Density profile

Consider an atmosphere described by the fluid equations in hydrostatic equilibrium, then

$$\vec{\nabla} P = -\rho \vec{\nabla} \phi. \tag{1}$$

To simplify the problem take the atmosphere to be spherically symmetric, such that we have a one dimensional problem. We may then rewrite hydrostatic equilibrium using $\vec{\nabla}\phi = -\frac{GM_{\rm enc}}{r^2}\rho(r)$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_{\rm enc}}{r^2}\rho(r) = -\frac{4\pi G\rho(r)}{r^2}\int_0^r \rho(r')\, r'^2\, \mathrm{d}r' = \frac{-GM_p}{r^2}\rho(r) - \frac{4\pi G}{r^2}\rho(r)\int_{R_p}^r \rho(r')\, r'^2\, \mathrm{d}r'. \tag{2}$$

Next assume the mass in the atmosphere is much less than the mass of the central object, M_p. This allows us to drop the second term which has to do with the gravitational pull on the atmosphere at r from the atmosphere below it. Thus we arrive at

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_p}{r^2}\rho(r). \tag{3}$$

Next we will take the atmosphere to be isentropic, with isentropic exponent γ, i.e., $P = K\rho^\gamma$. Thus we may solve for the density structure of an atmosphere under these assumptions

$$K\gamma\rho^{\gamma-1}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{GM_p}{r^2}\rho \quad \rightarrow \quad \int \rho^{\gamma-2}\, \mathrm{d}\rho = -\frac{GM_p}{\gamma K}\int \frac{1}{r^2}\, \mathrm{d}r. \tag{4}$$

It is at this point which we will select a reference location which we will assign initial conditions. Let's start by integrating from R_0 with local density ρ_0.

$$\begin{aligned}
\int_{\rho_0}^{\rho(r)} \rho^{\gamma-2}\, \mathrm{d}\rho &= -\frac{GM_p}{\gamma K}\int_{R_0}^r \frac{1}{r^2}\, \mathrm{d}r \\
\frac{1}{\gamma-1}\left(\rho(r)^{\gamma-1} - \rho_0^{\gamma-1}\right) &= \frac{GM_p}{\gamma K}\left(\frac{1}{r} - \frac{1}{R_0}\right) \\
\rho(r) &= \left[\frac{(\gamma-1)GM_p}{\gamma K}\left(\frac{1}{r} - \frac{1}{R_0}\right) + \rho_0^{\gamma-1}\right]^{\frac{1}{\gamma-1}}.
\end{aligned}$$

For an ideal gas, we can determine the constant K by noting $\frac{P}{\rho} = c_s^2$, where c_s is the isothermal sound speed. Thus $\dfrac{P_0}{\rho_0} = c_{s,0}^2 = K\rho_0^{\gamma-1} \rightarrow K = c_{s,0}^2\ \rho_0^{1-\gamma}$. Thus for ideal isentropic atmospheres of negligible mass

$$\rho(r) = \rho_0\left[\frac{(\gamma-1)GM_p}{\gamma\, c_{s,0}^2}\left(\frac{1}{r} - \frac{1}{R_0}\right) + 1\right]^{\frac{1}{\gamma-1}}. \tag{5}$$

It will be algebraically useful to define a "radius" $R_c = \dfrac{(\gamma-1)GM_p}{\gamma\, c_{s,0}^2}$. Then we can rewrite the density structure as

$$\rho(r) = \rho_0 \left[R_c \left(\frac{1}{r} - \left(\frac{1}{R_0} - \frac{1}{R_c} \right) \right) \right]^{\frac{1}{\gamma-1}}. \tag{6}$$

In this form two things become apparent. First when $\frac{1}{r} = \frac{1}{R_0} - \frac{1}{R_c}$ the density goes to zero. We will now define that radius as $R_z = \dfrac{R_0 R_c}{R_c - R_0}$, the zero radius. Second is to notice if $R_c \leq R_0$, then the density never goes to zero (easily seen as R_z being negative)! Noting this, R_c is called the critical radius, as it splits the atmospheres into two regimes; of course as written it is not very intuitive how our arbitrary radius R_0 and R_c are related to determine two fundamentally different regimes.

To explore this phenomena let's rewrite the inequality by unpacking R_c

$$\begin{aligned} R_c &\leq R_0, \\ \frac{(\gamma-1)GM_p}{\gamma\, c_{s,0}^2} &\leq R_0, \\ \frac{\gamma-1}{\gamma}\frac{GM_p}{R_0} &\leq c_{s,0}^2, \\ \frac{\gamma-1}{2\gamma} v_{\rm esc}^2 &\leq c_{s,0}^2, \end{aligned}$$

If we pick our thermal speed to be a factor of $\dfrac{\gamma-1}{2\gamma}$ greater than our escape speed, the planet's atmosphere extends out to infinity. We will call atmosphere's with $c_{s,0}^2 \leq \dfrac{\gamma-1}{2\gamma} v_{\rm esc}^2$, thermally bounded atmospheres, and otherwise thermally unbounded. Note our choice of R_0 remains arbitrary, but our choice of the temperature we assign at that point, $c_{s,0}^2$, determines the structure of the atmosphere.

To gain some perspective on what to expect in nature, let's consider Jupiter with a pure ideal hydrogen atmosphere: $M_J = 2 \times 10^{30}$g, $R_J = 7 \times 10^9$cm, then $c_{s,0}^2 \geq 2.76 \times 10^6$cm/s or equivalently $T \geq 9.2 \times 10^4$ K. Thus we really wouldn't expect thermally unbound atmospheres in hydrostatic equilibrium unless they were Jupiter sized stars.

By far the most convent way to write the density profile is

$$\rho(r) = \rho_0 \left[R_c \left(\frac{1}{r} - \frac{1}{R_z} \right) \right]^{\frac{1}{\gamma-1}} = \rho_0 \left[f(r) \right]^{\frac{1}{\gamma-1}}. \tag{7}$$

Since our gas was described by a polytropic relationship, specifically an isentrope, then we also know the pressure profile

$$P(r) = K\rho(r)^{\gamma} = c_{s,0}^2\, \rho_0\, \left[f(r) \right]^{\frac{\gamma}{\gamma-1}}. \tag{8}$$

The last equality held for an ideal gas, thus we can also use the ideal gas equation to obtain a temperature profile

$$T(r) = \frac{\mu}{k}\frac{P}{\rho} = \frac{\mu\, c_{s,0}^2}{k} \left[f(r) \right]^{\frac{\gamma}{\gamma-1} - \frac{1}{\gamma-1}} = T_0\, f(r). \tag{9}$$

Note that all quantities are dependent on the function $f(r)$ to various powers, as expected given the invoked relationships. Thus all quantities goes to zero R_z. Furthermore note that $\frac{\mathrm{d}f(r)}{\mathrm{d}r} = -\frac{R_c}{r^2} < 0$ for all r, so these quantities are strictly decreasing outwards. Reminder $f(R_0) = 1$ and $f(R_z) = 0$.

Scale Heights: F, G and H

In this section we will discuss scale heights in atmospheres, and come across three of interest: $F_n(r)$, $G(r)$ and $H_n(r)$. In the isothermal case, all three scale height functions $F_n(r)$, $H_n(r)$ and $G(r)$ are equal to each other, denoted as $H_{\mathrm{iso}}(r)$.

$$H_{iso}(r) = \frac{r^2\, kT(r)}{GM\mu} = \frac{\gamma - 1}{\gamma} r \left(1 - \frac{r}{R_z}\right). \tag{10}$$

Call the number of e-foldings of the quantity $X_n(r)$ between $r = a$ and $r = b$, $N_e(a, b)$. Take as the intuitive definition of $N_e(a, b)$ as

$$X_n(b) = X_n(a)\, e^{N_e(a,b)}. \tag{11}$$

Therefore we can solve for $N_e(a, b)$ and rewrite it as

$$N_e(a, b) = \log\left(\frac{X_n(b)}{X_n(a)}\right) = \int_{X_n(a)}^{X_n(b)} \mathrm{d}\ln X_n = \int_a^b \frac{\mathrm{d}\ln X_n(r)}{\mathrm{d}r}\, \mathrm{d}r. \tag{12}$$

To progress let's examine the integrand, $\frac{\mathrm{d}\ln X_n(r)}{\mathrm{d}r} = \frac{1}{X_n(r)}\frac{\mathrm{d}X_n(r)}{\mathrm{d}r}$. Dimensionally, it has inverse units of what one integrates with respect to, traditionally spatial distance. Mathematically, it is the infinitesimal relative change of the quantity $X_n(r)$, i.e., the absolute infinitesimal change in $X_n(r)$ normalized to it's current value.

This makes intuitive sense given the multiplicity law of exponentials and the poor man's limit interpretation of Riemann integrals. To illuminate that statement, think of $\frac{\mathrm{d}\ln X_n(r)}{\mathrm{d}r}\mathrm{d}r$ as, $\delta f(r)\mathrm{d}r$, the relative fraction of change to the function $X_n(r)$ at r by moving a distance $\mathrm{d}r$.

$$\int_a^b \frac{\mathrm{d}\ln X_n(r)}{\mathrm{d}r}\, \mathrm{d}r = \int_a^b \delta f(r)\, \mathrm{d}r \to \lim_{n\to\infty} \sum_{i=0}^{n} \delta f\left(a + i\,\Delta r_n\right) \Delta r_n. \tag{13}$$

Now going back to our definition of $N_e(a, b)$, and isolating the exponential, let's plug in this new interoperation and see what we get. I will drop the $\lim_{n\to\infty}$, as it is implicitly understood and define $\delta f_i = \delta f(a + i\,\Delta r_n)$,

$$\begin{aligned} \frac{X_n(b)}{X_n(a)} = e^{N_e(a,b)} \quad &\to \quad e^{\sum_{i=0}^{n} \delta f_i \Delta r_n}, \\ &= \quad \prod_{i=0}^{n} e^{\delta f_i \Delta r_n}, \\ &= \quad \prod_{i=0}^{n} (1 + \delta f_i \Delta r_n). \end{aligned}$$

Where the last line we taylor expanded the exponential, as in the limit of large n we can ignore higher order terms as $\Delta r_n \propto n^{-1}$. Thus we can see that δf is the fractional change to the quantity $X_n(a)$ for every step Δr_n; hopefully illuminating the logarithmic derivative.

Consider what we have found about this integrand so far, namely it has units of $[\mathrm{d}r]^{-1}$ and is a fractional change. This motivates us to define a reciprocal fractional change "height" as follows[1]

$$-\frac{1}{F_n(r)} \equiv \frac{\mathrm{d}\ln X_n(r)}{\mathrm{d}r} \quad \rightarrow \quad \frac{\mathrm{d}X_n(r)}{\mathrm{d}r} = -\frac{X_n(r)}{F_n(r)}. \tag{14}$$

From this it is clear that $F_n(r)$ is the scale length for an e-folding when $X_n(r)$ is of the form $X_n(r) = X_0 e^{(-r/F_n(r))}$. However, for general $X_n(r)$ this is not the case, and we must fall back on our fractional change interoperation—as is the case for an adiabatic atmosphere.[2] Focusing on our selected atmosphere, we previously noted all quantities are of the form $X_n(r) = X_0 f(r)^n$, where n is some power and X_0 is the quantities value at R_0. Then

$$\frac{\mathrm{d}X_n(r)}{\mathrm{d}r} = nX_0 f(r)^{n-1}\frac{\mathrm{d}f(r)}{\mathrm{d}r} = -nX_0 f(r)^{n-1}\frac{R_c}{r^2} = -\frac{nR_c}{r^2\,f(r)}X_n(r) = -\frac{X_n(r)}{F_n(r)}. \tag{15}$$

Therefore the reciprocal fractional change height is given as $F_n(r) = \frac{1}{n}\frac{r^2\,f(r)}{R_c} = \frac{r}{n}\left(1 - \frac{r}{R_z}\right)$. It is worth noting at this point that $F_n(r)$ is equal to $\gamma H_{\mathrm{iso}}(r)$, when $n = \frac{1}{\gamma-1}$, i.e., the density scale height. This means $F_\rho(r)$ is the scale height of an isothermal atmosphere with local quantities equal to those in our adiabatic atmosphere, except with a sound speed $\sqrt{\gamma}c_{s,0}$!

From the definition of $F_n(r)$ the number of e-foldings between $r = a$ and $r = b$ is given by

$$N_e(a,b) = \int_a^b -\frac{1}{F_n(r)}\mathrm{d}r = n\left[\log(r - R_z) - \log(r)\right]\big|_a^b = n\log\left(\frac{a}{b}\frac{b - R_z}{a - R_z}\right). \tag{16}$$

In general when $F_n(r)$ has r dependence, i.e., $X_n(r)$ is not strictly an exponential decay, then the distance given by $F_n(r)$ is not the distance in which the quantity will drop by one factor of e.[3] If we wanted a physical measure of the scale height, $H(r)$, in which was the actual distance from a given r at which the quantity $X_n(r)$ changes by a factor of s, we would need to solve the equation

$$X_n(r + H_n(r)) = sX_n(r). \tag{17}$$

Where s is the folding factor we are interested in, typically e^{-1} for the e-folding scale. Then

[1] The negative sign is to enforce $F_n(r)$ to be positive for outwardly decreasing functions, such as those in atmospheres

[2] In general we fit two points with an exponential decay, but that by no means is because it actually exponentially decays.

[3] Rather it is the distance at which an isothermal atmosphere, with conditions at r, would drop by a factor of e

$$\begin{aligned}
X_0[f(r+H_n(r))]^n &= sX_0[f(r)]^n \\
\left(\frac{1}{r+H_n(r)} - \frac{1}{R_z}\right) &= s^{\frac{1}{n}}\left(\frac{1}{r} - \frac{1}{R_z}\right) \\
H_n(r) &= \left(s^{\frac{1}{n}}\left(\frac{1}{r} - \frac{1}{R_z}\right) + \frac{1}{R_z}\right)^{-1} - r. \\
H_n(r) &= \frac{(1-s^{\frac{1}{n}})(R_z-r)}{s^{\frac{1}{n}}\left(\frac{R_z}{r}-1\right)+1}. \\
H_n(r) &= \frac{(1-s^{\frac{1}{n}})(R_z-r)}{s^{\frac{1}{n}}\left(R_z-r\right)+r}\, r. \\
H_n(r) &= \frac{R_z-r}{r+(s^{-\frac{1}{n}}-1)^{-1}R_z}\, r.
\end{aligned}$$

Note that as $s \to 0$, then $H_n(r) = R_z - r$, which is what we expect since this is the distance from r to the zero radius. As $s \to \infty$, $H_n(r) = -r$. This tells us that origin is infinitely dense, as one might have already notice.

The next way to think about a scale height will be the most useful for the optical depth, as it will get us exact results. We define the third scale height, $G_n(r)$, as follows

$$\int_r^\infty X_n(r')\,\mathrm{d}r' = X_n(r)G_n(r). \tag{18}$$

The physical meaning of this scale height is stated as, the height of a paralleled atmosphere of constant "density", $X(r)$, of equal "column density" to our actual atmosphere out to infinity.

This is very useful for optical depths, as this is reducing the integral for optical depth into simple function definitions. Unfortunately, for our atmosphere of interest this scale height is not composed of elementary functions. Since this scale height will almost exclusively be useful for column density, we will work with the specific case of $X_n(r) = \rho(r)$, therefore

$$\begin{aligned}
\int_r^\infty \rho(r')\,\mathrm{d}r' &= \rho_0\, R_c^{\frac{1}{\gamma-1}} \int_r^{R_z} \left(\frac{1}{r'} - \frac{1}{R_z}\right)^{\frac{1}{\gamma-1}}\ \mathrm{d}r', \\
&= \rho_0\, R_c^{\frac{1}{\gamma-1}} \int_r^{R_z} (r')^{-\frac{1}{\gamma-1}} \left(1 - \frac{r'}{R_z}\right)^{\frac{1}{\gamma-1}}\ \mathrm{d}r'. \\
\text{Variable substitution } x &= \frac{r'}{R_z}, \\
\int_r^\infty \rho(r')\,\mathrm{d}r' &= \rho_0 \left(\frac{R_c}{R_z}\right)^{\frac{1}{\gamma-1}} R_z \int_{\frac{r}{R_z}}^1 x^{-\frac{1}{\gamma-1}}\,(1-x)^{\frac{1}{\gamma-1}}\ \mathrm{d}x, \\
&= \rho_0 \left(\frac{R_c}{R_z}\right)^{\frac{1}{\gamma-1}} R_z \left[\int_0^1 x^{-\frac{1}{\gamma-1}}\,(1-x)^{\frac{1}{\gamma-1}}\ \mathrm{d}x - \int_0^{\frac{r}{R_z}} x^{-\frac{1}{\gamma-1}}\,(1-x)^{\frac{1}{\gamma-1}}\ \mathrm{d}x\right]. \\
\text{Note that } \beta(x,a,b) &= \int_0^x t^{a-1}(1-t)^{b-1}\mathrm{d}t, \text{ this is known as the incomplete Beta function.} \\
\int_r^\infty \rho(r')\,\mathrm{d}r' &= \rho_0 \left(\frac{R_c}{R_z}\right)^{\frac{1}{\gamma-1}} R_z \Bigg[\beta\left(1, 1-\frac{1}{\gamma-1}, 1+\frac{1}{\gamma-1}\right) \\
&\qquad -\beta\left(\frac{r}{R_z}, 1-\frac{1}{\gamma-1}, 1+\frac{1}{\gamma-1}\right)\Bigg].
\end{aligned}$$

Therefore we can read off $G(r)$, noting $\frac{R_z}{R_0} - 1 = \frac{R_z}{R_c}$,

$$G(r) = \left(\frac{R_z}{r} - 1\right)^{-\frac{1}{\gamma-1}} R_z \left[\beta\left(1-\frac{1}{\gamma-1}, 1+\frac{1}{\gamma-1}\right) - \beta\left(\frac{r}{R_z}, 1-\frac{1}{\gamma-1}, 1+\frac{1}{\gamma-1}\right)\right]. \quad (19)$$

Note the first incomplete Beta function reduces into a regular Beta function. Recalling that $\beta(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x)\Gamma(y)}$. For negative values of x in $\Gamma(x)$, use the recurrence relationship $\Gamma(x-1) = \frac{\Gamma(x)}{x-1}$, which doesn't help for negative integer $x-1$, as you will eventually recurrence to 0 leading to complex infinity. Then, we will want γ not of the form $\frac{n+1}{n}$, $n \in \mathbb{Z}^+$, which rules out relativistic ideal gases with $\gamma = 4/3$.

$$G(r) = \left(\frac{R_z}{r} - 1\right)^{-\frac{1}{\gamma-1}} R_z \left[\frac{\Gamma\left(1-\frac{1}{\gamma-1}\right)\Gamma\left(1+\frac{1}{\gamma-1}\right)}{\Gamma(2)} - \beta\left(\frac{r}{R_z}, 1-\frac{1}{\gamma-1}, 1+\frac{1}{\gamma-1}\right)\right]. \quad (20)$$

For the $\gamma = 5/3$ case

$$G(r) = \left(\frac{R_z}{r} - 1\right)^{-3/2} R_z \left[-\frac{3\pi}{2} - \beta\left(\frac{r}{R_z}, -\frac{1}{2}, \frac{5}{2}\right)\right]. \quad (21)$$

Admittedly this is not a huge improvement over actually numerically doing the integral, as we will now how to numerically evaluate the Beta function; however in principle we have reduced the scale height to "known" functions.

Numerical Stability

To model a hydrostatic adiabatic atmosphere numerical, we will first need to deal with the infinite core quantities, i.e., the quantities $X_n(r) \to \infty$ as $r \to 0$. To resolve this issue, we pick a inner most radius, r_0, under which we hardwire the quantities to $X_n(r_0)$. This immediately breaks the assumption of a hydrostatic adiabatic atmosphere and the atmosphere would evolve non-adiabatically; to prevent this, we artificially force the adiabatic hydrostatic solution from r_0 to r_b, the base of our organic atmosphere. The goal is the have the atmosphere above r_b evolve naturally, while keeping the interior of our atmosphere adiabatic and hydrostatic. This require the number of cells between r_0 and r_b, be at a minimum the number of cell used in your interpolation algorithm.

The values between r_b and r_0 should be thought of as the boundary conditions necessary establish a hydrostatic atmosphere. The values below r_0 are simply set if you cannot turn off the integration over this region, their values are set to keep the "boundary" conditions from drastically evolving over a time step.