}$$

Now we have a way of expressing the second term of (5), in terms of velocities. For the third term we can rewrite it in terms of the escape velocity.

$$v_{esc}^2(r) = \frac{2GM_*}{r}.$$

Thus (5) becomes

$$\begin{aligned}
\left(v - \frac{c_s^2}{v}\right)\frac{\partial v}{\partial r} &= \frac{2\,c_s^2}{r} - \frac{v_{esc}^2}{2r}, \\
\frac{1}{v}\frac{\partial v}{\partial r} &= \frac{1}{2r}\frac{4\,c_s^2 - v_{esc}^2}{(v^2 - c_s^2)}. \qquad (6)
\end{aligned}$$

From (6) we see that there is a critical point in the velocity field when $v_{esc}^2 = 4c_s^2$, or at a radius of $r = \frac{GM}{2c_s^2}$, except when the denominator is equally 0. Thus at the critical point there could be a non-zero velocity gradient if $v^2 = c_s^2$. Investigating this case we find that

$$\left(\frac{\partial v}{\partial r}\right)_{r_c} = \frac{\pm 2c_s^2}{GM_*}.$$

Analyzing (6) further we can get a feel for the various solutions. We can physically argue that the isothermal sound speed is less than the escape velocity below the critical radius.[1] Therefore at distances less than the critical radius the numerator is negative, and the gradient then depends on wether the velocity is greater than or less than the sound speed. If the velocity is less than the sound speed, below the critical radius the velocity field has a positive gradient–and vice versa. After the critical point the velocity will grow if it was initially greater than the sound speed or decrease if it was initially less than the sound speed, UNLESS it is the critical solutions, which hits the sound speed at the critical radius. This is the interesting case in which you get a transonic solution. See Fig. 1 visual representation of various solutions.

[1] At distances $r > \frac{GM_*\mu}{2kT}$, the isothermal sound speed is greater than the escape velocity. Since the sound speed is constant throughout the wind, is equal to the escape velocity at the critical distance, and escape velocity increases as you move in, then it is clear that the sound speed is less than the escape velocity below the critical radius.

Figure 3.1 Solutions of the momentum equation of an isothermal stellar wind with gas pressure and gravity in terms of v/a versus r/r_c. For this particular case $r_c = 5\ R_*$. The different curves are described in the text. Curve 1 (thick) is the transonic solution with increasing velocity through the critical point r_c where $v = a$.

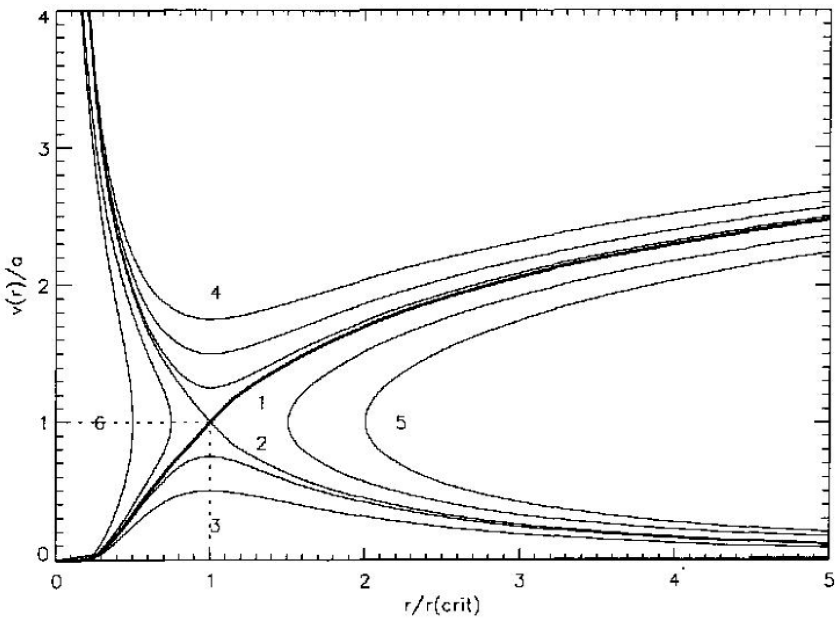

Figure 1: Taken from Introduction to Stellar Winds by Lamers & Casselli. See page 65 for further discussion of solutions.

3 Parting Questions

What is the minimum mass for a solar nebula? Consider a disc with surface density $\Sigma_g \sim 2 \times 10^3 (a/AU)^{-1}$ g/cm^2 (Note that our solar system has $(a/AU)^{-3/2}$, but most observations of other disk larger than 20 AU suggestion a power of -1). A dust to gas ratio of $f_d \sim 10^{-2}$, and the central body is accreting at a rate of $\dot{M} \sim (10^{-8} - 10^{-6}) M_\odot$/yr, the typical accretion rate of a young sun like star.

How much solid mass verse gas mass is there at $1/2a$, a and $2a$?

Consider Toomre's Q

$$Q = \frac{c_s \Omega}{\pi G \Sigma},$$

where c_s is the sound speed in the disc, Ω is the angular frequency of the disc, G is Newton's gravitational constant and Σ is the surface density. Note that $Q < 1$ is unstable to self gravity. Where to be unstable self gravity overcomes pressure. On large scales gravity overcomes pressure, and on small scales gravity wins out over tidal gravity.