Collisionless Boltzmann Equation

Let's consider a probability distribution function (*pdf*) of a single particle, f, in a phase space described by canonical coordinates $(\vec{q}, \vec{p})$. That is to say that $f(\vec{q}, \vec{p}, t)\mathrm{d}^3\vec{q}\ \mathrm{d}^3\vec{p}$ is the probability of that particle having $\vec{q} \in [\vec{q}, \vec{q} + \mathrm{d}^3\vec{q}]$ and $\vec{p} \in [\vec{p}, \vec{p} + \mathrm{d}^3\vec{p}]$ at time t. Note that by the inherent normalization of probability

$$\int \mathrm{d}^3\vec{q}\mathrm{d}^3\vec{p}\ f(q,p,t) = 1. \tag{1}$$

Restated, by assumption of the particles existence [first sentence], we can say it exist [it has some $(\vec{q}, \vec{p})$]. Thus we have seen that probability over all phase space must be conserved [1 is independent of time]. We can jump straight into a continuity equation for probability now, but it might be helpful to remind you of a physical situation invoking the same mathematical tools, to help ground this "abstraction". Recall that in continuum mechanics, in your run of the mill situations where mass is neither created or destroyed, that the total mass is conserved. Stated as

$$\int \mathrm{d}^3\vec{q}\ \rho(\vec{q}, t) = M. \tag{2}$$

That is to say that collecting all the mass over all space always adds up to M, independent of time. This is a less abstract and lower dimensional analogy to our *pdf*. From this "conservation of mass," we often derive or state as a known fact that

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \dot{\vec{q}}) = 0. \tag{3}$$

This is the continuity equation, and now carrying the analogy back to our higher dimensional *pdf* we say

$$\frac{\partial f}{\partial t} + \vec{\nabla}_q \cdot (f\dot{\vec{q}}) + \vec{\nabla}_p \cdot (f\dot{\vec{p}}) = 0. \tag{4}$$

To be clear about what I have almost asserted by analogy, this says, the change of our *pdf* at a given point $(\vec{q}, \vec{p})$ in phase space in time, $\partial_t f$, is equal to the negative divergence of the flux (an accumulation) of that *pdf* due to each canonical coordinate. Those terms are $-\partial_q(f\dot{q})$ for each spatial coordinate and, $-\partial_p(f\dot{p})$ for each momentum coordinate.

The unfamiliar flux driven by a "momentum velocity" might at first feel uncomfortable, to the uninitiated you might protest that this looks like an acceleration, not a velocity which talks about true spatial fluxes. The issue is that most of us experience life in spatial space. If we wish to understand this in a spatial manner, recall that a flux is the amount of something passing through a surface of some constant coordinate. Thus to do so you need a velocity with respect to that coordinate, $\dot{q}$, okay so we understand the spatial coordinates fine in this frame of thought. Our everyday spatial life tells us this is our traditional spatial velocity. If we wish to keep thinking like this then I have to insist that a surface of constant momentum is a surface with a velocity in our frame (okay not surprising). Thus to pass through this surface the particle will need to change it's velocity by accelerating, that is to say having a $\dot{p}$. Perhaps in the future it will be better to just thinking more abstractly, that to have a flux through a surface of any kind constant coordinate you need a velocity with coordinate (whatever that may be, spatial, momentum, Italian sausages, whatever).

Now let's use Hamilton's equations to rewrite out continuity equation using a subtle slight of math. Recall,

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{5}$$

Then our flux terms in the continuity equation become

$$\begin{aligned}
\tfrac{\partial}{\partial \vec{q}} \cdot (f\dot{\vec{q}}) + \tfrac{\partial}{\partial \vec{p}} \cdot (f\dot{\vec{p}}) &= \tfrac{\partial}{\partial \vec{q}} \cdot (f\tfrac{\partial H}{\partial \vec{p}}) - \tfrac{\partial}{\partial \vec{p}} \cdot (f\tfrac{\partial H}{\partial \vec{q}}) \\
&= \tfrac{\partial f}{\partial \vec{q}} \cdot \tfrac{\partial H}{\partial \vec{p}} - \tfrac{\partial f}{\partial \vec{p}} \cdot \tfrac{\partial H}{\partial \vec{q}} \\
&= \tfrac{\partial f}{\partial \vec{q}} \cdot \dot{\vec{q}} + \tfrac{\partial f}{\partial \vec{p}} \cdot \dot{\vec{p}} \ .
\end{aligned}$$

Putting all of this together we arrive at the anachronically named Vlasov equation

$$\frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \frac{\partial f}{\partial \vec{q}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} = 0. \tag{6}$$

This is usually known as the collisionless Boltzmann equation outside plasma physics. It is inherently collisions because we have assumed that there was only one particle, more on that shortly. Let's write out the equation if we stopped at the second line of the manipulation of the flux

$$\frac{\partial f}{\partial t} + \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial f}{\partial \vec{q}} - \frac{\partial H}{\partial \vec{q}} \cdot \frac{\partial f}{\partial \vec{p}} = 0. \tag{7}$$

This formulation is often useful we working in non-Cartisian coordinates and we do not wish to think about slightly tricker than usual $\dot{q}$ and $\dot{p}$. Notice that we can use the poisson bracket shorthand to rewrite it as

$$\frac{\partial f}{\partial t} + [f, H] = 0. \tag{8}$$

One last thought on the collisions Boltzmann equation is that the *pdf* is incompressible! Note we must be careful that we understand what being collisions means both mathematically and physically, coming very shortly. To see the incompressibility consider the Lagrangian derivative of the *pdf*

$$\frac{\mathrm{d}f}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \frac{\partial f}{\partial \vec{q}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} = 0. \tag{9}$$

If you are unfamiliar with the idea of incompressibility, mathematically is is as follows

$$\begin{aligned}
\tfrac{\mathrm{d}f}{\mathrm{d}t} \equiv \tfrac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \tfrac{\partial f}{\partial \vec{q}} + \dot{\vec{p}} \cdot \tfrac{\partial f}{\partial \vec{p}} &= \tfrac{\partial f}{\partial t} + \vec{\nabla}_{\vec{q}} \cdot \left(f\dot{\vec{q}}\right) + \vec{\nabla}_{\vec{p}} \cdot \left(f\dot{\vec{p}}\right) - f\left(\vec{\nabla}_{\vec{p}} \cdot \dot{\vec{p}}\right) - f\left(\vec{\nabla}_{\vec{q}} \cdot \dot{\vec{q}}\right) \\
&= -f\left(\vec{\nabla}_{\vec{p}} \cdot \dot{\vec{p}}\right) - f\left(\vec{\nabla}_{\vec{q}} \cdot \dot{\vec{q}}\right) \\
&= 0.
\end{aligned}$$

The first line is just rewriting the math, and second line uses the continuity equation (4), to set the first three terms to zero, and the last line is what we found from the Lagrangian derivative. Thus the sum of divergences of the coordinate velocities is zero ($\vec{\nabla} \cdot \vec{w} = 0$). This is called an incompressible flow since there is no change to f with respects to time in a Lagrangian description of the continuum [you ride around on a particle and it doesn't suddenly disappear or become twice as probable to be beneath you —it's just there].

Consider an inertial Caresian coordinate system, in which a single particle of mass m has the Hamiltonian, $H = \frac{p^2}{2m} + \Phi(\vec{x}, t)$. Then the Collisionless Boltzmann Equation takes the form

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial f}{\partial \vec{x}} - \frac{\partial \Phi}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{p}} = 0. \tag{10}$$

Collisional Boltzmann Equation

In the previous section we wrote a *pdf* for a single particle. What followed would have been the exact same if we had instead considered a collection of non-interacting particles. That's because each particle is still effected my external forces $\dot{\vec{p}} = \vec{F}$, and advection $\dot{\vec{q}} \cdot \vec{\nabla}$.

Shortcut to Boltzmann Equation

Just as the Lagrangian derivative is a trivial application of the chain rule, which we have applied physically meanings too, so is the Boltzmann equation. Recall that $f(q_1, ..., q_n, p_1, ..., p_n, t)$ and consider the total differential of f

$$\mathrm{d}f = \partial_t f \ \mathrm{d}t + \sum_{i=1}^{n} \partial_{q_i} f \ \mathrm{d}q_i + \partial_{p_i} f \ \mathrm{d}p_i. \tag{11}$$

Then finding the total derivative with respects to time is straight forward

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \partial_t f + \sum_{i=1}^{n} \partial_{q_i} f \ \frac{\mathrm{d}q_i}{\mathrm{d}t} + \partial_{p_i} f \ \frac{\mathrm{d}p_i}{\mathrm{d}t} \\ &= \partial_t f + \sum_{i=1}^{n} \partial_{q_i} f \ \dot{q}_i + \partial_{p_i} f \ \dot{p}_i \\ &= \partial_t f + \partial_{\vec{q}} f \cdot \dot{\vec{q}} + \partial_{\vec{p}} f \cdot \dot{\vec{p}} \ . \end{aligned}$$

This elementary mathematical manipulation has arrived at the Boltzmann equation exactly! However, to be instructive let me illuminate that for us. Standard notation for $\partial_{\vec{q}} = \vec{\nabla}$, then invoking Newton's second law $\dot{\vec{p}} = \vec{F}$, the change in momentum is the sum of all external forces (dubbed $\vec{F}$). Lastly let's recall our argument that the total time derivative of f at a point in phase space is due to particle diffusion, external forces on that particle and possibly collisions. Since collisions have not be written into f explicitly via coordinates, we say that total time derivative of f is equal to the change in f due to only these collisions. Rephrased one last time, the change in f due to the coordinates are accounted for in the total derivative, $\mathrm{d}f$, thus our approach to writing the total time derivative neglects non-coordinate effects. Therefore the total time derivative of f is not equal to zero, and we chose to say the total rate of change in f is equal to the partial rate of change in f due only to non-coordinate effects, such as collisions.

All this together spells out the physicist version of the equation

$$\frac{\partial f}{\partial t} + \vec{q} \cdot \vec{\nabla} f + \vec{F} \cdot \frac{\partial f}{\partial \vec{p}} = \left(\frac{\partial f}{\partial t} \right)_{collisions}. \tag{12}$$

Thus we see that the real power of the physicist was the non-trivialization of a trivial mathematical matter.